Time-harmonic Maxwell's equations in periodic waveguides

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Revised version of October 4, 2024

Abstract: We study Maxwell's equations with periodic coefficients in a closed waveguide. A functional analytic approach is used to formulate and to solve the radiation problem. We furthermore characterize the set of all bounded solutions to the homogeneous problem. The case of a compact perturbation of the medium is included, the scattering problem and the limiting absorption principle are discussed.

MSC: 35Q61

1. Introduction

Maxwell's equations describe electromagnetic waves with the two unknown fields E and H. When the frequency $\omega > 0$ is prescribed, one can use the time-harmonic equations. With two coefficients $\mu = \mu(x)$ (permeability) and $\varepsilon = \varepsilon(x)$ (permittivity), both depending on the spatial position $x \in \mathbb{R}^3$, the system reads

$$\begin{aligned} \operatorname{curl} E &= i\omega\mu\, H + f_h \,, \\ \operatorname{curl} H &= -i\omega\varepsilon\, E + f_e \,. \end{aligned} \tag{1.1}$$

Here, $f_e = f_e(x)$ models prescribed external currents and $f_h = f_h(x)$ a right-hand side in the E-equation. We include f_h for mathematical completeness and allow $\operatorname{div} f_h \neq 0$ in our mathematical analysis below. This will be technically helpful at a later point in the proofs. The inhomogeneities create the fields $E = E(x)$ and $H = H(x)$. We are interested in a waveguide geometry and treat the equations on a domain $\Omega = \mathbb{R} \times S \subset \mathbb{R}^3$, where $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain. We assume that $\varepsilon, \mu \in L^\infty(\Omega, \mathbb{R})$ are bounded from below by some constant $c_0 > 0$ and that ε and μ are 2π-periodic with respect to x_1. Along the exterior boundary, we consider a perfect conductor: $\nu \times E = 0$ on $\partial\Omega$ for the exterior unit normal vector $\nu = \nu(x)$.

1.1. **Literature.** Electromagnetic waves are described by Maxwell's equations, we refer to [14] for background and an overview over mathematical methods. In applications, one can often assume that the temporal frequency of solutions is fixed and uses the ansatz $u(x,t) = u(x)e^{-i\omega t}$. This ansatz leads from Maxwell's equations to the time-harmonic Maxwell system (1.1), just as it leads from the wave equation to the Helmholtz equation. Solutions of both, the time-harmonic Maxwell system and the Helmholtz equation, describe waves in a medium. When the underlying domain in unbounded, one typically has to complement the system with a radiation condition. The physically relevant condition is, loosely speaking, that energy should be transported to infinity.

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The Helmholtz equation is, in some particular situation, a special case of the Maxwell system. It is therefore not surprising that the appearing phenomena and the analytical methods are very similar.

Of particular interest are waveguides with periodic coefficients. An analysis of the corresponding Helmholtz problem, along with the adequate radiation conditions, is given in [10]. We note that a more functional analytic approach for the same problem was developed in [15, 16], simplified proofs and strengthened results can be found in [17]. These approaches are all containing also results on a limiting absorption principle. Even though the constructions are different, the approaches are all based on the Floquet-Bloch transform, we refer to [18, 21] for more material on this technique.

Some closely related publications, still for the Helmholtz equation, are the following: A semi-infinite waveguide is treated in [13], once more, alongside a limiting absorption principle. An existence result that does not use the Floquet-Bloch transform is given in [25] without limiting absorption principle. Another form of the radiation condition in terms of Bloch waves is developed in [20]. A related approach is studied in [7]. In periodic media, domain truncation can be used to derive an equivalent formulation of the problem on a bounded domain, by using Dirichlet-to-Neumann boundary conditions; for this powerful method we mention [9, 11] and refer to references therein.

Regarding the Maxwell system, we are not aware of investigations of the radiation problem in waveguides with periodic media. One subject of research are scattering problems in the time-harmonic setting, see [3, 6], or [2] for a study with Bloch waves. Another subject are numerical methods in waveguide geometries, e.g., in [12]. A nonlinear material (the material cannot be given simply by factors ε and μ) is studied in a geometry $(0, h) \times \mathbb{R}^2$ in [27].

Another topic about the Maxwell system is the regularity of solutions, we refer to [1, 8] for related results. We mention that such topics are closely related to compactness issues, which are very important in our approach.

Finally, there is the large topic of homogenization. In this context, one studies periodic media in the limit that the periodicity length converges to zero. This is an interesting limit which leads to effective theories which can have very surprising features such as negative index materials [4, 19, 22]. A related topic is the question whether or not waves, described by Maxwell's equations, can pass through thin layers of material with small holes, see [5, 26]. We note that these works study either bounded domains or assume boundedness of the solution sequence; in this sense, the results at hand can help to find homogenization limits in more general situations.

Outline of this article. We will show the existence and uniqueness of solutions to (1.1). Our results are based on Assumption 3.1, a non-degeneracy assumption on the frequency; such an assumption is standard in the study of radiation problems. The Maxwell system is written in a weak form, then re-written with the help of the Floquet-Bloch transformation and finally written as an abstract equation in Banach spaces, see (2.10). The latter equation is solved with a functional analytic result of [17], see Theorem 3.3. This abstract theorem can be applied directly to find bounded solutions to (1.1) when the right-hand side satisfies an orthogonality condition, see Theorem 3.5. To solve the problem for general right-hand sides, one has to introduce a radiation condition, which is done in Definition 3.7. The radiation problem is solved in Theorem 3.9 as a quite direct consequence of Theorem 3.5.

Other results are the characterization of bounded solutions to the homogeneous problem, see Theorem 4.1, and the solution of the problem with locally perturbed periodic media, see Theorem 5.2.

1.2. **Variational formulation of the problem.** Let us first make clear how we understand system (1.1). We search for $E, H \in L^2_{\text{loc}}(\Omega, \mathbb{C}^3)$ such that the distributional rotation of H satisfies $\operatorname{curl} H \in L^2_{\text{loc}}(\Omega, \mathbb{C}^3)$. We interpret the first equation of (1.1) in the distributional sense and demand

$$\int_\Omega E \cdot \operatorname{curl} \psi = \int_\Omega [i\omega\mu\, H + f_h] \cdot \psi \qquad \text{for every} \quad \psi \in C^1_c(\bar{\Omega}, \mathbb{C}^3)\,. \tag{1.2}$$

The space of test-functions demands that ψ has compact support. This implies, on the one hand, that ψ is supported on a set $\Omega_R := (-R, R) \times S$ for some $R > 0$. On the other hand, the values on lateral boundaries are free, formula (1.2) therefore encodes also the boundary condition $E \times \nu = 0$ on $\partial\Omega$.

The second equation of (1.1) is interpreted as an equality of L^2_{loc}-functions. In order to illustrate the symmetry in the equations, we note that it is equivalent to:

$$\int_\Omega \operatorname{curl} H \cdot \psi = \int_\Omega [-i\omega\varepsilon\, E + f_e] \cdot \psi \qquad \text{for every} \quad \psi \in C^0_c(\bar{\Omega}, \mathbb{C}^3)\,. \tag{1.3}$$

In contrast to (1.2), the rotation in (1.3) does not act on the test-function; this means that no boundary condition for H is explicitly encoded. Nevertheless, we will see below that the coupled system contains an implicit boundary condition for H.

The unknown E can be eliminated from the equations to obtain a single equation for the only unknown $u := H$. For an arbitrary function $\phi \in C^1_c(\bar{\Omega}, \mathbb{C}^3)$, we use $\psi := \varepsilon^{-1} \operatorname{curl} \bar{\phi}$ in (1.3); the bar over a function denotes complex conjugation. Replacing the integral over $-i\omega E \cdot \operatorname{curl} \bar{\phi}$ with the help of (1.2), we obtain the following weak formulation of system (1.1):

$$\int_\Omega \left\{ \frac{1}{\varepsilon} \operatorname{curl} u \cdot \operatorname{curl} \bar{\phi} - \omega^2 \mu\, u\, \bar{\phi} \right\} = \int_\Omega \left\{ \frac{1}{\varepsilon} f_e \operatorname{curl} \bar{\phi} - i\omega f_h\, \bar{\phi} \right\} \tag{1.4}$$

for every test-function $\phi \in C^1_c(\bar{\Omega}, \mathbb{C}^3)$. We note the following: When (1.4) holds and both u and $\operatorname{curl} u$ are of class $L^2_{\text{loc}}(\Omega, \mathbb{C}^3)$, by density of smooth functions, relation (1.4) holds also when ϕ and $\operatorname{curl} \phi$ are of class $L^2(\Omega, \mathbb{C}^3)$ and have a bounded support.

1.3. **Function spaces.** In the following, we will assume that f_e and f_h are vector fields with good decay properties, more precisely,

$$f_e, f_h \in L^2_*(\Omega, \mathbb{C}^3) := \left\{ f \in L^2(\Omega, \mathbb{C}^3) \,\middle|\, \int_\Omega (1 + x_1^2)^2\, |f(x)|^2 dx < \infty \right\}. \tag{1.5}$$

When we are interested in solutions to (1.4) with a decay for $|x_1| \to \infty$, we seek for

$$u \in H(\operatorname{curl}, \Omega) := \left\{ u \in L^2(\Omega, \mathbb{C}^3) \middle| \operatorname{curl} u \in L^2(\Omega, \mathbb{C}^3) \right\}, \tag{1.6}$$

and demand that (1.4) holds for all ϕ in the same space. When we are interested in radiating solutions, we seek for u in the space

$$H_{\text{loc}}(\operatorname{curl}, \Omega) := \left\{ u : \Omega \to \mathbb{C}^3 \,\middle|\, \forall R > 0 :\ u|_{\Omega_R} \in H(\operatorname{curl}, \Omega_R) \right\}, \tag{1.7}$$

where we used again $\Omega_R = (-R, R) \times S$. In this setting, we demand (1.4) for all $\phi \in H(\operatorname{curl}, \Omega)$ with compact support in $\bar{\Omega}$. The remainder of this text is devoted to these two variants of the variational problem (1.4).

1.4. **Comments on the system.** The reader might be more familiar with the strong form of system (1.1), which can be read off from (1.4):

$$\operatorname{curl}\left(\frac{1}{\varepsilon}\operatorname{curl} H\right) - \omega^2\mu\, H = \operatorname{curl}\left(\frac{1}{\varepsilon} f_e\right) - i\omega f_h\,. \tag{1.8}$$

We include a comment on the implicit boundary conditions for H: Let us assume that the right-hand side f_h is L^2-orthogonal to all gradients; this encodes $\operatorname{div} f_h = 0$ and the boundary condition $f_h \cdot \nu = 0$ on $\partial\Omega$. In this situation, using a gradient $\phi = \nabla p$ as test-function in (1.4), we find that also the function $\mu u = \mu H$ is orthogonal to gradients. This encodes $\operatorname{div}(\mu H) = 0$ in the domain and $\mu H \cdot \nu = 0$ on the boundary. In particular: solving the equations with $f_h = 0$, the solution satisfies automatically the boundary condition $H \cdot \nu = 0$ on $\partial\Omega$.

On the equivalence of the two descriptions: We have shown that every solution of the original problem solves the variational problem (1.4). Vice versa, let $H = u \in H_{loc}(\operatorname{curl}, \Omega)$ be a solution to (1.4). We define the electric field by $E := i(\operatorname{curl} H - f_e)/(\omega\varepsilon)$; with this definition, (1.3) is satisfied. We note that $E \in L^2_{loc}(\Omega, \mathbb{C}^3)$ is satisfied and hence the definition of E together with (1.4) implies (1.2). Additionally, because of $i\omega\mu\, H + f_h \in L^2_{loc}(\Omega, \mathbb{C}^3)$, (1.2) also yields $\operatorname{curl} E \in L^2_{loc}(\Omega, \mathbb{C}^3)$ and we conclude that (1.1) is also solved strongly, i.e., as two equalities in $L^2_{loc}(\Omega, \mathbb{C}^3)$. We note that this implies also that system (1.1) is satisfied pointwise almost everywhere.

It is sometimes convenient to have the boundary condition for E encoded in the function space. We use[1]

$$H_0(\operatorname{curl}, \Omega) := \left\{ E \in H(\operatorname{curl}, \Omega) \,\middle|\, \int_\Omega E \cdot \operatorname{curl} \psi = \int_\Omega \operatorname{curl} E \cdot \psi \quad \forall \psi \in H(\operatorname{curl}, \Omega) \right\}$$

Relation (1.2) is equivalent to: There holds $E \in H_0(\operatorname{curl}, \Omega)$ and the first equation of (1.1) is satisfied.

Our main results regard existence and uniqueness of solutions to (1.4). Our approach is quite similar to the one in [17], where we treated the Helmholtz equation. Since that article is quite detailed and contains proofs of all relevant tools (in particular Floquet-Bloch transformations and the fundamental functional analysis result), we focus here on those aspects of the analysis that are different for Maxwell's equations.

In general, the solution to the radiation problem will not be of class $L^2(\Omega)$. It is therefore not clear how the Floquet-Bloch transform can be helpful in the construction of solutions. Indeed, it can be helpful because of our two-step construction of solutions, constructing first (in a special case) bounded solutions in Theorem 3.5, and then solving the general radiation problem in Theorem 3.9.

2. The Floquet-Bloch transformed equation

2.1. **Application of the Floquet-Bloch transform.** We use the Floquet-Bloch transformation in the x_1-variable. It transforms a function $u = u(x)$, $x \in \mathbb{R} \times S$ into a function $\hat{u}$. The transformed function has two arguments, $\hat{u} = \hat{u}(x, \alpha)$, where x ranges in the periodicity cell, $x \in W := (0, 2\pi) \times S$, and α ranges in a unit interval, $\alpha \in I := [-1/2, 1/2]$. The two arguments are related by the fact that, for every $\alpha \in I$, the map $W \ni x \mapsto \hat{u}(x, \alpha)$ is an α-quasiperiodic function (the definition is given below).

[1] when no confusion with the space of $(\alpha = 0)$-quasiperiodic functions is possible

The transformation is a bounded linear map

$$\mathcal{F}_{\mathrm{FB}} : L^2(\Omega) \to L^2\big(W \times I\big)\,, \qquad u \mapsto \hat{u}\,. \tag{2.1}$$

For smooth functions u with compact support, writing $x = (x_1, \tilde{x})$ for the argument, the transformation is defined by the formula

$$\hat{u}((x_1, \tilde{x}), \alpha) \;:=\; \sum_{\ell \in \mathbb{Z}} u((x_1 + 2\pi\ell, \tilde{x}))\, e^{-i\ell 2\pi\alpha}\,. \tag{2.2}$$

The operator $\mathcal{F}_{\mathrm{FB}}$ is a unitary operator. We recall that [17] contains more details of the construction and proofs.

We have to introduce function spaces that are adapted to Maxwell's equations. We need spaces of periodic and of α-quasiperiodic functions. In order to formulate periodicity on W, we start from

$$\begin{aligned} H_{\mathrm{per,loc}}(\mathrm{curl}, \Omega) := \big\{ u : \Omega \to \mathbb{C}^3 \,\big|\, & u \text{ is } 2\pi\text{-periodic in } x_1\,, \\ & \forall R > 0 :\; u|_{\Omega_R} \in H(\mathrm{curl}, \Omega_R) \big\}\,. \end{aligned} \tag{2.3}$$

This space allows to introduce periodic functions on W,

$$H_{\mathrm{per}}(\mathrm{curl}, W) := \{ u|_W \mid u \in H_{\mathrm{per,loc}}(\mathrm{curl}, \Omega) \}\,, \tag{2.4}$$

and the space of α-quasiperiodic functions

$$H_\alpha(\mathrm{curl}, W) := \big\{ u|_W \;\big|\; [x \mapsto u(x) e^{-i\alpha x_1}] \in H_{\mathrm{per,loc}}(\mathrm{curl}, \Omega) \big\}\,. \tag{2.5}$$

We equip the space $H_\alpha(\mathrm{curl}, W)$ with the inner product

$$\langle u, \phi \rangle_{H_\alpha(\mathrm{curl}, W)} := \left\langle \frac{1}{\varepsilon} \,\mathrm{curl}\, u, \mathrm{curl}\, \phi \right\rangle_{L^2(W)} + \langle \mu u, \phi \rangle_{L^2(W)}\,. \tag{2.6}$$

The Floquet-Bloch transform is an isomorphism from $H(\mathrm{curl}, \Omega)$ to a space that we write as $L^2(I, H_\alpha(\mathrm{curl}, W))$, elements of the latter are maps $w : I \to H(\mathrm{curl}, W)$ with $w(\alpha) \in H_\alpha(\mathrm{curl}, W)$ for almost every $\alpha \in I$.

For functions with decay, this allows an equivalent formulation of the variational problem. A function $u \in H(\mathrm{curl}, \Omega)$ solves (1.4) if, and only if, its transformation satisfies $\hat{u} \in L^2(I, H_\alpha(\mathrm{curl}, W))$ and

$$\begin{aligned} &\int_W \left\{ \frac{1}{\varepsilon} \,\mathrm{curl}\, \hat{u}(\cdot, \alpha) \cdot \mathrm{curl}\, \bar{\phi} - \omega^2 \mu\, \hat{u}(\cdot, \alpha) \cdot \bar{\phi} \right\} \\ &\qquad = \int_W \left\{ \frac{1}{\varepsilon} (\mathcal{F}_{\mathrm{FB}} f_e)(\cdot, \alpha) \cdot \mathrm{curl}\, \bar{\phi} - i\omega (\mathcal{F}_{\mathrm{FB}} f_h)(\cdot, \alpha) \cdot \bar{\phi} \right\} \end{aligned} \tag{2.7}$$

holds for every $\phi \in H_\alpha(\mathrm{curl}, W)$ and for almost every $\alpha \in I$. The proof is as that of Lemma 2.1 in [17]. Let us indicate the relevant calculation with the first term, where we use, in the first equality, the unitarity of $\mathcal{F}_{\mathrm{FB}}$ and, in the second equality, that multiplication with a 2π-periodic function can be taken out of the Floquet-Bloch transformation (compare the definition in (2.2)), and that differential operators (such as curl) commute with the Floquet-Bloch transformation (here, we write $\check{\phi}$ for the test function on Ω and set $\phi = \mathcal{F}_{\mathrm{FB}}(\check{\phi})$):

$$\begin{aligned} \left\langle \frac{1}{\varepsilon} \,\mathrm{curl}\, u, \mathrm{curl}\, \check{\phi} \right\rangle_{L^2(\Omega)} &= \int_I \big\langle \mathcal{F}_{\mathrm{FB}}(\varepsilon^{-1} \,\mathrm{curl}\, u)(\cdot, \alpha), \mathcal{F}_{\mathrm{FB}}(\mathrm{curl}\, \check{\phi})(\cdot, \alpha) \big\rangle_{L^2(W)}\, d\alpha \\ &= \int_I \big\langle \varepsilon^{-1}(\cdot) \,\mathrm{curl}\, \hat{u}(\cdot, \alpha), \mathrm{curl}\, \phi(\cdot, \alpha) \big\rangle_{L^2(W)}\, d\alpha\,. \end{aligned}$$

Since the test-function $\phi(\cdot, \alpha)$ can be chosen arbitrarily, repeating the calculation for the other terms of (1.4), we arrive at (2.7) for almost every α.

2.2. **Re-writing the equation with a family of operators.** We want to write the variational problem (2.7) with the help of operators. Since we want to construct operators that are defined on an α-independent function space, we transform all equations to the space of periodic functions (which is the space corresponding to $\alpha = 0$). We use $X := H_{\text{per}}(\text{curl}, W)$ with the inner product of (2.6).

We note that a function $x \mapsto U(x)$ is α-quasiperiodic in x_1 if, and only if, the function $x \mapsto U(x)e^{-i\alpha x_1}$ is periodic in x_1. Instead of using $\hat{u}(\cdot, \alpha) \in H_\alpha(\text{curl}, W)$ as an unknown, we seek for $v(\cdot) := \hat{u}(\cdot, \alpha)e^{-i\alpha x_1} \in X$.

For a given $v \in X$, the left-hand side of (2.7) (with the replacements $\hat{u}(\cdot, \alpha) = v(\cdot)e^{i\alpha x_1}$ and $\phi(\cdot, \alpha) = \varphi(\cdot)e^{i\alpha x_1}$) defines an anti-linear form in φ, a map $X \to \mathbb{C}$. By the Riesz theorem on Hilbert spaces, this form can be represented by an element $L_\alpha v$ via the scalar product in X. We obtain a bounded linear operator $L_\alpha : X \to X$, defined by the relation

$$\langle L_\alpha v, \varphi \rangle_X = \int_W \left\{ \frac{1}{\varepsilon} \operatorname{curl}(v e^{i\alpha x_1}) \cdot \operatorname{curl} \overline{(\varphi e^{i\alpha x_1})} - \omega^2 \mu \, v \cdot \bar{\varphi} \right\} \tag{2.8}$$

for all $v, \varphi \in X$. Similarly, we represent the right-hand side of (2.7) with an element $y_\alpha \in X$,

$$\langle y_\alpha, \varphi \rangle_X = \int_W \left\{ \frac{1}{\varepsilon} (\mathcal{F}_{\text{FB}} f_e)(\cdot, \alpha) \cdot \operatorname{curl} \overline{(\varphi e^{i\alpha x_1})} - i\omega (\mathcal{F}_{\text{FB}} f_h)(\cdot, \alpha) \cdot \overline{(\varphi e^{i\alpha x_1})} \right\} . \tag{2.9}$$

With these representations, the original problem (1.4) is solved in $H(\text{curl}, \Omega)$ when we find, for almost every $\alpha \in I$, a solution $v(\cdot, \alpha) \in X$ of

$$L_\alpha v(\cdot, \alpha) = y_\alpha \,, \tag{2.10}$$

and if this family of solution satisfies $v \in L^2(I, X)$. This concludes the transformation of the equation, we arrived at a representation as in (2.14) of [17].

So far, we introduced the Floquet-Bloch transform and the abstract formulation of the system, in the next section we will derive existence results. In order to make clear that we indeed construct solutions to the Maxwell system, let us outline the overall procedure of the existence proof:

- Solve the abstract problem (2.10) for almost every $\alpha \in I$. This yields periodic functions $v = v(\cdot, \alpha)$. They are found with abstract functional analysis in Section 3.
- Construct from v the α-quasiperiodic counterparts $\hat{u}(\cdot, \alpha)$. By construction of (2.10), these counterparts solve (2.7).
- The (inverse) Floquet-Bloch transformation of $\hat{u}(\cdot, \alpha)$ provides the solution u to (1.4), see the text after (2.7).

Our next aim is to analyze the properties of the operator L_α. Later on, the dependence on α will be crucial. By contrast, in this section, we study, for a fixed $\alpha \in I$, the operator L_α. The main result will be that L_α is a self-adjoint Fredholm operator with index 0. We recall that ε and μ are also fixed, real, and of class $L^\infty(W)$ with a positive lower bound.

Two equivalent operators. The space of periodic functions is X and the operator on this space is L_α. The space of α-quasiperiodic functions is $\tilde{X}_\alpha := H_\alpha(\text{curl}, W)$, defined in (2.5). Every periodic function $u \in X$ can be transformed to an α-quasiperiodic function $\tilde{u} \in \tilde{X}_\alpha$, defined as $\tilde{u}(x) := u(x)e^{i\alpha x_1}$. This transformation defines an isomorphism between X and $\tilde{X}_\alpha$.

We must check the properties of the operator $L_\alpha : X \to X$ of (2.8). For fixed α, it is actually easier to perform proofs in the space of α-quasiperiodic functions. Indeed, we recall that the operator L_α was actually defined through the transformation of a problem in the space of α-quasiperiodic functions, compare (2.7). The variational problem $\langle L_\alpha u, \varphi\rangle_X = \int_W f \cdot \bar\varphi$ is identical to $\langle \tilde L_\alpha \tilde u, \tilde\varphi\rangle_{\tilde X_\alpha} = \int_W \tilde f \cdot \overline{\tilde\varphi}$ with the functions $\tilde u(x) = u(x)e^{i\alpha x_1}$ and $\tilde f(x) = f(x)e^{i\alpha x_1}$ for the operator $\tilde L_\alpha : \tilde X_\alpha \to \tilde X_\alpha$ defined by

$$\langle \tilde L_\alpha \tilde u, \tilde\varphi\rangle_{\tilde X_\alpha} = \int_W \left\{ \frac{1}{\varepsilon} \operatorname{curl} \tilde u \cdot \operatorname{curl} \overline{\tilde\varphi} - \omega^2 \mu\, \tilde u \cdot \overline{\tilde\varphi} \right\} \tag{2.11}$$

for all $\tilde u, \tilde\varphi \in \tilde X_\alpha$.

We claim that the operator L_α is a Fredholm operator if, and only if, $\tilde L_\alpha$ is Fredholm, and that the index of the two operators coincides. Indeed, when the kernel of L_α is spanned by $u_1, ..., u_M$, then the transformed functions defined as $\tilde u_m(x) := u_m(x)e^{i\alpha x_1}$ span the kernel of $\tilde L_\alpha$, and vice versa. Similarly, when $u_1, ..., u_M$ span a complement of $L_\alpha(X)$, then the transformed functions span a complement of $\tilde L_\alpha(\tilde X_\alpha)$.

Lemma B.1 of Appendix B analyzes the operator $\tilde L_\alpha$ of (2.11) and provides the following result.

Proposition 2.1 (Properties of L_α). *Let $\varepsilon, \mu : \Omega \to \mathbb{R}$ be as described after* (1.1). *For $\alpha \in I = [-1/2, 1/2]$ we consider, on the space $X = H_{\mathrm{per}}(\mathrm{curl}, W)$, the operator $L_\alpha : X \to X$ of* (2.8). *Then L_α is a self-adjoint Fredholm operator with index* 0.

3. Existence of solutions to the Maxwell radiation problem

From now on and until the start of Subsection 3.6, we consider the homogeneous equations, i.e., $f_e = f_h = 0$. We recall that ε and μ are real and have a positive lower bound.

3.1. Physical conservation laws: Poynting vector and energy flux.

We provide a sloppy exposition of physical background. We emphasize that our mathematical analysis is independent of physical arguments. The mathematics is based on an integration by parts which yields that the flux quantity in (3.3) is independent of η.

Physically, for a solution (E, H) of the Maxwell system, the Poynting vector $P := \frac{1}{2} E \times \bar H$ describes the flow of energy. More precisely, through a cross-section $\Gamma_r := \{r\} \times S$ of the waveguide with normal vector $e_1 = (1, 0, 0) \in \mathbb{R}^3$, the quantity $\frac{1}{2} \operatorname{Re} \int_{\Gamma_r} (E \times \bar H) \cdot e_1$ is the energy flux through Γ_r. This motivates to study, for a position $r \in \mathbb{R}$ and a solution H, the real-valued flux quantity

$$\mathcal{F}_r \ := \ -2 \operatorname{Im} \int_{\Gamma_r} \frac{1}{\varepsilon} (\operatorname{curl} H \times \bar H) \cdot e_1 \,. \tag{3.1}$$

Conservation of energy is reflected by the fact that the flux $\mathcal{F}_r$ is independent of the position r.

Let us sketch the argument for smooth coefficients μ and ε and a classical solution H of the Maxwell system (1.4) (below, we provide rigorous derivations for weak solutions): For two positions $-\infty < s < r < \infty$ and the domain $\Omega_{s,r} := (s, r) \times S$ with characteristic function χ we use the test-function $\phi = u\,\chi$ in (1.4). Assuming that s and r are chosen so that f_e and f_h vanish in $\Omega_{s,r}$, the right-hand side vanishes. On the left-hand side we integrate by parts and take the imaginary part. The bulk term is real because of the strong equation $\operatorname{curl}(\varepsilon^{-1} \operatorname{curl} H) - \omega^2 \mu H = 0$, whence the imaginary part of this term vanishes. There remain only boundary terms, they coincide with $\mathcal{F}_r - \mathcal{F}_s$. The result of this calculation is that $\mathcal{F}_r$ is independent of r.

Our next aim is to prove this fact in a generalized setting for weak solutions.

3.2. **Generalization and weak description.** We consider two weak solutions u and v to the homogeneous problem, i.e., both u and v solve

$$\int_\Omega \left\{ \frac{1}{\varepsilon} \operatorname{curl} u \cdot \operatorname{curl} \bar{\phi} - \omega^2 \mu\, u\, \bar{\phi} \right\} = 0 \qquad \text{for every} \quad \phi \in H^1_{\mathrm{cpt}}(\Omega)\,. \tag{3.2}$$

We consider an arbitrary weight function, $\eta : \mathbb{R} \to \mathbb{R}$ bounded with compact support and with $\int_\mathbb{R} \eta = 1$. We interpret η also as a function on Ω by setting $\eta(x) := \eta(x_1)$. We introduce the weighted average of the complex flux,

$$\mathcal{F}_{u,v} \;:=\; i \int_\Omega \eta\, \frac{1}{\varepsilon}\, [(\operatorname{curl} u \times \bar{v}) - (\operatorname{curl} \bar{v} \times u)] \cdot e_1\,. \tag{3.3}$$

Conservation of energy is now reflected by the fact that the expression $\mathcal{F}_{u,v}$ is independent of the weight η.

Let us prove this important observation by considering two weight functions η_1 and η_2 as above. The difference $\eta_1 - \eta_2$ satisfies $\int_\mathbb{R} (\eta_1 - \eta_2) = 0$. By the latter property, the primitive of the difference, defined as $\vartheta(t) := \int_{-\infty}^t (\eta_1 - \eta_2)(s)\, ds$, is a Lipschitz function with compact support. Once more, we interpret ϑ also as a function on Ω by setting $\vartheta(x) := \vartheta(x_1)$.

For the solution u of (3.2), we use the test-function $\phi = v\vartheta$; the result is of the form $I_u = 0$ for some integral I_u that contains u (and v and ϑ). Independently, we consider the solution v of (3.2) and use the test-function $\phi = u\vartheta$. The result is of the form $I_v = 0$ for some other integral I_v that contains v (and u and ϑ). Up to a complex conjugation, the interesting equation is obtained by taking the difference, we consider $I_u - \bar{I}_v = 0$.

In the expression $I_u - \bar{I}_v$ all terms without derivatives of ϑ cancel. There remain only terms that contain a derivative of $\vartheta(x)$, which is $\nabla\vartheta(x) = (\eta_1 - \eta_2)(x)\, e_1$. The result is the expression of (3.3) vanishes when η is replaced by $\eta_1 - \eta_2$. This proves that $\mathcal{F}_{u,v}$ is indeed independent of η.

Motivated by this calculation, we introduce two sesquilinear forms on functions $u, \phi \in H(\operatorname{curl}, W)$ for $W = (0, 2\pi) \times S$:

$$Q(u, \phi) \;:=\; i \int_W \frac{1}{\varepsilon} \left[(\operatorname{curl} u \times \bar{\phi}) - (\operatorname{curl} \bar{\phi} \times u) \right] \cdot e_1\,. \tag{3.4}$$

The form Q is hermitian, since ε is real. We emphasize that the arguments of Q are not necessarily periodic functions; indeed, we will typically use α-quasiperiodic functions as arguments. Solutions u, v of (3.2) satisfy

$$Q(u, v) \;=\; 2\pi \mathcal{F}_{u,v}\,. \tag{3.5}$$

3.3. **Non-degeneracy of Q.** We will impose the following physically meaningful assumption. For arbitrary $\alpha \in I = [-1/2, 1/2]$, we define the space of α-quasiperiodic solutions to the homogeneous problem as

$$Y^\alpha := \{ \phi \in H_\alpha(\operatorname{curl}, W) \,|\, \phi \text{ satisfies (1.4) for } f_h = f_e = 0 \}\,. \tag{3.6}$$

We emphasize that every element in $\phi \in H_\alpha(\operatorname{curl}, W)$ can be extended to an α-quasiperiodic function on Ω by demanding, for every $x_1 \in (0, 2\pi)$ and every $k \in 2\pi\mathbb{Z}$ that $\phi(x_1 + k, x_2, x_3) = \phi(x_1, x_2, x_3)\, e^{i\alpha k}$. This was actually the basis of the definition in (2.5). We will oftentimes identify a function in $H_\alpha(\operatorname{curl}, W)$ with its quasiperiodic extension to Ω.

Of special interest are those numbers $\alpha \in I$ (quasimomenta) that are related to non-trivial quasiperiodic solutions to the homogeneous Maxwell problem on Ω. We collect these critical values of α in the set

$$\mathcal{A} \;:=\; \{\alpha \in [-1/2, 1/2] \mid Y^\alpha \neq \{0\}\}\,. \tag{3.7}$$

Assumption 3.1 (Energy transport of quasiperiodic solutions)**.** *We assume that, for every $\alpha \in \mathcal{A}$ and every $0 \neq \phi \in Y^\alpha$, the map $Q(\cdot, \phi) : Y^\alpha \to \mathbb{C}$ does not vanish identically.*

To clearify the requirement: We demand that, for every $\alpha \in \mathcal{A}$ and every non-vanishing $\phi \in Y^\alpha$, there exists $\psi \in Y^\alpha$ such that $Q(\psi, \phi) \neq 0$. The assumption is satisfied when every (non-trivial) quasiperiodic homogeneous solution to the Maxwell system is transporting energy, either to the left or to the right.

Comments on Assumption 3.1. We say that a frequency ω is an exceptional frequency when Assumption 3.1 is not satisfied for ω. We conjecture that the set of exceptional frequencies is a countable set. When ω is an exceptional frequency, our methods do not work; whether or not existence can be shown nonetheless is not clear. On the other hand: Simple scalar examples show that a limiting absorption principle does not hold for an exceptional frequency.

By definition of L_α, there holds: $v \in \ker L_\alpha$ implies $\bar{v} \in \ker L_{-\alpha}$. In particular, the following symmetry holds: When α is a critical value, also $-\alpha$ is a critical value.

The value $\alpha = 1/2$ is special: A function U in $H_\alpha = H_{1/2}$ is antisymmetric in the sense that $U|_{x_1=2\pi} = -U|_{x_1=0}$. For $\alpha = 1/2$, the two spaces H_α and $H_{-\alpha}$ coincide, $\ker L_\alpha$ and $\ker L_{-\alpha}$ represent the same space of antisymmetric solutions. In order not to count these solutions twice, we introduce the (possibly) reduced set

$$\mathcal{A}_* \;:=\; \mathcal{A} \cap (-1/2, 1/2]\,. \tag{3.8}$$

We note in passing that $Q(v, v) = -Q(\bar{v}, \bar{v})$ implies that, when v is a right-going wave, $\bar{v}$ is a left-going wave, and vice versa. Under our assumption, the number of right-going waves is therefore the same as the number of left-going waves.

3.4. **The derivative of L_α.** For the space $X = H_{\mathrm{per}}(\mathrm{curl}, W)$ and $L_\alpha : X \to X$ of (2.8) we can compute the derivative of L_α with respect to α.

Lemma 3.2 (Derivative of L_α)**.** *The derivative $\partial_\alpha L_\alpha : X \to X$ is given by Q in the sense that*

$$\langle \partial_\alpha L_\alpha v, \varphi\rangle_X = Q\left(v\, e^{i\alpha x_1}, \varphi\, e^{i\alpha x_1}\right) \tag{3.9}$$

for every $v, \varphi \in X$. In particular, when Assumption 3.1 holds, the derivative $\partial_\alpha L_\alpha$ is non-degenerate on the kernel of L_α for every $\alpha \in \mathcal{A}$.

Proof. For arbitrary $v, \varphi \in X$ we can calculate

$$\begin{aligned}
\langle \partial_\alpha L_\alpha v, \varphi\rangle_X &= \partial_\alpha \langle L_\alpha v, \varphi\rangle_X \\
&= \partial_\alpha \int_W \frac{1}{\varepsilon}\,\mathrm{curl}(v e^{i\alpha x_1}) \cdot \mathrm{curl}\,\overline{(\varphi e^{i\alpha x_1})} - \omega^2 \mu\, v \cdot \bar{\varphi} \\
&= i \int_W \frac{1}{\varepsilon}\left\{\mathrm{curl}(v\, x_1\, e^{i\alpha x_1}) \cdot \mathrm{curl}\,\overline{(\varphi\, e^{i\alpha x_1})} - \mathrm{curl}(v\, e^{i\alpha x_1}) \cdot \mathrm{curl}\,\overline{(\varphi\, x_1\, e^{i\alpha x_1})}\right\} \\
&= i \int_W \frac{1}{\varepsilon}\left\{(e_1 \times v\, e^{i\alpha x_1}) \cdot \mathrm{curl}\,\overline{(\varphi\, e^{i\alpha x_1})} - (e_1 \times \overline{\varphi\, e^{i\alpha x_1}}) \cdot \mathrm{curl}(v\, e^{i\alpha x_1})\right\} \\
&= Q\left(v\, e^{i\alpha x_1}, \varphi\, e^{i\alpha x_1}\right)\,.
\end{aligned}$$

This provides (3.9). $\square$

3.5. **Functional analysis.** When Assumption 3.1 is satisfied, Proposition 2.1 provides a Fredholm property of the family $(L_\alpha)_\alpha$, and Lemma 3.2 implies the non-degeneracy of the α-derivatives. These properties imply that the family $(L_\alpha)_\alpha$ is a *regular C^1-family of operators* in the sense of Definition 3.1 of [17]. For regular C^1-families of operators, the subsequent result was shown in [17]. We use the same notation, with only one minimal change: We have to replace the variable $\varepsilon > 0$ by $\delta > 0$, because in Maxwell's equations, ε is the name of the permittivity. We therefore write $I_\delta := (-1/2-\delta, 1/2+\delta)$ for the enlarged unit interval. Theorem 3.2 of [17] provides:

Theorem 3.3 (Functional analysis)**.** *Let Assumption 3.1 hold. The family L_α has the following properties.*

1. *The set of critical numbers is finite: For a number $J \in \mathbb{N}$ (we allow $J = 0$ for an empty set $\mathcal{A}_*$) and values $\{\alpha_j \,|\, 0 < j \le J\}$ holds*

$$\mathcal{A}_* \;=\; \{\alpha \in (-1/2, 1/2] \,|\, \ker(L_\alpha) \neq \{0\}\} \;=\; \{\alpha_j \,|\, 0 < j \le J\}\,. \tag{3.10}$$

2. *For some $\delta > 0$, let $I_\delta \ni \alpha \mapsto y_\alpha$ be a C^1-family of right-hand sides with the property that $y_\alpha \in L_\alpha(X)$ holds for every $\alpha \in I_\delta$ with* $\ker(L_\alpha) \neq \{0\}$. *Then the family of solutions*

$$I_\delta \setminus \mathcal{A} \ni \alpha \mapsto u_\alpha := (L_\alpha)^{-1}(y_\alpha)$$

can be continued to a C^0-family on I_δ. For some constant $C > 0$, which is independent of the family $(y_\alpha)_\alpha$, there holds

$$\sup_{\alpha \in I} \|u_\alpha\|_X \;\le\; C \sup_{\alpha\in I} \big[\|y_\alpha\|_X + \|\partial_\alpha y_\alpha\|_X\big]\,. \tag{3.11}$$

The theorem implies that there exists a finite number of quasimomenta $(\alpha_j)_{1\le j\le J}$, corresponding to propagative wave numbers. They are characterized by the fact that the kernel $\ker(L_{\alpha_j}) = \{\varphi \,|\, L_{\alpha_j}\varphi = 0\}$ is not trivial. The kernels are finite dimensional. We introduce, for every $0 < j \le J$, the space of α_j-quasiperiodic propagating modes

$$Y_j \;:=\; \big\{\phi \in H_{\alpha_j}(\mathrm{curl}, W) \,|\, \phi \text{ satisfies (1.4) for } f_h = f_e = 0\big\}\,. \tag{3.12}$$

As explained after (3.6), we consider elements in Y_j also as α_j-quasiperiodic functions on Ω. There holds $\ker(L_{\alpha_j}) = \big\{\phi e^{-i\alpha_j x_1} \,|\, \phi \in Y_j\big\}$, we denote the dimension by $m_j := \dim Y_j = \dim\ker(L_{\alpha_j})$. In every space Y_j we choose a basis $\{\phi_{1,j}, \dots, \phi_{m_j,j}\} \subset Y_j$ as follows: We fix an inner product $\langle\cdot,\cdot\rangle_{Y_j}$ and consider the self-adjoint eigenvalue problem to find $\lambda \in \mathbb{R}$ and $0 \neq \phi \in Y_j$ such that

$$Q(\phi, \cdot) \;=\; \lambda\, \langle \phi, \cdot\rangle_{Y_j}\,.$$

We denote the eigenvalues by $\lambda_{\ell,j}$, $\ell = 1, \dots, m_j$, and the eigenfunctions by $\phi_{\ell,j}$, $\ell = 1, \dots, m_j$, normalized such that

$$\langle \phi_{\ell,j}, \phi_{\ell',j}\rangle_{Y_j} \;=\; \delta_{\ell,\ell'}\,, \quad \ell, \ell' = 1, \dots, m_j\,.$$

The following lemma translates Lemma 3.4 of [17] to the Maxwell system. We recall the short and simple proof.

Lemma 3.4 (Orthogonality)**.** *The spaces Y_j are orthogonal with respect to the form Q: There holds $Q(u, v) = 0$ for $u \in Y_j$ and $v \in Y_{j'}$ whenever $j \neq j'$.*

Proof. We evaluate the expression $\mathcal{F}_{u,v}$ of (3.3) for two weights $\eta_1(\cdot) = \eta(\cdot)$ and $\eta_2(\cdot) = \eta(\cdot + 2\pi e_1)$. We evaluate $\mathcal{F}_{u,v}$ with the weight η_2 by a substitution, using that u is α_j quasiperiodic and v is $\alpha_{j'}$ quasiperiodic. The fact that $\mathcal{F}_{u,v}$ is independent of the weight η implies $\mathcal{F}_{u,v} = e^{2\pi i\alpha_j} e^{-2\pi i\alpha_{j'}} \mathcal{F}_{u,v}$. This implies $\mathcal{F}_{u,v} = 0$ and thus $Q(u,v) = 0$. □

In order to shorten formulas in the subsequent text, we re-arrange the basis functions into a set $\{\phi_\ell \,|\, \ell = 1,\dots,L\}$ where $L := \sum_{j\in J} m_j$. In the following, when we write ϕ with a single index as in the expression "ϕ_ℓ", we refer to one of the above functions $\phi_{k,j}$. With the corresponding eigenvalues λ_ℓ we can formulate: Assumption 3.1 is satisfied if, and only if, $Q(\phi_\ell,\phi_\ell) = \lambda_\ell \neq 0$ for all $\ell = 1,\dots,L$. Lemma 3.4 provides, in particular, that the family of functions $(\phi_\ell)_\ell$ is linearly independent.

3.6. **Solutions with decay,** $u \in H(\mathrm{curl},\Omega)$**.** We can now formulate the existence and uniqueness result in $H(\mathrm{curl},\Omega)$. This result gives the unique existence of a decaying solution for right-hand sides that satisfy an orthogonality condition.

Theorem 3.5 (Existence and uniqueness of $H(\mathrm{curl},\Omega)$-solutions)**.** *Let S, ε, μ, ω be as described after* (1.1) *and let Assumption 3.1 hold. We assume that the right-hand side functions $f_e, f_h \in L^2_*(\Omega,\mathbb{C}^3)$ are orthogonal to the propagating modes $(\phi_\ell)_\ell$ in the following sense:*

$$\int_\Omega \left\{ \frac{1}{\varepsilon} f_e \cdot \mathrm{curl}\, \overline{\phi_\ell} - i\omega f_h \cdot \overline{\phi_\ell} \right\} = 0 \quad \text{for every} \quad \ell = 1,\dots,L\,. \tag{3.13}$$

Then there exists a unique solution $u \in H(\mathrm{curl},\Omega)$ of (1.4)*. There exists a constant $C > 0$, independent of f_e and f_h, such that $\|u\|_{H(\mathrm{curl},\Omega)} \le C\left(\|f_e\|_{L^2_*(\Omega)} + \|f_h\|_{L^2_*(\Omega)}\right)$.*

Proof. Step 1: Solvability. We use L_α of (2.8) and y_α of (2.9). We solve, for $\alpha \in I$, the family of equations $L_\alpha v(\cdot,\alpha) = y_\alpha$ of (2.10). This is done with Theorem 3.3, Part 2., the only point that we have to verify is $y_\alpha \in L_\alpha(X)$. This is clear for every $\alpha \notin \mathcal{A}$ since, in this case, $L_\alpha(X) = X$. It remains to check it for the critical values α_j. Once this is checked, estimate (3.11) yields also that the solutions satisfy $v \in L^2(I,X)$, even $v \in L^\infty(I,X)$.

For fixed $j \le J$ and fixed $\ell \le m_j$, we consider the basis function $\varphi_{\ell,j} \in \ker(L_{\alpha_j})$ and the α_j-quasiperiodic function $\phi_{\ell,j}(x) := \varphi_{\ell,j}(x) e^{i\alpha_j x_1} \in Y_j$. For notational convenience, we perform the calculations for the case $f_h = 0$. Starting with the orthogonality information (3.13), we find

$$\begin{aligned}
0 &= \int_\Omega \frac{1}{\varepsilon} f_e \cdot \mathrm{curl}\, \overline{\phi_{\ell,j}} \\
&= \sum_{m\in\mathbb{Z}} \int_W \varepsilon(x)^{-1} f_e(x + 2\pi m e_1)\, \mathrm{curl}\, \overline{\phi_{\ell,j}(x + 2\pi m e_1)}\, dx \\
&= \sum_{m\in\mathbb{Z}} \int_W \varepsilon(x)^{-1} f_e(x + 2\pi m e_1)\, e^{-i2\pi\alpha_j m}\, \mathrm{curl}\, \overline{\phi_{\ell,j}(x)}\, dx \\
&= \int_W \varepsilon(x)^{-1} (\mathcal{F}_{\mathrm{FB}} f_e)(x,\alpha_j)\, \mathrm{curl}\, \overline{\phi_{\ell,j}(x)}\, dx \\
&= \langle y_{\alpha_j}, \varphi_{\ell,j} \rangle_X\,,
\end{aligned}$$

where the last step uses the definition of y_{α_j}. The calculation provides that y_{α_j} is orthogonal to the kernel of L_{α_j}. Since L_{α_j} is a self-adjoint Fredholm operator with index 0 by Proposition 2.1, we conclude that y_{α_j} is in the range of L_{α_j} (kernel and range are orthogonal for self-adjoint operators, the vanishing index implies that kernel and range span the entire space). This yields $y_\alpha \in L_\alpha(X)$.

The result $y_\alpha \in L_\alpha(X)$ for $\alpha = 1/2$ implies the same result for $\alpha = -1/2$, because the operators L_α and the right-hand sides y_α coincide. To have the family y_α defined for all $\alpha \in \mathbb{R}$, one can extend the family periodically outside $I = [-1/2, 1/2]$. The calculations for $f_h \neq 0$ are completely analogous.

Step 2: Bounds on the solution. Theorem 3.3 implies a bound (uniform in α) for $v_\alpha := v(\cdot, \alpha)$, namely $\sup_\alpha \|v_\alpha\|_X \leq C \sup_\alpha [\|y_\alpha\|_X + \|\partial_\alpha y_\alpha\|_X]$. We note that

$$\begin{aligned} \|y_\alpha\|_X &= \sup\{\langle y_\alpha, \phi\rangle_X \mid \|\phi\|_X = 1\} \\ &\leq C\left(\|(\mathcal{F}_{\rm FB} f_e)(\cdot,\alpha)\|_{L^2(W)} + \|(\mathcal{F}_{\rm FB} f_h)(\cdot,\alpha)\|_{L^2(W)}\right) \end{aligned}$$

and

$$\begin{aligned} \|\partial_\alpha y_\alpha\|_X &= \sup\left\{\frac{d}{d\alpha}\langle y_\alpha, \phi\rangle_X \,\middle|\, \|\phi\|_X = 1\right\} \\ &\leq C \sum_{\#\in\{e,h\}} \left(\|(\mathcal{F}_{\rm FB} f_\#)(\cdot,\alpha)\|_{L^2(W)} + \left\|\frac{\partial}{\partial\alpha}(\mathcal{F}_{\rm FB} f_\#)(\cdot,\alpha)\right\|_{L^2(W)}\right). \end{aligned}$$

As in [17], Theorem 3.8, these terms can be estimated by $\sum_{\#\in\{e,h\}} \|f_\#\|_{L^2_*(\Omega)}$; the argument for this fact exploits that good decay properties of $f_\#$ imply good regularity properties of $\mathcal{F}_{\rm FB} f_\#$. We can therefore define u by the inverse Floquet-Bloch transform,

$$u(x + 2\pi\ell e_1) := \int_I v(x,\alpha)\, e^{i(x_1 + 2\pi\ell)\alpha}\, d\alpha\,, \quad x \in W\,,\ \ell \in \mathbb{Z}\,.$$

As discussed above, $u \in H(\mathrm{curl}, \Omega)$ is a solution of (1.4). Furthermore, since $\mathcal{F}_{\rm FB}$ is an isometry,

$$\|u\|^2_{H(\mathrm{curl},\Omega)} = \|v\|^2_{L^2(I,X)} \leq \sup_\alpha \|v(\cdot,\alpha)\|^2_X \leq C\left(\|f_e\|^2_{L^2_*(\Omega)} + \|f_h\|^2_{L^2_*(\Omega)}\right).$$

This provides the a priori estimate. □

3.7. **The radiation problem.** In order to formulate the radiation problem, we introduce cut-off functions ρ_+ and ρ_-.

Definition 3.6 (Cut-off functions $\rho_\pm$)**.** *We say that $\rho_+, \rho_- : \mathbb{R} \to \mathbb{R}$ are admissible cut-off functions when they are Lipschitz-continuous, satisfy $\rho_\pm(x_1) \in [0,1]$ for every $x_1 \in \mathbb{R}$, and the limiting behavior is*

$$\begin{aligned} &\rho_+(x_1) \to 1 \ \textit{ for } \ x_1 \to \infty\,, \quad \rho_+(x_1) \to 0 \ \textit{ for } \ x_1 \to -\infty\,, \\ &\rho_-(x_1) \to 0 \ \textit{ for } \ x_1 \to \infty\,, \quad \rho_-(x_1) \to 1 \ \textit{ for } \ x_1 \to -\infty\,. \end{aligned}$$

Moreover, we demand specific decay properties: $1-\rho_+ \in L^2(\mathbb{R}_{>0})$ and $\rho_+ \in L^2(\mathbb{R}_{<0})$, and, analogously, $1 - \rho_- \in L^2(\mathbb{R}_{<0})$ and $\rho_- \in L^2(\mathbb{R}_{>0})$. Additionally, for the derivative, we demand $\partial_{x_1}\rho_\pm \in L^2(\mathbb{R})$.

We fix admissible cut-off functions $\rho_\pm$ as in Definition 3.6. For every $\ell \leq L$, we say that the mode ϕ_ℓ is right-going when $Q(\phi_\ell, \phi_\ell) > 0$, we say that it is left-going when $Q(\phi_\ell, \phi_\ell) < 0$. Note that, when Q is non-degenerate, these are the only possible cases. For every ℓ such that ϕ_ℓ is right-going, we set $\rho_\ell := \rho_+$, and for every ℓ for which ϕ_ℓ is left-going, we set $\rho_\ell := \rho_-$. As with other functions in one real variable, we regard also $\rho_\pm$ as functions on Ω through $\rho_\pm(x) := \rho_\pm(x_1)$.

Definition 3.7 (Propagating part and radiation condition)**.** *For fixed cut-off functions ρ_ℓ as above, we introduce the following decomposition of solutions u.*

(i) Propagating part. *For complex coefficients $(a_\ell)_{1\leq\ell\leq L}$, we say that*

$$u^{\rm prop} := \sum_{\ell=1}^{L} a_\ell\, \rho_\ell\, \phi_\ell \tag{3.14}$$

is the propagating wave function corresponding to $a \in \mathbb{C}^L$*.*

(ii) Radiation condition. *We say that a solution* $u \in H_{\mathrm{loc}}(\mathrm{curl}, \Omega)$ *of* (1.4) *satisfies the radiation condition, when there exists* $a \in \mathbb{C}^L$ *such that, with the corresponding propagating wave function* u^{prop} *of* (3.14)*, there holds*

$$u^{\mathrm{rad}} := u - u^{\mathrm{prop}} \in H(\mathrm{curl}, \Omega)\,. \tag{3.15}$$

Some comments on the choice of the scalar product on Y_j are appropriate at this point. Recall that we chose an inner product on Y_j after equation (3.12). The basis functions $(\phi_\ell)_\ell$ and, hence, the radiation condition of Definition 3.7, depend on the choice of this scalar product. In general, also the solution depends on the scalar product; this is in agreement with physics since different limiting absorption settings lead, in general, to different radiation conditions and different solutions. This is discussed in the scalar case in [17], see also Subsection 6.1.

Remark 3.8 (Choice of cut-off functions)**.** *Below we show, for fixed cut-off functions* $\rho_\pm$*, the existence and uniqueness of a radiating solution. Nevertheless, when* $\rho_\pm$ *is replaced by another admissible pair* $\tilde\rho_\pm$*, the coefficients* $(a_\ell)_\ell$ *remain unchanged as can be seen with formula* (3.17)*.*

For the two corresponding solutions $u = u^{\mathrm{rad}} + u^{\mathrm{prop}}$ *and* $\tilde u = \tilde u^{\mathrm{rad}} + \tilde u^{\mathrm{prop}}$*, the difference is a solution to the homogeneous problem and it satisfies* $u - \tilde u = (u^{\mathrm{rad}} - \tilde u^{\mathrm{rad}}) + (u^{\mathrm{prop}} - \tilde u^{\mathrm{prop}}) = (u^{\mathrm{rad}} - \tilde u^{\mathrm{rad}}) + \sum_{\ell=1}^{L} a_\ell\,(\rho_\ell - \tilde\rho_\ell)\,\phi_\ell \in H(\mathrm{curl}, \Omega)$*. We can therefore conclude* $u = \tilde u$ *from Theorem 3.5.*

Without loss of generality we can therefore choose in the following $\rho_\pm$ *such that, for some* $R > 0$*, there holds* $\partial_{x_1}\rho_\pm(x_1) = 0$ *for* $|x_1| > R$*.*

Theorem 3.9 (Existence and uniqueness of solutions to the radiation problem)**.** *Let* S*,* ω*,* ε*,* μ *be as described after* (1.1) *and let* f_e *and* f_h *be as in* (1.5)*. Let Assumption 3.1 be satisfied and let admissible cut-off functions* $(\rho_\ell)_\ell$ *be chosen as in Definition 3.6. Then* (1.4) *has a unique solution* $u \in H_{\mathrm{loc}}(\mathrm{curl}, \Omega)$ *satisfying the radiation condition of Definition 3.7. For a positive constant* $C = C(S, \varepsilon, \mu, \omega, \rho_\pm)$ *the solution components* u^{prop}*,* u^{rad}*, and* $a \in \mathbb{C}^L$ *of the radiation condition satisfy*

$$\|u^{\mathrm{rad}}\|_{H(\mathrm{curl},\Omega)} + \|u^{\mathrm{prop}}|_W\|_{H(\mathrm{curl},W)} + \|a\|_{\mathbb{C}^L} \le C\,\left(\|f_e\|_{L^2_*(\Omega)} + \|f_h\|_{L^2_*(\Omega)}\right)\,. \tag{3.16}$$

The coefficients $(a_\ell)_{1\le\ell\le L}$ *are given by*

$$a_\ell \;=\; \frac{2\pi i}{|Q(\phi_\ell,\phi_\ell)|}\left(\langle \varepsilon^{-1} f_e, \mathrm{curl}\,\phi_\ell\rangle_{L^2(\Omega)} - \langle i\omega f_h, \phi_\ell\rangle_{L^2(\Omega)}\right)\,. \tag{3.17}$$

Proof. The strategy of the proof is very direct: We want to apply Theorem 3.5 to find u^{rad} as described in the radiation problem. In order to apply Theorem 3.5, we must make sure that the right-hand side of the u^{rad}-problem satisfies the orthogonality condition (3.13). As mentioned in Remark 3.8, we can assume that the cut-off functions $\rho_\pm$ are chosen such that they are constant outside of some bounded interval, i.e., the functions ρ_ℓ' have compact support.

We want to find coefficients $a = (a_\ell)_\ell$, such that, with u^{prop} defined by a, the equation for u^{rad} has a right-hand side that satisfies the orthogonality condition. We will see that the coefficients a of (3.17) have this property.

Step 1: Equation for u^{rad}. We derive, for fixed coefficients $(a_\ell)_\ell$, the equation for $u^{\mathrm{rad}} = u - u^{\mathrm{prop}} = u - \sum_{\ell=1}^{L} a_\ell\,\rho_\ell\,\phi_\ell$. Equation (1.4) yields, for ϕ with compact support,

$$\int_\Omega \left\{ \frac{1}{\varepsilon} \operatorname{curl} u^{\mathrm{rad}} \cdot \operatorname{curl} \bar\phi - \omega^2 \mu\, u^{\mathrm{rad}} \cdot \bar\phi \right\}$$
$$= - \sum_{\ell=1}^L a_\ell \int_\Omega \left\{ \frac{1}{\varepsilon} \operatorname{curl}(\rho_\ell\, \phi_\ell) \cdot \operatorname{curl} \bar\phi - \omega^2 \mu\, \rho_\ell\, \phi_\ell \cdot \bar\phi \right\}$$
$$+ \int_\Omega \left\{ \frac{1}{\varepsilon} f_e \cdot \operatorname{curl} \bar\phi - i\omega f_h \cdot \bar\phi \right\} .$$

We next re-write the terms in the first integral on the right-hand side. We exploit the homogeneous equation which is satisfied by ϕ_ℓ. With this aim, we calculate

$$\int_\Omega \frac{1}{\varepsilon} \operatorname{curl}(\rho_\ell\, \phi_\ell) \cdot \operatorname{curl} \bar\phi = \int_\Omega \frac{1}{\varepsilon} \rho_\ell \operatorname{curl} \phi_\ell \cdot \operatorname{curl} \bar\phi + \int_\Omega \frac{1}{\varepsilon} \rho_\ell' \, (e_1 \times \phi_\ell) \cdot \operatorname{curl} \bar\phi$$
$$= \int_\Omega \frac{1}{\varepsilon} \operatorname{curl} \phi_\ell \cdot \operatorname{curl}(\rho_\ell \bar\phi) + \int_\Omega \frac{1}{\varepsilon} \rho_\ell' \, e_1 \cdot \left[\phi_\ell \times \operatorname{curl} \bar\phi - \bar\phi \times \operatorname{curl} \phi_\ell \right] .$$

We recall that ϕ_ℓ solves (1.4) with vanishing right-hand side; the test function is the compactly supported function $\rho_\ell \phi$. This leads to the cancelling of lower order terms and we arrive at the following equation for u^{rad}:

$$\int_\Omega \left\{ \frac{1}{\varepsilon} \operatorname{curl} u^{\mathrm{rad}} \cdot \operatorname{curl} \bar\phi - \omega^2 \mu\, u^{\mathrm{rad}} \cdot \bar\phi \right\}$$
$$= - \sum_{\ell=1}^L a_\ell \int_\Omega \frac{1}{\varepsilon} \rho_\ell' \, e_1 \cdot \left[\phi_\ell \times \operatorname{curl} \bar\phi - \bar\phi \times \operatorname{curl} \phi_\ell \right] \tag{3.18}$$
$$+ \int_\Omega \left\{ \frac{1}{\varepsilon} f_e \cdot \operatorname{curl} \bar\phi - i\omega f_h \cdot \bar\phi \right\} .$$

We note that the terms containing ρ_ℓ' are compactly supported, accordingly, the right-hand side of this equation has the property of fast decay that is required in Theorem 3.5.[2]

Step 2: Orthogonality condition. It remains to choose $(a_\ell)_\ell$ such that the orthogonality condition (3.13) is satisfied. We use, for arbitrary $k \le L$, the test function $\phi := \phi_k$ in (3.18). Using $\mathcal{F}_{u,v}$ of (3.3), the integrals containing ρ_ℓ' are equal to

$$\int_\Omega \frac{1}{\varepsilon} \rho_\ell' \, e_1 \cdot \left[\phi_\ell \times \operatorname{curl} \bar\phi_k - \bar\phi_k \times \operatorname{curl} \phi_\ell \right] = -i\, \mathcal{F}_{\phi_\ell, \phi_k} \operatorname{sign} \int_{\mathbb{R}} \rho_\ell'(x_1)\, dx_1$$
$$= -\frac{i}{2\pi} \left| Q(\phi_\ell, \phi_k) \right| ,$$

where we used (3.5) and $\int_{\mathbb{R}} \rho_\ell' = \operatorname{sign}(Q(\phi_\ell, \phi_\ell))$. The right-hand side of (3.18) for $\phi := \phi_k$ is therefore

$$\sum_{\ell=1}^L a_\ell \frac{i}{2\pi} |Q(\phi_\ell, \phi_k)| + \int_\Omega \left\{ \frac{1}{\varepsilon} f_e \cdot \operatorname{curl} \bar\phi_k - i\omega f_h \cdot \bar\phi_k \right\} .$$

Using finally $|Q(\phi_\ell, \phi_k)| = |Q(\phi_k, \phi_k)| \delta_{k,\ell}$, we conclude that, with $(a_\ell)_\ell$ chosen as in (3.17), the right-hand side of (3.18) vanishes for every $\phi := \phi_k$. We can apply Theorem 3.5 to find u^{rad}. This yields a solution of the Maxwell radiation problem together with the estimates of (3.16). □

[2] The right-hand side in the equation for u^{rad} contains the curl of the test-function, but it also contains the test-function itself. At this point we see that it was useful to include the right-hand side f_h in the original system.

4. Two spaces of homogeneous solutions

The space Y. Let us first recall the spaces that were used in the above constructions, based on Assumption 3.1. The space Y_j of (3.12) consists of α_j-quasiperiodic homogeneous solutions, we recall $\alpha_j \in (-1/2, 1/2]$. When we take the span of all the Floquet modes, we obtain the space

$$(4.1) \qquad Y := \bigoplus_{j=1}^{J} Y_j \subset H(\mathrm{curl}, W)\,, \quad \text{identified with} \quad Y \subset H_{\mathrm{loc}}(\mathrm{curl}, \Omega)\,.$$

The identification is done by considering every α_j-quasiperiodic function on W also as an α_j-quasiperiodic function on Ω. The space Y has the basis $\{\phi_\ell \,|\, \ell = 1, \dots, L\}$ with the orthogonality property $Q(\phi_\ell, \phi_{\ell'}) = 0$ for $\ell \neq \ell'$.

We observe that, by the orthogonality of Y_j with Y_i for $i \neq j$ (see Lemma 3.4) and by the choice of a basis in each Y_j, the functions ϕ_ℓ are linearly independent. In particular, the dimension of the space Y is $L \geq 0$.

The space B. Let us consider another space, the space B of bounded solutions. That space was extensively used in [25] (where it was named X). In order to impose a boundedness property, we introduce the norm $\|U\|_{sL} := \sup_{r \in 2\pi\mathbb{Z}} \|U|_{W_r}\|_{L^2(W_r)}$ for functions $U \in L^2_{\mathrm{loc}}(\Omega)$, where $W_r = (r, r + 2\pi) \times S$. The space of bounded homogeneous solutions is defined as

$$(4.2) \qquad B := \{U \in H_{\mathrm{loc}}(\mathrm{curl}, \Omega) \,|\, U \text{ solves (1.4) for } f_e = f_h = 0\,,\ \|U\|_{sL} < \infty\}\,.$$

It is clear that every quasiperiodic homogeneous solution is a bounded homogeneous solution: $Y \subset B$. Our aim is to show that the spaces Y and B actually coincide. When this is shown, we know that every bounded homogeneous solution of Maxwell's equations is a linear combination of Floquet modes.

Theorem 4.1 (Characterization of bounded homogeneous solutions)**.** *When Assumption 3.1 holds, the spaces Y of* (4.1) *and B of* (4.2) *coincide:*

$$(4.3) \qquad Y = B\,.$$

Proof. Step 1: Preparations. The inclusion $Y \subset B$ is clear, since every Floquet mode has finite sL-norm. We know that Y has the dimension $\dim Y = L$ since Y is spanned by $(\phi_\ell)_{1 \leq \ell \leq L}$. In order to show $B = Y$, it is therefore sufficient to show $\dim B \leq L$.

In this proof, we use a solution to the Maxwell radiation problem. For an arbitrary $M > 0$ we choose the cut-off functions $\rho_\pm$ so that the support of $\rho'_\pm$ is contained in $(-M, M)$. For arbitrary $R > M$, we furthermore use the piecewise affine cut-off function $\vartheta_R : \mathbb{R} \to [0, 1]$ with $\vartheta_R(s) = 1$ for every $s \in [-R, R]$, $\vartheta_R(s) = 0$ for $|s| \geq R + 2\pi$, and linearly affine on $[-R - 2\pi, -R]$ and on $[R, R + 2\pi]$. We interpret ϑ_R also as a function on Ω by setting $\vartheta_R(x) := \vartheta_R(x_1)$.

Let us recall the statements of Theorem 3.9. The theorem provides, for f_e and f_h in the space $L^2_*(\Omega, \mathbb{C}^3)$ defined in (1.5), a solution $u = u^{\mathrm{prop}} + u^{\mathrm{rad}}$, where u^{prop} is given by coefficients $(a_\ell)_\ell$ that are determined in (3.17). The coefficients $(a_\ell)_\ell$ depend linearly and continuously on f_e and f_h. We will use these facts for $f = f_h$ and $f_e = 0$.

We consider an arbitrary element $U \in B$. Our aim is to show that U can be written as a linear combination of the functions $(\phi_\ell)_{1 \leq \ell \leq L}$. When this is achieved, we know $\dim B \leq L$ and the proof of the theorem is complete.

Step 2: The scalar product of an element $U \in B$ and a test-function f. We choose a test-function $f = f_h \in L^2_*(\Omega)$ with compact support. This proof is based on the evaluation of the $L^2(\Omega)$-scalar product $\langle U, f\rangle$.

We use Theorem 3.9 and consider the solution $u = u^{\mathrm{prop}} + u^{\mathrm{rad}}$ of system (1.4), the radiation condition introduces coefficients $(a_\ell)_{1\le\ell\le L}$. The right-hand side of the system is $f_h = f$ and $f_e = 0$. Using $U\vartheta_R$ as a test-function in (1.4), we find

$$(4.4)\quad \int_\Omega \left\{\frac{1}{\varepsilon}\operatorname{curl} u\cdot \operatorname{curl}(\bar U\vartheta_R) - \omega^2\mu\, u\cdot\bar U\vartheta_R\right\} = -i\omega\int_\Omega f\cdot\bar U\vartheta_R = -i\omega\int_\Omega f\cdot\bar U\,,$$

where the last equality holds for R large enough such that the support of f is contained in Ω_R.

We now re-write the left-hand side of (4.4). We want to exploit that it coincides, up to a complex conjugation and terms involving derivatives of ϑ_R, with the weak equation for U (which has the right-hand side $f_e = f_h = 0$) for the test-function $u\vartheta_R$. With a calculation as in the proof of Theorem 3.9, Step 1, we find that the left-hand side of (4.4) coincides with

$$\frac{1}{2\pi}\int_{W_R}\frac{1}{\varepsilon}\left[(\operatorname{curl} u\times\overline{U}) - (\operatorname{curl}\overline{U}\times u)\right]\cdot e_1 \\ - \frac{1}{2\pi}\int_{W_{-R-2\pi}}\frac{1}{\varepsilon}\left[(\operatorname{curl} u\times\overline{U}) - (\operatorname{curl}\overline{U}\times u)\right]\cdot e_1\,,$$

where $W_r = (r, r+2\pi)\times S$.

We now use the decomposition $u = u^{\mathrm{prop}} + u^{\mathrm{rad}}$. We write $G(R)$ for all contributions of u^{rad}. In the limit $R\to\infty$ holds $G(R)\to 0$, since contributions of u^{rad} vanish by the decay of u^{rad}. The contributions of $u^{\mathrm{prop}} = \sum_{\ell=1}^L a_\ell\,\rho_\ell\,\phi_\ell$ are independent of R for $R > M$. The left-hand side of (4.4) reads

$$G(R) + \sum_{\ell=1}^L a_\ell\,\operatorname{sign} Q(\phi_\ell,\phi_\ell)\,\frac{1}{2\pi}\int_{W^\ell(R)}\frac{1}{\varepsilon}\left[(\operatorname{curl}\phi_\ell\times\overline{U}) - (\operatorname{curl}\overline{U}\times\phi_\ell)\right]\cdot e_1\,,$$

where $W^\ell(R) = W_R$ for ℓ with $\lambda_\ell = Q(\phi_\ell,\phi_\ell) > 0$, and $W^\ell(R) = W_{-R-2\pi}$ for ℓ with $\lambda_\ell = Q(\phi_\ell,\phi_\ell) < 0$. Altogether, we obtain from (4.4) the relation

$$(4.5)\qquad G(R) + \sum_{\ell=1}^L c_\ell^R\,a_\ell = -i\omega\,\langle f, U\rangle_{L^2(\Omega)}\,,$$

where the complex numbers c_ℓ^R depend on U and R, but not on f. These numbers are bounded by the boundedness property of U. We choose a subsequence $R\to\infty$ such that all the coefficients c_ℓ^R converge along the subsequence, $c_\ell^R\to c_\ell$ for every ℓ. In the limit $R\to\infty$, relation (4.5) yields

$$(4.6)\qquad \langle f, U\rangle_{L^2(\Omega)} = \frac{i}{\omega}\sum_{\ell=1}^L c_\ell\,a_\ell\,.$$

By inserting a_ℓ from (3.17), we obtain

$$(4.7)\qquad \langle f, U\rangle_{L^2(\Omega)} = \sum_{\ell=1}^L c_\ell\,\frac{2\pi i}{|Q(\phi_\ell,\phi_\ell)|}\langle f,\phi_\ell\rangle_{L^2(\Omega)}\,.$$

Since f was chosen arbitrarily, this relation determines U. We find

$$U = \sum_{\ell=1}^{L} c_\ell \, \frac{2\pi i}{|Q(\phi_\ell, \phi_\ell)|} \phi_\ell \,. \tag{4.8}$$

This shows that the arbitrarily chosen element $U \in B$ can be written as a linear combination of the functions $(\phi_\ell)_{1\le\ell\le L}$ and concludes the proof. □

5. A compactly perturbed medium

This section is devoted to more complex media, we recall that a medium is represented by the coefficients ε and μ. We assume that outside a compact subset of $\bar{\Omega}$, the coefficients coincide with 2π-periodic functions (as considered above). But, within a central region in Ω, the coefficients can be arbitrary positive functions. We denote this situation as "a compactly perturbed medium". Let us turn to the concise mathematical description.

For some $R > 0$, we use the cylinder $Z := \Omega_R = (-R, R) \times S$ and assume

$$\varepsilon_{\text{pert}} = \varepsilon - q_\varepsilon \,, \qquad \mu_{\text{pert}} = \mu - q_\mu \,, \qquad \operatorname{supp}(q_\varepsilon) \,,\ \operatorname{supp}(q_\mu) \subset Z \,, \tag{5.1}$$

where ε and μ have a positive upper and lower bound and are 2π-periodic. We always demand that Assumption 3.1 holds for ε and μ. Regarding the perturbed coefficients, we also assume that $\varepsilon_{\text{pert}}$ and μ_{pert} are of class $L^\infty(\Omega)$ and have a positive lower bound. System (1.1) with the coefficients $\varepsilon_{\text{pert}}$ and μ_{pert} can be written as

$$\begin{aligned} \operatorname{curl} H + i\omega\varepsilon\, E &= i\omega q_\varepsilon\, E + f_e \,, \\ -\operatorname{curl} E + i\omega\mu\, H &= i\omega q_\mu\, H - f_h \,. \end{aligned} \tag{5.2}$$

We seek for solutions (E, H) of this system. In Section 3, we imposed only a radiation condition on H. In the formulation (5.2) as a coupled system, one would rather expect a radiation condition for both variables as formulated in (5.3) below. Actually, the two formulations are equivalent since, outside a compact set, the field E can be calculated from H by taking the curl.

We follow the procedure of Definition 3.7. We use the cut-off functions ρ_ℓ of Definition 3.6 and assume that the support of $\partial_{x_1}\rho_\pm$ is contained in Z. Here, regarding the equation as a system, the propagating solutions to the homogeneous system with periodic coefficients are denoted as $\phi_\ell = (E_\ell, H_\ell)$. The radiation condition demands that, for some coefficients $(a_\ell)_{1\le\ell\le L} \in \mathbb{C}^L$, the solution (E, H) has the form

$$(E, H) = (E^{\text{rad}}, H^{\text{rad}}) + \sum_{\ell=1}^{L} a_\ell \, \rho_\ell \, (E_\ell, H_\ell) \,, \tag{5.3}$$

and that the first part satisfies $(E^{\text{rad}}, H^{\text{rad}}) \in H_0(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega)$.

Our first aim is to write this system in a compact form. In this section, we use the notations $X := H_0(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega)$ and $Y := L^2(\Omega, \mathbb{C}^3) \times L^2(\Omega, \mathbb{C}^3)$ and

$$u^{\text{rad}} := (E^{\text{rad}}, H^{\text{rad}}) \in X \,, \quad f := (f_e, -f_h) \in Y \,, \quad \phi_\ell := (E_\ell, H_\ell) \,.$$

We introduce the operators $D : X \to Y$ and $\xi, \mathcal{Q} : Y \to Y$,

$$D := \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix} , \quad \xi := \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} , \quad \mathcal{Q} := \begin{pmatrix} q_\varepsilon & 0 \\ 0 & q_\mu \end{pmatrix} .$$

With this notation, equation (5.2) takes the form

$$(D + i\omega\, \xi) u = i\omega\, \mathcal{Q} u + f \,. \tag{5.4}$$

The propagating waves are solutions to the homogeneous system; in the new notation, this condition reads $(D + i\omega\,\xi)\phi_\ell = 0$ for every $\ell \le L$.

With the decomposition (5.3) of u, we can write the differential equation (5.4) in terms of the unknown $u^{\rm rad}$. Using the abbreviation $\varphi_\ell := (D + i\omega\,\xi)(\rho_\ell \phi_\ell)$, the equation takes the form

$$(D + i\omega\,\xi)u^{\rm rad} \;+\; \sum_{\ell=1}^{L} a_\ell\, \varphi_\ell \;=\; i\omega\,\mathcal{Q}u^{\rm rad} \;+\; i\omega \sum_{\ell=1}^{L} a_\ell\, \mathcal{Q}(\rho_\ell\phi_\ell) \;+\; f\,. \tag{5.5}$$

We note that the functions $\mathcal{Q}u^{\rm rad}$ and $\mathcal{Q}(\rho_\ell\phi_\ell)$ are supported in $Z = \Omega_R$ since the application of $\mathcal{Q}$ is a multiplication with a function with that support. Furthermore, φ_ℓ has support in Z since ϕ_ℓ is a solution and $\nabla\rho_\ell$ has support in Z. On f we assume the fast decay $f_e, f_h \in L^2_*(\Omega, \mathbb{C}^3)$.

It is our first aim to show a Fredholm property: if (5.5) posesses no nontrivial solution $(u^{\rm rad}, a)$ for $f = 0$, then (5.5) posesses a unique solution $(u^{\rm rad}, a) \in X \times \mathbb{C}^L$ for every right-hand side $f \in L^2_*(\Omega, \mathbb{C}^3) \times L^2_*(\Omega, \mathbb{C}^3)$. Since (5.5) is equivalent to (5.2), this shows also the Fredholm property for the original Maxwell system in a perturbed medium.

We recall from Theorem 3.9 that, in the unperturbed case $\mathcal{Q} = 0$, system (5.5) has a unique solution pair $(u^{\rm rad}, a)$.

5.1. **Helmholtz decompositions and reformulation.** We use several Helmholtz decompositions. In Appendix A, the decompositions $H_0(\mathrm{curl}, \Omega) = D^{(E)} \oplus G^{(E)}$ for the electric field, and $H(\mathrm{curl}, \Omega) = D^{(H)} \oplus G^{(H)}$ for the magnetic field are given; they differ in the weight (ε and μ, respectively) and in the boundary condition for the potential. When we apply these decompositions to the product space $X = H_0(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$, we find the decomposition

$$X = X_D \oplus X_G := \left(D^{(E)} \times D^{(H)}\right) \oplus \left(G^{(E)} \times G^{(H)}\right)\,. \tag{5.6}$$

Accordingly, an element $u^{\rm rad} \in X$ is written as $u^{\rm rad} = u^{\rm rad}_D + u^{\rm rad}_G$. An element $u^{\rm rad}_G \in X_G$ is of the form $u^{\rm rad}_G = (\nabla\psi_E, \nabla\psi_H)$ with $\psi_E \in H^1_0(\Omega)$ and $\psi_H \in \dot H^1(\Omega)$ where we refer to Appendix A for the definition of the space $\dot H^1(\Omega)$.

Regarding right-hand sides and the space Y, we need another decomposition. Below, we introduce a decomposition $Y = Y_D \oplus Y_G$ and write

$$f = (f_e, -f_h) \in Y := L^2(\Omega, \mathbb{C}^3)^2 = Y_D \oplus Y_G\,. \tag{5.7}$$

Accordingly, we write an element $f \in Y$ as $f = f^D + f^G$. On Y, we use the scalar product defined by $\langle f, g\rangle = \langle (f_e, -f_h), (g_e, -g_h)\rangle = \int_\Omega \varepsilon^{-1} f_e\, \bar g_e + \mu^{-1} f_h\, \bar g_h$. The subspace Y_G is defined as a space of gradients; more precisely, an element $f^G \in Y_G$ has the form $f^G = (\varepsilon\nabla\psi_E, \mu\nabla\psi_H)$ with $\psi_E \in H^1_0(\Omega)$ and $\psi_H \in \dot H^1(\Omega)$. The subspace Y_D is the orthogonal complement of Y_G in Y. In particular, $f^D \in Y_D$ has the property that its components satisfy $\int_\Omega f^D_e \cdot \nabla\psi_E = 0$ and $\int_\Omega f^D_h \cdot \nabla\psi_H = 0$ for all $\psi_E \in H^1_0(\Omega)$ and $\psi_H \in \dot H^1(\Omega)$.

In Equation (5.5), we write $u^{\rm rad} = u^{\rm rad}_D + u^{\rm rad}_G$ for the unknown. Furthermore, we project the equation to the two subspaces Y_D (with projection π^Y_D) and Y_G (with projection π_G^Y). We find the following system, equivalent to (5.5):

$$\begin{aligned}\pi_D^Y(D+i\omega\,\xi)u_D^{\mathrm{rad}} \;+\; \pi_D^Y(D+i\omega\,\xi)u_G^{\mathrm{rad}} \;+\; \sum_{\ell=1}^L a_\ell\,\pi_D^Y\varphi_\ell \\ =\; i\omega\,\pi_D^Y\mathcal{Q}(u_D^{\mathrm{rad}}+u_G^{\mathrm{rad}}) \;+\; i\omega\sum_{\ell=1}^L a_\ell\,\pi_D^Y\mathcal{Q}(\rho_\ell\phi_\ell) \;+\; \pi_D^Y f\,,\end{aligned} \tag{5.8}$$

$$\begin{aligned}\pi_G^Y(D+i\omega\,\xi)u_D^{\mathrm{rad}} \;+\; \pi_G^Y(D+i\omega\,\xi)u_G^{\mathrm{rad}} \;+\; \sum_{\ell=1}^L a_\ell\,\pi_G^Y\varphi_\ell \\ =\; i\omega\,\pi_G^Y\mathcal{Q}(u_D^{\mathrm{rad}}+u_G^{\mathrm{rad}}) \;+\; i\omega\sum_{\ell=1}^L a_\ell\,\pi_G^Y\mathcal{Q}(\rho_\ell\phi_\ell) \;+\; \pi_G^Y f\,.\end{aligned} \tag{5.9}$$

These equations simplify considerably when we use the following facts: (i) $Du_G^{\mathrm{rad}} = 0$, since the curl of a gradient vanishes. (ii) $\pi_D^Y(\xi u_G^{\mathrm{rad}}) = 0$ and $\pi_G^Y(\xi u_G^{\mathrm{rad}}) = \xi u_G^{\mathrm{rad}}$, because of $\xi u_G^{\mathrm{rad}} \in Y_G$. (iii) $\pi_D^Y(\xi u_D^{\mathrm{rad}}) = \xi u_D^{\mathrm{rad}}$ and $\pi_G^Y(\xi u_D^{\mathrm{rad}}) = 0$ because of $\xi u_D^{\mathrm{rad}} \in Y_D$. (iv) $\pi_D^Y Du_D^{\mathrm{rad}} = Du_D^{\mathrm{rad}}$ and $\pi_G^Y Du_D^{\mathrm{rad}} = 0$ since every curl is L^2-orthogonal to gradients. In this last point, one has to be careful: The first entry is of the form $\langle \operatorname{curl} H^{\mathrm{rad}}, \varepsilon\nabla\psi_E\rangle_{1/\varepsilon} = \langle \operatorname{curl} H^{\mathrm{rad}}, \nabla\psi_E\rangle_{L^2} = 0$ because of $\psi_E \in H_0^1(\Omega)$. The second entry is of the form $\langle -\operatorname{curl} E^{\mathrm{rad}}, \mu\nabla\psi_H\rangle_{1/\mu} = \langle -\operatorname{curl} E^{\mathrm{rad}}, \nabla\psi_H\rangle_{L^2} = 0$ because of the boundary condition $E^{\mathrm{rad}} \times \nu = 0$, encoded in $E^{\mathrm{rad}} \in H_0(\operatorname{curl},\Omega)$.

Omitting the corresponding terms, system (5.8)–(5.9) takes the equivalent form

$$\begin{aligned}(D+i\omega\,\xi)u_D^{\mathrm{rad}} \;+\; \sum_{\ell=1}^L a_\ell\,\pi_D^Y\varphi_\ell \\ =\; i\omega\,\pi_D^Y\mathcal{Q}u_D^{\mathrm{rad}} \;+\; i\omega\,\pi_D^Y\mathcal{Q}u_G^{\mathrm{rad}} \;+\; i\omega\sum_{\ell=1}^L a_\ell\,\pi_D^Y\mathcal{Q}(\rho_\ell\phi_\ell) \;+\; \pi_D^Y f\,,\end{aligned} \tag{5.10}$$

$$\begin{aligned}i\omega\,\pi_G^Y(\xi u_G^{\mathrm{rad}}) \;+\; \sum_{\ell=1}^L a_\ell\,\pi_G^Y\varphi_\ell \\ =\; i\omega\,\pi_G^Y\mathcal{Q}u_G^{\mathrm{rad}} \;+\; i\omega\,\pi_G^Y\mathcal{Q}u_D^{\mathrm{rad}} \;+\; i\omega\sum_{\ell=1}^L a_\ell\,\pi_G^Y\mathcal{Q}(\rho_\ell\phi_\ell) \;+\; \pi_G^Y f\,.\end{aligned} \tag{5.11}$$

We will see that (5.10) is a Maxwell system, while (5.11) is a Poisson problem.

5.2. **Solution of** (5.11)**.** Our aim is to solve the second equation for u_G^{rad}, assuming that u_D^{rad} and $(a_\ell)_\ell$ are given. We have to study the problem

$$i\omega\,\pi_G^Y((\xi-\mathcal{Q})u_G^{\mathrm{rad}}) \;=\; g \;\in\; Y_G \;\subset\; L^2(\Omega,\mathbb{C}^3)^2\,. \tag{5.12}$$

Solving (5.12) means that we want to achieve that the Y_G-part of the function $i\omega\,(\xi-\mathcal{Q})u_G^{\mathrm{rad}} - g$ vanishes. This is the case if, and only if, $i\omega\,(\xi-\mathcal{Q})u_G^{\mathrm{rad}} - g$ is Y-orthogonal to the subspace Y_G. Equation (5.12) is therefore identical to

$$\int_\Omega (i\omega(\xi-\mathcal{Q})u_G^{\mathrm{rad}})\cdot(\nabla\varphi_E,\nabla\varphi_H) = \int_\Omega g\cdot(\nabla\varphi_E,\nabla\varphi_H) \tag{5.13}$$

for all $(\varphi_E,\varphi_H) \in H_0^1(\Omega)\times\dot H^1(\Omega)$. Recalling $u_G^{\mathrm{rad}} = (\nabla\psi_E,\nabla\psi_H)$ shows that this is a Poisson problem for (ψ_E,ψ_H), which is uniquely solvable because of strict positivity of $\xi-\mathcal{Q}$, see Lemma C.1. Note that we impose a Dirichlet boundary condition for

ψ_E, while we impose no boundary condition for ψ_H (leading to a Neumann boundary condition).

The construction provides a solution operator for equation (5.12),

$$\mathcal{G} : Y_G \ni g \mapsto u_G^{\mathrm{rad}} = (\nabla\psi_E, \nabla\psi_H) \in X_G \,. \tag{5.14}$$

This operator is linear and bounded.

5.3. **Solution of** (5.10)**.** We know that the solution of (5.11) is given by

$$u_G^{\mathrm{rad}} = \mathcal{G}\left(i\omega\, \pi_G^Y \mathcal{Q} u_D^{\mathrm{rad}} + i\omega \sum_{\ell=1}^{L} a_\ell\, \pi_G^Y \mathcal{Q}(\rho_\ell \phi_\ell) + \pi_G^Y f - \sum_{\ell=1}^{L} a_\ell\, \pi_G^Y \varphi_\ell \right),$$

and can insert this expression into (5.10). Writing now $\tilde{u}_D$ for the unknown, the remaining equation reads

(5.15)
$$\begin{aligned}(D + i\omega\,\xi)\tilde{u}_D \;+\; \sum_{\ell=1}^{L} a_\ell\, \pi_D^Y \varphi_\ell \;=\; i\omega\, \pi_D^Y \mathcal{Q}\tilde{u}_D \;+\; i\omega \sum_{\ell=1}^{L} a_\ell\, \pi_D^Y \mathcal{Q}(\rho_\ell\phi_\ell) \;+\; \pi_D^Y f \\ +\; i\omega\, \pi_D^Y \mathcal{Q}\mathcal{G}\left(i\omega\, \pi_G^Y \mathcal{Q}\tilde{u}_D + i\omega \sum_{\ell=1}^{L} a_\ell\, \pi_G^Y \mathcal{Q}(\rho_\ell\phi_\ell) + \pi_G^Y f - \sum_{\ell=1}^{L} a_\ell\, \pi_G^Y \varphi_\ell \right).\end{aligned}$$

Our next aim is to construct a map that is closely related to this equation. We want to map a pair $(u_D^{\mathrm{old}}, a^{\mathrm{old}}) \in L^2(Z,\mathbb{C}^3)^2 \times \mathbb{C}^L$ to a new pair $(u_D^{\mathrm{new}}, a^{\mathrm{new}}) \in L^2(Z,\mathbb{C}^3)^2 \times \mathbb{C}^L$. Let us give the construction of the map. In a first step we seek for $(\tilde{u}_D^{\mathrm{new}}, a^{\mathrm{new}}) \in X_D \times \mathbb{C}^L$ that solves

$$(D + i\omega\xi)\tilde{u}_D^{\mathrm{new}} \;+\; \sum_{\ell=1}^{L} a_\ell^{\mathrm{new}}\, \pi_D^Y \varphi_\ell \;=\; \tilde{f} \tag{5.16}$$

in Ω, where the right-hand side is defined as

$$\begin{aligned}\tilde{f} := i\omega\, \pi_D^Y \mathcal{Q} u_D^{\mathrm{old}} \;-\; \omega^2\, \pi_D^Y \mathcal{Q}\mathcal{G}\pi_G^Y \mathcal{Q} u_D^{\mathrm{old}} +\; i\omega \sum_{\ell=1}^{L} a_\ell^{\mathrm{old}}\, \pi_D^Y \mathcal{Q}(\rho_\ell\phi_\ell) \\ +\; i\omega\, \pi_D^Y \mathcal{Q}\mathcal{G}\left(i\omega \sum_{\ell=1}^{L} a_\ell^{\mathrm{old}}\, \pi_G^Y \mathcal{Q}(\rho_\ell\phi_\ell) - \sum_{\ell=1}^{L} a_\ell^{\mathrm{old}}\, \pi_G^Y \varphi_\ell \right).\end{aligned} \tag{5.17}$$

We claim that the existence and uniqueness result of Theorem 3.9 allows to solve equation (5.16). Theorem 3.9 provides a solution $\tilde{u} = \tilde{u}^{\mathrm{rad}} + \tilde{u}^{\mathrm{prop}}$ of $(D + i\omega\xi)\tilde{u} = \tilde{f}$ (we show below that $\tilde{f} \in L^2_*(\Omega,\mathbb{C}^3)^2$ is satisfied). Then $(D + i\omega\xi)\tilde{u}^{\mathrm{prop}} = \sum_{\ell=1}^{L} a_\ell^{\mathrm{new}}\, \varphi_\ell$ for some coefficients a_ℓ^{new}. Decomposing $\tilde{u}^{\mathrm{rad}} = \tilde{u}_D + \tilde{u}_G$, the pair $(\tilde{u}_D, a_\ell^{\mathrm{new}})$ solves (5.16). Indeed, $(D + i\omega\xi)\tilde{u}_G \in Y_G$, since the curl of a gradient vanishes and ξ maps from X_G to Y_G. Therefore, $(D + i\omega\xi)\tilde{u}_D = \pi_D^Y(D + i\omega\xi)\tilde{u}_D = \pi_D^Y(D + i\omega\xi)\tilde{u}^{\mathrm{rad}} = \pi_D^Y\tilde{f} - \pi_D^Y(D + i\omega\xi)\tilde{u}^{\mathrm{prop}} = \tilde{f} - \pi_D^Y(D + i\omega\xi)\tilde{u}^{\mathrm{prop}}$, which shows that $(\tilde{u}_D, a_\ell^{\mathrm{new}})$ solves (5.16).

The comparison of (5.16) with (5.15) shows that we added the superscripts "new" on the left-hand side and the superscripts "old" on the right-hand side. Furthermore, we have omitted the terms containing f in the definition of $\tilde{f}$. The function $u_D^{\mathrm{new}} \in L^2(Z,\mathbb{C}^3)$ is defined as the restriction of $\tilde{u}_D^{\mathrm{new}}$ to Z,

$$u_D^{\mathrm{new}} := \tilde{u}_D^{\mathrm{new}}|_Z \,.$$

This concludes the construction of the iteration operator

$$\mathcal{T} : L^2(Z,\mathbb{C}^3)^2 \times \mathbb{C}^L \ni (u_D^{\text{old}}, a^{\text{old}}) \mapsto (u_D^{\text{new}}, a^{\text{new}}) \in L^2(Z,\mathbb{C}^3)^2 \times \mathbb{C}^L \,. \tag{5.18}$$

Below, we will show that $\mathcal{T}$ is compact. While such a property is not unusual for solution maps, it can only be expected when the underlying domain is bounded. This is why the restriction to the bounded set Z is crucial in our construction.

We note that the definition of $\tilde{f}$ in (5.17) uses the input variable u_D^{old} only with a factor $\mathcal{Q}$. This shows that the restriction of the unknown to the domain Z is not related to a loss of relevant information for the iteration.

Lemma 5.1 (The iteration operator)**.** *We assume, as before, that ε and μ are real, of class $L^\infty(W)$, with a positive lower bound. Then the operator $\mathcal{T}$ of* (5.18) *is well-defined and compact.*

Proof. The fact that $\mathcal{T}$ is well-defined relies on a property of π_D^Y. In the definition of $\tilde{f}$ we have to evaluate the projection π_D^Y for an argument that is a compactly supported function. When we identify a function that is supported on Z with its trivial extension, we can regard $L^2(Z,\mathbb{C}^3)^2$ as a subspace of $L^2(\Omega,\mathbb{C}^3)^2$. With this convention, we claim that

$$\pi_D^Y \;:\; L^2(Z,\mathbb{C}^3)^2 \;\to\; L^2_*(\Omega,\mathbb{C}^3)^2 \tag{5.19}$$

is a bounded linear operator. Once this is shown, by definition of $\tilde{f}$ in (5.17), it is clear that $\mathcal{T}$ is well-defined.

In order to show (5.19), we have to recall the construction of π_D^Y. The projection acts on a function $f = (f_e, f_h)$, and the two components are treated independent of each other. Regarding the first component, we want to find, for $f_e \in L^2(Z,\mathbb{C}^3)$ with support in Z, the projection onto $D^{(E)}$. This projection is given by $f_e - \varepsilon\nabla\psi$, where $\psi \in H_0^1(\Omega,\mathbb{C})$ solves the problem

$$\int_\Omega \varepsilon\,\nabla\psi\cdot\nabla\bar{\phi} = \int_\Omega f_e\cdot\nabla\bar{\phi} \quad \text{for all } \phi\in H_0^1(\Omega)\,.$$

Solutions ψ of this equation have exponential decay of $\nabla\psi$ for $|x_1|\to\infty$, which is shown in Lemma C.1 of Appendix C. In that lemma, also the Neumann problem is treated and, hence, an analogous result holds for the projection of f_h. These two facts provide (5.19).

Compactness. In order to show the compactness of $\mathcal{T}$, we have to recall the last step in the construction of $\mathcal{T}$: A function $u_D^{\text{old}} \in L^2(Z,\mathbb{C}^3)$ is, in the main part of the construction, mapped to a function $\tilde{u}_D^{\text{new}} \in H_{\text{loc}}(\text{curl},\Omega)$, which is the solution of a radiation problem. When we restrict $\tilde{u}_D^{\text{new}}$ to any bounded subdomain $\Omega_R = (-R,R)\times S$, then the map $L^2(Z,\mathbb{C}^3) \ni u_D^{\text{old}} \mapsto \tilde{u}_D^{\text{new}}|_{\Omega_R} \in H(\text{curl},\Omega_R)$ is bounded.

We choose R large such that $\bar{Z}$ is contained in Ω_R, and a cut-off function $\eta\in C^2(\Omega)$ depending only on x_1 such that $\eta(x)=1$ for $x\in Z$ and $\eta(x)=0$ for $x\in\Omega\setminus\Omega_R$. With this construction, the map

$$\mathcal{T}' : L^2(Z,\mathbb{C}^3) \ni u_D^{\text{old}} \mapsto \tilde{u}_D^{\text{new}}\,\eta \in L^2(\Omega_R,\mathbb{C}^3)$$

is bounded. Furthermore, for bounded arguments u_D^{old}, both, the curl of the right-hand side $\tilde{u}_D^{\text{new}}\,\eta$ and the divergence of $\xi\tilde{u}_D^{\text{new}}\,\eta$ are bounded in $L^2(\Omega_R)$. This follows from the facts that the divergence of $\xi\tilde{u}_D^{\text{new}}$ vanishes and that the curl of $\tilde{u}_D^{\text{new}}$ (that is, $D\tilde{u}_D^{\text{new}}$) is bounded in L^2. The first component of $\tilde{u}_D^{\text{new}}\,\eta$ satisfies a (tangential) Dirichlet condition on all boundaries of Ω_R, the second component satisfies a Neumann condition. Lemma A.2 can be used to obtain compactness of the second (the magnetic) component of u, Lemma A.2 with the spaces of (c) yields compactness of the first (the electric) component. We note that the lemma remains valid when $W = (0, 2\pi) \times S$ is replaced by Ω_R. This yields the compactness of $\mathcal{T}'$, and hence also the compactness of $\mathcal{T}$, which is given by the further restriction of the function to Z. □

In order to include the given right-hand side f of (5.7) into the equation, we finally define $(\tilde{u}_D^\circ, a^\circ) \in X \times \mathbb{C}^L$ as the unique solution of

$$(D + i\omega\xi)\tilde{u}_D^\circ \; + \; \sum_{\ell=1}^L a_\ell^\circ \, \pi_D^Y \varphi_\ell \; = \; \pi_D^Y f \; + \; i\omega \, \pi_D^Y \mathcal{Q}\mathcal{G}\pi_G^Y f \tag{5.20}$$

on Ω. We refer to the discussion after (5.16) for the fact that the existence and uniqueness Theorem 3.9 provides a solution to this equation. We set $u_D^\circ := \tilde{u}_D^\circ|_Z$.

Let us not forget that we are interested in the Maxwell equations in a locally perturbed periodic medium, i.e., in (5.2). The first part of this section was devoted to an equivalent re-formulation of this system. The result was that (5.2) is equivalent to equation (5.15). We now claim that (5.15) is equivalent to the following problem for $(u_D, a) \in L^2(Z, \mathbb{C}^3)^2 \times \mathbb{C}^L$:

$$(u_D, a) \; = \; (u_D^\circ, a^\circ) \; + \; \mathcal{T}(u_D, a)\,. \tag{5.21}$$

Let us verify this claim. To this end, we first consider a solution (u_D, a) to (5.21). We use $(\tilde{u}_D^\circ, a^\circ)$ as constructed in (5.20) and the function $(\tilde{u}_D^{\rm new}, a^{\rm new})$ from the definition of $\mathcal{T}(u_D, a)$ for $(u_D^{\rm old}, a^{\rm old}) := (u_D, a)$. We claim that $(\tilde{u}_D, a) := (\tilde{u}_D^\circ + \tilde{u}_D^{\rm new}, a^\circ + a^{\rm new})$ is a solution of (5.15).

In the subsequent calculation, we use the definition of $(\tilde{u}_D, a)$ in the first equation, the definition of $(\tilde{u}_D^\circ, a^\circ)$ and (5.16) in the second equation, and the definition of $\tilde{f}$ in the last equation. We obtain

$$\begin{aligned}
(D + i\omega\xi)\tilde{u}_D \; + \; \sum_{\ell=1}^L a_\ell \, \pi_D^Y \varphi_\ell \; &= \; (D + i\omega\xi)(\tilde{u}_D^{\rm new} + \tilde{u}_D^\circ) \; + \; \sum_{\ell=1}^L (a_\ell^{\rm new} + a_\ell^\circ) \, \pi_D^Y \varphi_\ell \\
&= \; \tilde{f} \; + \; \pi_D^Y f \; + \; i\omega \, \pi_D^Y \mathcal{Q}\mathcal{G}\pi_G^Y f \\
&= \; i\omega \, \pi_D^Y \mathcal{Q} u_D^{\rm old} \; + \; \omega^2 \, \pi_D^Y \mathcal{Q}\mathcal{G}\pi_G^Y \mathcal{Q} u_D^{\rm old} \; + \; i\omega \sum_{\ell=1}^L a_\ell^{\rm old} \, \pi_D^Y \mathcal{Q}(\rho_\ell \phi_\ell) \\
&\quad - \; i\omega \, \pi_D^Y \mathcal{Q} \sum_{\ell=1}^L a_\ell^{\rm old} \mathcal{G}\Big(i\omega \, \pi_G^Y \mathcal{Q}(\rho_\ell \phi_\ell) - \pi_G^Y \varphi_\ell \Big) \; + \; \pi_D^Y f \; + \; i\omega \, \pi_D^Y \mathcal{Q}\mathcal{G}\pi_G^Y f\,.
\end{aligned}$$

From (5.21) we find $u_D^{\rm old} = \tilde{u}_D^{\rm new}|_Z + \tilde{u}_D^\circ|_Z = \tilde{u}_D|_Z$ and $a_\ell^{\rm old} = a_\ell^{\rm new} + b_\ell^\circ = a_\ell$. This shows that $(\tilde{u}_D, a)$ is a solution of (5.15).

Vice versa, let $(\tilde{u}_D, a)$ be a solution of (5.15). By the definition of $\mathcal{T}$, it is clear that (u_D, a) with $u_D := \tilde{u}_D|_Z$ is a solution of (5.21).

We have shown that the system with a locally perturbed medium is equivalent to (5.21), which is of Fredholm type: The operator that acts on the unknown (u_D, a) is of the form $\mathrm{id} - \mathcal{T}$ (identity plus compact) by Lemma 5.1. We therefore obtain Fredholm's alternative for equation (5.5), and thus for (5.2). In particular, there holds: If the homogeneous system admits only the trivial solution $(u^{\rm rad}, a) = (0, 0)$, then system (5.2) has a unique solution $(u^{\rm rad}, a) \in X \times \mathbb{C}^L$ for every right-hand side.

In the proof of the subsequent theorem, we show that any solution $(u^{\rm rad}, a) \in X \times \mathbb{C}^L$ of the homogeneous system (5.2) is necessarily radiating, i.e. $a = 0$. This property implies that Fredholm's alternative can be formulated in a strengthened way: If the homogeneous system admits only the trivial solution in L^2-spaces, then system (5.2) has a unique solution $(u^{\rm rad}, a) \in X \times \mathbb{C}^L$ for every right-hand side.

Theorem 5.2 (Fredholm alternative for perturbed media)**.** *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $\omega > 0$. Let the coefficients $\mu, \varepsilon \in L^\infty(\Omega)$ have positive lower bounds and let them be given as compact perturbations of periodic functions with positive lower bounds. We demand that Assumption 3.1 is satisfied for the periodic medium. Furthermore, let the homogeneous perturbed system, i.e.* (5.2) *for $f_e = f_h = 0$, admit only the trivial solution in $H_0(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$. (In other words: Let ω^2 be not in the point spectrum of $\mu_{\rm pert}^{-1} \mathrm{curl}(\varepsilon_{\rm pert}^{-1} \mathrm{curl})$.) Then there exists a unique radiating solution to* (5.2) *for every (f_e, f_h) as in Theorem 3.9.*

Proof. We consider a solution $(u^{\rm rad}, a) \in X \times \mathbb{C}^L$ of the homogeneous system (5.2) and want to prove $a = 0$. In this proof, we will work again with the magnetic field only. To avoid confusion with the pair $u = (E, H)$, we set $v^{\rm rad} = H^{\rm rad}$ and $v^{\rm prop} = \sum_{\ell=1}^L a_\ell \rho_\ell \, \phi_\ell$. It is our aim to show that $v^{\rm prop}$ vanishes.

The form Q was defined in (3.4), we modify this definition and set

$$\tilde{Q}_r(u, w) := i \int_{W_r} \frac{1}{\varepsilon_{\rm pert}} \left[(\mathrm{curl}\, u \times \bar{w}) - (\mathrm{curl}\, \bar{w} \times u) \right] \cdot e_1 \, ,$$

where $W_r = (r, r + 2\pi) \times S$. Note that we made the position r variable and that we replaced the periodic coefficient ε by the perturbed coefficient $\varepsilon_{\rm pert}$. As in (3.3) for the periodic case, one shows that $\tilde{Q}_r(u, w)$ is independent of $r \in \mathbb{R}$ for solutions u and w of $\mathrm{curl}(\varepsilon_{\rm pert}^{-1} \mathrm{curl}\, u) - \omega^2 \mu_{\rm pert} u = 0$.

We insert $v = v^{\rm rad} + v^{\rm prop}$ in $\tilde{Q}_r(v, v)$. Because of $v^{\rm rad} \in H(\mathrm{curl}, \Omega)$ and the uniform (with respect to r) boundedness of $\|v^{\rm prop}\|_{H(\mathrm{curl}, W_r)}$ we conclude that

$$\begin{aligned} \tilde{Q}_r(v, v) - \tilde{Q}_r(v^{\rm prop}, v^{\rm prop}) = i \int_{W_r} \frac{1}{\varepsilon_{\rm pert}} \Big[& (\mathrm{curl}\, v^{\rm rad} \times \overline{v^{\rm rad}}) - (\mathrm{curl}\, \overline{v^{\rm rad}} \times v^{\rm rad}) \\ + \; & (\mathrm{curl}\, v^{\rm rad} \times \overline{v^{\rm prop}}) - (\mathrm{curl}\, \overline{v^{\rm prop}} \times v^{\rm rad}) \\ + \; & (\mathrm{curl}\, v^{\rm prop} \times \overline{v^{\rm rad}}) - (\mathrm{curl}\, \overline{v^{\rm rad}} \times v^{\rm prop}) \Big] \cdot e_1 \end{aligned}$$

tends to zero as $|r| \to \infty$. We consider $r \geq R$ such that $\varepsilon_{\rm pert}$ coincides with ε in W_r. From

$$\tilde{Q}_r(\phi_{\ell,j}, \phi_{\ell',j'}) \;=\; e^{i2\pi r(\alpha_j - \alpha_{j'})} \, Q(\phi_{\ell,j}, \phi_{\ell',j'}) \;=\; \delta_{j,j'} \, \delta_{\ell,\ell'} \, Q(\phi_{\ell,j}, \phi_{\ell,j})$$

follows

$$\tilde{Q}_r(v^{\rm prop}, v^{\rm prop}) \;=\; \sum_{j=1}^{J} \sum_{\ell : \lambda_{\ell,j} > 0} |a_{\ell,j}|^2 \, Q(\phi_{\ell,j}, \phi_{\ell,j}) \;\geq\; 0 \, .$$

In the same way, for $r \leq -R - 2\pi$, we have $\tilde{Q}_r(v^{\rm prop}, v^{\rm prop}) \leq 0$. Since $\tilde{Q}_r(v, v)$ is constant with respect to r, we conclude that $\tilde{Q}_r(v, v)$ has to vanish. This also implies $a_{\ell,j} = 0$ for all ℓ, j and hence $v^{\rm prop} = 0$. □

6. Concluding remarks

6.1. **Limiting absorption principle.** In a limiting absorption principle, one studies the original problem, in our case (1.1) with real coefficients ε and μ, and adds a term that introduces physical absorption of energy. In our setting, a natural choice is to replace ε by the complex (still x-dependent and 2π-periodic) parameter $\varepsilon + i\eta\sigma/\omega$, where $\eta > 0$ is a small real number and $\sigma \in L^\infty(\Omega)$ is a positive 2π-periodic function describing the conductivity of the medium. With this choice, η is a parameter for (small) ohmic losses in the system.

The new system is solvable with the Lax-Milgram lemma, the new sesquilinear form is coercive for positive η. Denoting the corresponding solutions by E^η and H^η, the relevant questions are: (i) Are the fields E^η and H^η in every compact subset of $\bar\Omega$ bounded in L^2? (ii) Are all weak limits E and H solutions to the original problem (1.1) and do they satisfy the radiation condition? We say that a limiting absorption principle holds, when both questions can be answered in an affermative sense.

For the Helmholtz equation in the waveguide geometry, a limiting absorption principle has been derived in several works, we mention once more [10], [13], and [17]. We note that [17] provides also the following fact: the form of the radiation condition and also the solution can depend on the choice of σ. In view of these results for the Helmholtz equation and, moreover, the method of proof in [17], we expect that it is not difficult to show that the limiting absorption principle holds also in the above sense for the Maxwell system (1.1). It is interesting to note that in this case the inner product $\langle \cdot, \cdot \rangle_{Y_j}$ used in the construction of the orthogonal basis $\{\phi_{\ell,j} \,|\, \ell = 1, \dots, m_j\}$ in Y_j is given by

$$\langle u, v \rangle_{Y_j} \;=\; \frac{1}{\omega} \int_W \frac{\sigma}{\varepsilon^2} \operatorname{curl} u \cdot \operatorname{curl} \bar{v} \,.$$

In particular, this inner product – and thus the radiation condition – depends on the conductivity σ.

6.2. **Scattering problem.** Based on the results of Section 5, one can also study a scattering problem. Given a mode $(E^{\mathrm{inc}}, H^{\mathrm{inc}}) = (E_{\ell,j}, H_{\ell,j})$, which is interpreted as an incoming field, one is interested in a corresponding solution in the perturbed medium, which is given by $\varepsilon_{\mathrm{pert}}$ and μ_{pert}. The problem is to determine the total field $(E^{\mathrm{tot}}, H^{\mathrm{tot}}) = (E^{\mathrm{inc}}, H^{\mathrm{inc}}) + (E^s, H^s)$ which satisfies (5.2) for $(f_e, f_h) = (0, 0)$ such that the scattered field (E^s, H^s) satisfies the radiation condition. We observe that (E^s, H^s) solves (5.2) for the (compactly supported) right-hand side

$$(f_e, f_h) := (i\omega q_\varepsilon E^{\mathrm{inc}}, -i\omega q_\mu H^{\mathrm{inc}}) \,.$$

Theorem 5.2 can be applied and provides the solution (E^s, H^s).

We furthermore observe that also the situation of Section 4 can be considered for the perturbed system (5.2). By almost the same arguments as in the proof of Theorem 4.1, one can show that the space

$$\big\{(E, H) \in H_{0,loc}(\mathrm{curl}, \Omega) \times H_{loc}(\mathrm{curl}, \Omega) \,|\, (E, H) \text{ solves (5.2) for } (f_e, f_h) = (0, 0) \,,\ \|E\|_{sL} + \|H\|_{sL} < \infty\big\}$$

of bounded solutions coincides with the space spanned by the total fields $(E^{\mathrm{tot}}_{\ell,j}, H^{\mathrm{tot}}_{\ell,j})$ of the above remark.

Appendix A. Helmholtz decompositions and compactness

A.1. **The unbounded domain Ω.** In this part we investigate Helmholtz decompositions in $H(\mathrm{curl}, \Omega)$ and $H_0(\mathrm{curl}, \Omega)$. The letter "D" is used for the space with a condition on the divergence, the letter "G" is used for the space that is related to gradients. For the field E, we introduce

$$D^{(E)} \;:=\; \left\{ E \in H_0(\mathrm{curl}, \Omega) \,\middle|\, \int_\Omega \varepsilon\, E \cdot \nabla\psi = 0 \text{ for all } \psi \in H^1_0(\Omega) \right\} ,$$

$$G^{(E)} \;:=\; \big\{ E \in H_0(\mathrm{curl}, \Omega) \,\big|\, \exists \psi \in H^1_0(\Omega) :\; E = \nabla\psi \big\} \,.$$

In contrast to the use of the letter ψ in the main part of this article, we now use the letter for scalar functions, $\psi : \Omega \to \mathbb{C}$. The above definitions are such that the spaces are orthogonal in the space $L^2(\Omega,\mathbb{C}^3)$ with the weighted scalar product $\langle u,v\rangle = \int_\Omega \varepsilon\, u\cdot\bar v$. Furthermore, since the (distributional) curl of a gradient always vanishes, $\operatorname{curl}(\nabla\psi)=0$, the two subspaces are also orthogonal in $H(\operatorname{curl},\Omega)$ with the scalar product $\langle u,v\rangle = \int_\Omega \varepsilon\, u\cdot\bar v + \operatorname{curl} u\cdot \operatorname{curl}\bar v$. By construction, $D^{(E)}$ is the $H(\operatorname{curl},\Omega)$-orthogonal complement of $G^{(E)}$, we therefore have an orthogonal decomposition $H_0(\operatorname{curl},\Omega) = D^{(E)}\oplus G^{(E)}$.

The decomposition implies the following for solutions (E,H) of (1.3): Let $f_e\in L^2(\Omega)$ satisfy $\int_\Omega f_e\cdot\nabla\psi = 0$ for every $\psi\in H^1_0(\Omega,\mathbb{C})$. In this case, since the left-hand side of (1.3) vanishes for $\phi=\nabla\psi\in G^{(E)}$, we find $E\in D^{(E)}$.

An analogous decomposition can be made for the magnetic field. We must be very careful in the definition of $G^{(H)}$ since the space of gradients of $H^1(\Omega)$-potentials is not closed in $H(\operatorname{curl},\Omega)$. Loosely speaking, every limit of gradients is again (locally) a gradient, but its potential may fail to be in the space $L^2(\Omega)$. We therefore work with the Hilbert space where integrability is demanded only for the derivatives, $\dot H^1(\Omega) := \{v\in L^2_{\mathrm{loc}}(\Omega,\mathbb{C}) | \nabla v\in L^2(\Omega)\}$, equipped with the inner product $\langle u,v\rangle := \langle u,v\rangle_{L^2(W)} + \langle\nabla u,\nabla v\rangle_{L^2(\Omega)}$. We use the closed subspaces

$$\begin{aligned} D^{(H)} &:= \left\{ H\in H(\operatorname{curl},\Omega) \,\middle|\, \int_\Omega \mu\, H\cdot\nabla\psi = 0 \text{ for all } \psi\in\dot H^1(\Omega)\right\},\\ G^{(H)} &:= \left\{ H\in H(\operatorname{curl},\Omega) \,\middle|\, \exists\psi\in\dot H^1(\Omega):\ H=\nabla\psi\right\}. \end{aligned}$$

In the above definition, it is important to use $\dot H^1(\Omega)$ in order to achieve that $G^{(H)}$ is a closed subspace. Let us verify that, indeed, $G^{(H)}$ is closed. With this aim, we consider a sequence $(\psi_j)_j$ with $\psi_j\in\dot H^1(\Omega)$ such that $(\nabla\psi_j)_j$ is convergent in $H(\operatorname{curl},\Omega)$. Upon subtracting a constant $c_j\in\mathbb{C}$ and using the sequence $\psi_j - c_j$, we can assume the normalization $\int_W \psi_j = 0$. For this sequence and for arbitrary $R>0$, the restrictions $\psi_j|_{\Omega_R}$ are a Cauchy sequence in $H^1(\Omega_R)$ by Poincaré's inequality. This implies that $(\psi_j)_j$ converges to some $\psi\in H^1_{loc}(\Omega)$ locally. Since $(\nabla\psi_j)$ converges in $L^2(\Omega,\mathbb{C}^3)$ to some F, there holds $F=\nabla\psi$ and $\psi\in\dot H^1(\Omega)$. We conclude that $G^{(H)}$ is closed as a subspace of $H(\operatorname{curl},\Omega)$ and of $L^2(\Omega,\mathbb{C}^3)$.

The spaces $D^{(H)}$ and $G^{(H)}$ are orthogonal in $L^2(\Omega,\mathbb{C}^3)$ with the weighted scalar product $\langle u,v\rangle = \int_\Omega \mu\, u\cdot\bar v$ and in $H(\operatorname{curl},\Omega)$ with the scalar product $\langle u,v\rangle = \int_\Omega \mu\, u\cdot \bar v + \operatorname{curl} u\cdot\operatorname{curl}\bar v$.

Let $f_h\in L^2(\Omega)$ satisfy $\int_\Omega f_h\cdot\nabla\psi = 0$ for all $\psi\in\dot H^1(\Omega,\mathbb{C})$; this is encoding that f_h is divergence-free with $\nu\cdot f_h = 0$ on $\partial\Omega$. Since the left-hand side of (1.2) vanishes for $\phi=\nabla\psi\in G^{(H)}$, we find $H\in D^{(H)}$.

The condition $H\in D^{(H)}$ includes a boundary condition for H. The fact that μH is L^2-orthogonal to gradients (without condition on the boundary values of the potential) is the weak form of $\nabla\cdot(\mu H)=0$ and $\nu\cdot(\mu H)=0$ on $\partial\Omega$. Since μ is a scalar, we find $\nu\cdot H = 0$ on $\partial\Omega$.

A.2. **The bounded domain W.** We study now Helmholtz decompositions for the bounded domain $W=(0,2\pi)\times S$, noting that all results remain valid for domains of the form $W=(r_1,r_2)\times S$. Here, we are interested in different boundary conditions. In particular, we have to investigate the boundary condition of α-quasiperiodicity.

For fixed $\alpha\in\mathbb{R}$, we use the spaces $H_\alpha(\operatorname{curl},W)$ and $H_{0,\alpha}(\operatorname{curl},W) := \{u\in H_\alpha(\operatorname{curl},W) \,|\, \nu\times u = 0 \text{ on } (0,2\pi)\times\partial S\}$. We are interested in different pairs of function spaces (X,Y). The four choices of interest are (a) $\big(H(\operatorname{curl},W), H^1(W)\big)$, (b) $\big(H_0(\operatorname{curl},W), H^1_0(W)\big)$, (c) $\big(H_{0,\alpha}(\operatorname{curl},W), H^1_{0,\alpha}(W)\big)$ or (d) $\big(H_\alpha(\operatorname{curl},W), H^1_\alpha(W)\big)$.

In the following, the weight function ρ is either ε or μ. In all these cases, we have the decomposition $X = D \oplus G$ with

$$
\begin{aligned}
D &:= \left\{ u \in X \,\middle|\, \int_W \rho\, u \cdot \nabla\psi = 0 \text{ for all } \psi \in Y \right\}, \\
G &:= \{ u \in X \mid \exists \psi \in Y :\ u = \nabla\psi \} .
\end{aligned}
\tag{A.1}
$$

The decomposition is orthogonal with respect to the scalar products $\langle u, v\rangle = \int_W \rho\, u \cdot \bar v$ in $L^2(W, \mathbb{C}^3)$ and $\langle u, v\rangle = \int_W \rho\, u \cdot \bar v + \operatorname{curl} u \cdot \operatorname{curl} \bar v$ in $H(\operatorname{curl}, W)$. In the following, our aim is to show that, in any of these cases, the space D is compactly imbedded in $L^2(W, \mathbb{C}^3)$.

We start by showing compactness in Case (a). The following lemma and its proof are almost identical to Theorem 4.7 in [23], a small difference concerns the spaces X and X_0 in [23] which contain more information on the boundary data. We include the proof for convenience and also since we expand the result afterwards.

Lemma A.1 (Compactness)**.** *Let $\rho \in L^\infty(W)$ be bounded below by some positive constant. Then the space*

$$
D \;=\; \left\{ u \in H(\operatorname{curl}, W) \,\middle|\, \int_W \rho\, u \cdot \nabla\psi = 0 \textit{ for all } \psi \in H^1(W) \right\}
$$

is compactly imbedded in $L^2(W, \mathbb{C}^3)$.

Proof. Step 1: The case $\rho \equiv 1$. We first treat the special case $\rho \equiv 1$, in which the space of interest is $D = D_1$ with

$$
D_1 \;:=\; \left\{ u \in H(\operatorname{curl}, W) \,\middle|\, \int_W u \cdot \nabla\psi = 0 \text{ for all } \psi \in H^1(W) \right\}.
$$

Compactness of D_1 can be derived from classical results. Let $(u_j)_{j\in\mathbb{N}}$ be a bounded sequence in $H(\operatorname{curl}, W)$ with $u_j \in D_1$ for every $j \in \mathbb{N}$. In particular, the sequence $\operatorname{curl} u_j$ is bounded in $L^2(W, \mathbb{C}^3)$. Because of $u_j \in D_1$, there also holds that $\operatorname{div} u_j = 0$ in W and $\nu \cdot u_j = 0$ on ∂W. The control of $\operatorname{curl} u_j$ and $\operatorname{div} u_j$ together with the boundary conditions imply compactness of the sequence u_j in $L^2(W, \mathbb{C}^3)$ by Theorem 3.47 of [23]. Compare also Theorem 3.1 of [24]. This shows compactness of D_1 in $H(\operatorname{curl}, W)$.

Step 2: The general case. We now consider a general coefficient $\rho \in L^\infty(W)$ and use further Helmholtz decompositions. Just as D and G are defined as subspaces of X, we can analogously define subspaces in $L^2(W, \mathbb{C}^3)$,

$$
\begin{aligned}
D_{L^2} &:= \left\{ u \in L^2(W, \mathbb{C}^3) \,\middle|\, \int_W \rho\, u \cdot \nabla\psi = 0 \text{ for all } \psi \in H^1(W) \right\}, \\
G_{L^2} &:= \left\{ v \in L^2(W, \mathbb{C}^3) \,\middle|\, \exists \psi \in H^1(W) :\ v = \nabla\psi \right\} .
\end{aligned}
$$

These are closed subspaces of $L^2(W, \mathbb{C}^3)$. We observe that D_{L^2} is the orthogonal complement of G_{L^2} with respect to the (ρ-dependent) inner product $\langle \rho u, v\rangle_{L^2(W)}$ in $L^2(W, \mathbb{C}^3)$. It is important to observe that the ρ-dependent norms in $H_\alpha(\operatorname{curl}, W)$ and $L^2(W, \mathbb{C}^3)$ are equivalent to the norms for $\rho \equiv 1$.

We will use the two different decompositions as orthogonal direct sums: $X = D \oplus G$ for the space of functions with a curl, and, for L^2-functions,

$$
L^2(W, \mathbb{C}^3) = D_{L^2} \;\oplus\; G_{L^2} .
$$

All corresponding projections are bounded; the boundedness of a projection is independent of the choice of the scalar product (with or without the factor ρ).

In order to show compactness, we consider once more a bounded sequence $(u_j)_{j\in\mathbb{N}}$ in $H(\mathrm{curl}, W)$ with $u_j \in D$ for every $j \in \mathbb{N}$. We decompose u_j with respect to the decomposition that corresponds to $\rho = 1$, that is, using $H(\mathrm{curl}, W) = D_1 \oplus G_1$, where $H(\mathrm{curl}, W)$ is equipped with the scalar product related to $\rho = 1$:

$$u_j \;=\; v_j \;+\; \nabla\psi_j \quad \text{with } v_j \in D_1 \text{ and } \psi_j \in H^1(W,\mathbb{C})\,.$$

Since u_j is bounded in $H(\mathrm{curl}, W)$ and since projections are bounded, the sequence $v_j \in D_1$ is bounded in $H(\mathrm{curl}, W)$. Step 1 yields that there exists a subsequence, again denoted by v_j, which converges in $L^2(W,\mathbb{C}^3)$. With this knowledge, we now read the previous decomposition in the form

$$v_j \;=\; u_j \;-\; \nabla\psi_j\,,$$

and note that this is a decomposition of v_j in $L^2(W,\mathbb{C}^3) = D_{L^2} \;\oplus\; G_{L^2}$; we exploit here that D is contained in D_{L^2}. Since v_j converges in $L^2(W,\mathbb{C}^3)$ and since the projection onto D_{L^2} is bounded in the space $L^2(W,\mathbb{C}^3)$, we conclude that u_j converges in $L^2(W,\mathbb{C}^3)$. This concludes the proof. □

The choice of function spaces that was given as (b) can be treated with exactly the same arguments. We obtain that the space

$$\left\{ u \in H_0(\mathrm{curl}, W) \,\middle|\, \int_W \rho\, u \cdot \nabla\psi = 0 \text{ for all } \psi \in H^1_0(W) \right\}$$

is compactly imbedded in $L^2(W,\mathbb{C}^3)$.

Our next aim is to generalize these result. Instead of demanding $\mathrm{div}(\rho u) = 0$, we want to impose only a boundedness property.

Lemma A.2 (Improved compactness)**.** *Let $\rho \in L^\infty(W)$ be real with positive lower bound. Let $(u_j)_j$ be a bounded sequence in $X = H(\mathrm{curl}, W)$ such that, for a sequence $(f_j)_j$ that is bounded in $L^2(W)$, there holds*

$$\int_W \rho\, u_j \cdot \nabla\varphi = \int_W f_j\, \varphi \tag{A.2}$$

for every $\varphi \in Y = H^1(W)$. Then there exists a subsequence $j \to \infty$ and a limit function u such that $u_j \to u$ in $L^2(W,\mathbb{C}^3)$.

The result remains valid for other pairs (X, Y), e.g.: (b) $\big(H_0(\mathrm{curl}, W), H^1_0(W)\big)$, (c) $\big(H_{0,\alpha}(\mathrm{curl}, W), H^1_{0,\alpha}(W)\big)$, (d) $\big(H_\alpha(\mathrm{curl}, W), H^1_\alpha(W)\big)$.

Proof. On the two spaces $H(\mathrm{curl}, W)$ and $L^2(W,\mathbb{C}^3)$ we define again inner products $\int_W \mathrm{curl}\, u \cdot \mathrm{curl}\, \bar v + \rho\, u \cdot \bar v$ and $\int_W \rho\, u \cdot \bar v$, respectively. We use the decomposition $H(\mathrm{curl}, W) = D \oplus G$ with D as in Lemma A.1 (that is: (A.1) for the choice (a)). We decompose u_j as

$$u_j \;=\; v_j \;+\; \nabla\psi_j \quad \text{with } v_j \in D \text{ and } \psi_j \in H^1(W)\,.$$

Since v_j is obtained as a projection of u_j, the sequence v_j is bounded in $H(\mathrm{curl}, W)$. Additionally, the sequence lies in D. Lemma A.1 provided the compactness of the subspace D, hence we find a subsequence $j \to \infty$ and a limit function v such that $v_j \to v$ in $L^2(W,\mathbb{C}^3)$.

The functions $\psi_j \in H^1(W)$ solve the problem

$$\int_W \rho\, \nabla\psi_j \cdot \nabla\varphi = \int_W \rho\,(u_j - v_j) \cdot \nabla\varphi = \int_W f_j \varphi \quad \text{for all } \varphi \in H^1(W)\,.$$

We consider the solution operator $f_j \mapsto \nabla\psi_j$ of this problem. By the Lax-Milgram lemma, the solution operator is a bounded operator $\big(H^1(W)\big)' \to L^2(W,\mathbb{C}^3)$ on the dual space $\big(H^1(W)\big)'$ of $H^1(W)$. As $H^1(W)$ is compactly imbedded in $L^2(W)$, the embedding $L^2(W) \to \big(H^1(W)\big)'$ is also compact. This compactness allows to choose a subsequence $j \to \infty$ and a limit function $f \in L^2(W, \mathbb{C}^3)$ such that $f_j \to f$ is a strong convergence in $\big(H^1(W)\big)'$. The corresponding solutions $\nabla\psi_j$ are then strongly convergent in $L^2(W, \mathbb{C}^3)$.

The strong convergences of v_j and $\nabla\psi_j$ imply the strong convergence of u_j. This concludes the proof in the case $(X, Y) = (H(\mathrm{curl}, W), H^1(W, \mathbb{C}))$.

Case (b) with the pair $(X, Y) = \big(H_0(\mathrm{curl}, W), H_0^1(W)\big)$ can be shown with exactly the same arguments, compare the remark after the proof of Lemma A.1.

We now turn to the cases (c) and (d) which involve functions with quasiperiodic boundary conditions at the flat boundaries $\{0\} \times S$ and $\{2\pi\} \times S$. We start by choosing a cut-off function $\eta \in C^\infty(\mathbb{R})$ with $\eta(x_1) = 1$ for $x_1 \in [0, 2\pi]$ and $\eta(x_1) = 0$ for $x_1 \notin [-1, 2\pi + 1]$. Let now u_j be a bounded sequence in X as demanded in the lemma. We identify u_j and f_j with their quasiperiodic extensions to $\mathbb{R} \times S$. We consider the truncated sequence $\tilde{u}_j(x) := \eta(x_1)\, u_j(x)$. This truncated sequence satisfies homogeneous Dirichlet conditions on the boundaries $\{x \,|\, x_1 = -2\pi\}$ and $\{x \,|\, x_1 = 4\pi\}$ of the cylinder $\tilde{W} := (-2\pi, 4\pi) \times S$. For the truncated sequence we observe, for an arbitrary α-quasiperiodic test-function φ,

$$\begin{aligned}\int_{\tilde{W}} \rho\, \tilde{u}_j \cdot \nabla\varphi &= \int_{\tilde{W}} \rho\, u_j \cdot \nabla(\eta\,\varphi) - \int_{\tilde{W}} \rho\, u_j \cdot \nabla\eta\ \varphi = \int_{\tilde{W}} f_j\, \eta\, \varphi - \int_{\tilde{W}} \rho\, u_j \cdot \nabla\eta\ \varphi \\ &= \int_{\tilde{W}} [f_j - \rho\, u_j \cdot \nabla\eta]\ \varphi\,.\end{aligned}$$

The sequence $\tilde{f}_j := f_j - \rho\, u_j \cdot \nabla\eta$ is bounded in $L^2(\tilde{W})$. We can therefore apply the compactness result for case (b) with the domain $\tilde{W}$ and obtain the convergence of a subsequence $\tilde{u}_j$ in $L^2(\tilde{W})$. This implies, along this subsequence, also the convergence of u_j in $L^2(W)$. □

For use in Appendix B, we formulate the following special case of our results.

Corollary A.3. *Let $\mu \in L^\infty(W)$ be bounded below by some positive constant and let $\alpha \in [-1/2, 1/2]$ be a quasimoment. Then the space*

$$\left\{ u \in H_\alpha(\mathrm{curl}, W) \,\middle|\, \int_W \mu\, u \cdot \nabla\psi = 0 \text{ for all } \psi \in H_\alpha^1(W) \right\}$$

is compactly imbedded in $L^2(W, \mathbb{C}^3)$.

Appendix B. Properties of the operator curl curl

In this appendix, we study an operator on a space of α-quasiperiodic functions. The parameter $\alpha \in I$ is fixed throughout this appendix and any dependence on α is suppressed. Correspondingly, for notational convenience, we suppress the tilde that was used in (2.11) and write L instead of $\tilde{L}_\alpha$. In this appendix, we study the space $H_\alpha(\mathrm{curl}, W)$, equipped with the scalar product

$$\langle u, \varphi \rangle_{H(\mathrm{curl},W)} = \int_W \left\{ \frac{1}{\varepsilon} \,\mathrm{curl}\, u \cdot \mathrm{curl}\, \bar{\varphi} + \mu\, u \cdot \bar{\varphi} \right\} \tag{B.1}$$

and the operator $L : H_\alpha(\mathrm{curl}, W) \to H_\alpha(\mathrm{curl}, W)$, defined by

$$\langle Lu, \varphi \rangle_{H(\mathrm{curl},W)} = \int_W \left\{ \frac{1}{\varepsilon} \,\mathrm{curl}\, u \cdot \mathrm{curl}\, \bar{\varphi} - \omega^2 \mu\, u \cdot \bar{\varphi} \right\} \tag{B.2}$$

for all $u, \varphi \in H_\alpha(\mathrm{curl}, W)$. As above, $W = (0, 2\pi) \times S$ and $\varepsilon, \mu \in L^\infty(\Omega)$ are real valued and have a positive lower bound.

The previous subsection provides with case (d) the Helmholtz decomposition $H_\alpha(\mathrm{curl}, W) = D \oplus G$ with

$$\begin{aligned} D &= \left\{ u \in H_\alpha(\mathrm{curl}, W) \,\middle|\, \int_W \mu\, u \cdot \nabla\psi = 0 \text{ for all } \psi \in H^1_\alpha(W) \right\} , \\ G &= \left\{ v \in H_\alpha(\mathrm{curl}, W) \,\middle|\, \exists \psi \in H^1_\alpha(W) : \; v = \nabla\psi \right\} . \end{aligned}$$

The subspace D is the $H(\mathrm{curl}, W)$-orthogonal complement of G.

Lemma B.1 (Fredholm property)**.** *The operator L is a self-adjoint Fredholm operator with index* 0.

Proof. The definition of L in (B.2) is symmetric, this implies that L is self-adjoint.

Step 1: Expressing L in the decomposition $H_\alpha(\mathrm{curl}, W) = D \oplus G$. We claim that $Lu \in D$ holds for every $u \in D$. This follows when we show that Lu is orthogonal to G. Let therefore $v = \nabla\psi \in G$ be arbitrary. We calculate

$$\langle Lu, v\rangle_{H(\mathrm{curl},W)} = \langle Lu, \nabla\psi\rangle_{H(\mathrm{curl},W)} = -\omega^2 \langle \mu u, \nabla\psi\rangle_{L^2(W)} = 0 \,,$$

using the definition of D. This shows $L|_D : D \to D$. The same calculation can be performed for G: Let $v \in G$ be arbitrary, we consider $Lv \in X$ and a test-function $u \in D$ to calculate

$$\langle Lv, u\rangle_{H(\mathrm{curl},W)} = -\omega^2 \langle \mu v, u\rangle_{L^2(W)} = -\omega^2 \langle v, \mu u\rangle_{L^2(W)} = 0 \,,$$

where we exploited that μ is real. This shows $L|_G : G \to G$. We can therefore write L on the space $H_\alpha(\mathrm{curl}, W) = D \oplus G$ in the form

$$L = \begin{pmatrix} L|_D & 0 \\ 0 & L|_G \end{pmatrix} . \tag{B.3}$$

Step 2: Fredholm property of $L|_D$. We claim that $L|_D : D \to D$ is a Fredholm operator with index 0. To prove this, we show a stronger property, namely that $K := L - \mathrm{id}$ is a compact operator $D \to D$. The equation $Ku = F$ is equivalent to $Lu = u + F$, which can be written as

$$\langle Lu, \varphi\rangle_{H(\mathrm{curl},W)} = \langle u, \varphi\rangle_{H(\mathrm{curl},W)} + \langle F, \varphi\rangle_{H(\mathrm{curl},W)} \quad \text{for all } \varphi \in H_\alpha(\mathrm{curl}, W) \,.$$

This, by definition of L, is equivalent to

$$\int_W \frac{1}{\varepsilon} \mathrm{curl}\, u \cdot \mathrm{curl}\, \bar\varphi - \omega^2 \mu\, u \cdot \bar\varphi = \int_W \frac{1}{\varepsilon} \mathrm{curl}\, u \cdot \mathrm{curl}\, \bar\varphi + \mu\, u \cdot \bar\varphi + \langle F, \varphi\rangle_{H(\mathrm{curl},W)}$$

for all $\varphi \in H_\alpha(\mathrm{curl}, W)$, and hence also equivalent to

$$\int_W \mu\, (1 + \omega^2)\, u \cdot \bar\varphi = -\langle F, \varphi\rangle_{H(\mathrm{curl},W)} \quad \text{for all } \varphi \in H_\alpha(\mathrm{curl}, W) \,. \tag{B.4}$$

In order to show compactness of K, we consider a bounded sequence $u_j \in D$ and the images $F_j := Ku_j$. Corollary A.3 provides compactness of D in $L^2(W, \mathbb{C}^3)$, hence, up to the choice of a subsequence (not relabeled), we can assume $u_j \to u$ in $L^2(W, \mathbb{C}^3)$. Because of (B.4), the sequence F_j consists of the Riesz representations of $\langle -\mu(1 + \omega^2)\, u_j, \cdot\rangle_{L^2(W)}$ in $H(\mathrm{curl}, W)$. This implies the convergence $F_j \to F$ in $H(\mathrm{curl}, W)$. Since we have found a convergent subsequence of the images F_j, we have verified the compactness of K.

Step 3: Fredholm property of $L|_G$. Let us investigate the action of L on the subspace G. We consider $Lu = F$ with $u \in G$ and, hence, $F \in G$. The equation is reads $\langle Lu, \varphi\rangle_{H(\text{curl},W)} = \langle F, \varphi\rangle_{H(\text{curl},W)}$ for every $\varphi \in H_\alpha(W)$. By definition of L and by definition of the scalar product on $H(\text{curl}, W)$, this is equivalent to $-\int_W \omega^2 \mu\, u \cdot \bar\varphi = \int_W \mu\, F \cdot \bar\varphi$. We find that, on G, the operator L is nothing but the multiplication $u \mapsto -\omega^2 u$. This operator has a continuous inverse and is therefore a Fredholm operator with index 0.

Step 2 and Step 3 imply that L is Fredholm with index 0. □

Appendix C. Decay of solutions to Poisson problems

The subsequent lemma is a very general statement about solutions of Poisson problems in unbounded cylindrical domains. In this appendix, we only assume the following: The dimension is $d \geq 2$, the set $S \subset \mathbb{R}^{d-1}$ is a bounded Lipschitz domain, the unbounded domain is $\Omega = \mathbb{R} \times S$, the coefficient is a map $\rho : \Omega \to \mathbb{R}$ of class $L^\infty(\Omega)$ with a positive lower bound. In particular, we do not assume that ρ is periodic in direction x_1.

In our results on compactly perturbed media, we use the lemma with the coefficient function ρ being the permittivity ε or the permeability μ. For shorter formulas, we use here segments of unit length instead of segments of length 2π.

Lemma C.1 (Decay of Poisson solutions)**.** *Let $\rho : \Omega \to \mathbb{R}$ be of class $L^\infty(\Omega)$ with a positive lower bound. Let $R > 0$ be fixed, we consider right-hand sides g that are supported in $\Omega_R = (-R, R) \times S$. For arbitrary $r > 0$ let W_r be the segment $W_r = (r, r+1) \times S$. Then there exists a constant $C = C(\rho, R)$ and a decay rate $\delta = \delta(\rho, R)$ such that the following holds: For every function $g : \Omega \to \mathbb{C}$ that is supported in Ω_R, the two Poisson problems below have solutions which decay exponentially. More precisely:*

Dirichlet problem: *Every solution $v \in H^1_0(\Omega, \mathbb{C})$ of*

$$\int_\Omega \rho\, \nabla v \cdot \nabla \bar\varphi = \int_\Omega g \cdot \nabla \bar\varphi \qquad \forall\ \varphi \in H^1_0(\Omega) \tag{C.1}$$

satisfies the exponential decay estimate:

$$\int_{W_r} |\nabla v|^2 \leq C \|g\|^2_{L^2(\Omega_R)}\, e^{-\delta |r|} \qquad \forall r \in \mathbb{R}\,. \tag{C.2}$$

Neumann problem: *Every solution $v \in \dot H^1(\Omega) := \{v \in L^2_{\text{loc}}(\Omega) | \nabla v \in L^2(\Omega)\}$ of the Neumann problem*

$$\int_\Omega \rho\, \nabla v \cdot \nabla \bar\varphi = \int_\Omega g \cdot \nabla \bar\varphi \qquad \forall\ \varphi \in \dot H^1(\Omega)\,. \tag{C.3}$$

satisfies the exponential decay estimate (C.2).

Proof. Step 1: Existence of solutions. The Dirichlet problem can be solved in standard function spaces with the Lax-Milgram lemma. On the space $H^1_0(\Omega)$ we use the sesquilinear form $b(v, \varphi) := \int_\Omega \rho \nabla v \cdot \nabla \bar\varphi$. This form is coercive since $H^1_0(\Omega)$ permits a Poincaré inequality $\int_\Omega |v|^2 \leq C_P \int_\Omega |\nabla v|^2$. The right-hand side of (C.1) defines a linear form on $\varphi \in H^1_0(\Omega)$. We find existence and uniqueness of solutions $v \in H^1_0(\Omega, \mathbb{C})$ together with an estimate $\|v\|_{H^1(\Omega)} \leq C \|g\|_{L^2(\Omega_R)}$.

For the Neumann problem (C.3) one has to modify the function space. We use the space $\dot H^1(\Omega) := \{v \in L^2_{\text{loc}}(\Omega, \mathbb{C}) | \nabla v \in L^2(\Omega)\}$ with the squared norm $\|v\|^2 := \int_{\Omega_R} |v|^2 + \int_\Omega |\nabla v|^2$, and consider once more the sesquilinear form $b(v, \varphi) := \int_\Omega \rho \nabla v \cdot \nabla \bar\varphi$. In order to find a solution v, we restrict b to a closed subspace of $\dot H^1(\Omega)$, namely to $\dot H^1_*(\Omega) := \left\{ v \in \dot H^1(\Omega) \,\middle|\, \int_{\Omega_R} v = 0 \right\}$. The Poincaré inequality on Ω_R for vanishing averages of u implies that b is coercive on $\dot H^1_*(\Omega)$; the Lax-Milgram lemma yields a solution $v \in \dot H^1_*(\Omega)$ of

$$b(v,\psi) = \int_\Omega g \cdot \nabla\psi \quad \text{for all } \psi \in \dot H^1_*(\Omega)\,. \tag{C.4}$$

An arbitrary function $\varphi \in \dot H^1(\Omega)$ can be decomposed, for some constant $c \in \mathbb{C}$, as $\varphi = c + \psi$ with $\psi \in \dot H^1_*(\Omega)$. This allows to check (C.3): For arbitrary $\varphi = c + \psi \in \dot H^1(\Omega)$ with $\psi \in \dot H^1_*(\Omega)$ holds

$$\int_\Omega \rho\, \nabla v \cdot \nabla \varphi = \int_\Omega \rho\, \nabla v \cdot \nabla\psi = b(v,\psi) = \int_\Omega g \cdot \nabla\bar\psi = \int_\Omega g \cdot \nabla\bar\varphi\,.$$

Lax-Milgram yields also the uniqueness of ∇v and, hence, uniqueness up to constants of v. We emphasize that, in general, v will not be of class $L^2(\Omega)$, for any choice of the constant.

Step 2: Exponential decay for the Dirichlet problem. We consider a solution $v \in H^1_0(\Omega)$ of (C.1). All constants in this proof will be independent of v and g, they depend on ρ only through the upper and lower bounds of ρ.

For arbitrary $r > 0$, we use the domain $\Sigma_r := (r,\infty) \times S \subset \Omega$. We perform all arguments for $r > 0$, the arguments for $r < 0$ are analogous.

We note that the test-function $\varphi = v$ leads to the equality $\int_\Omega \rho |\nabla v|^2 = \int_\Omega g \cdot \nabla\bar v$. We now modify the test-function and construct φ as follows: $\varphi(x) := v(x)$ for $x_1 < r$, $\varphi(x) := v(x)(r+1-x_1)$ for $r \le x_1 \le r+1$, $\varphi(x) := 0$ for $x_1 > r+1$. The definition implies $\varphi \in H^1_0(\Omega)$, hence φ is an admissible test-function. For $r > R$ we find, using that the support of g is contained in Ω_R,

$$\begin{aligned}\int_\Omega \rho\, |\nabla v|^2 = \int_\Omega g \cdot \nabla\bar v = \int_\Omega g \cdot \nabla\bar\varphi \; &= \; \int_\Omega \rho\, \nabla v \cdot \nabla\bar\varphi \\ &= \int_{\Omega\setminus\Sigma_r} \rho\, |\nabla v|^2 + \int_{W_r} \rho\, \nabla v \cdot [(\nabla \bar v)(r+1-x_1) - \bar v\, e_1]\,.\end{aligned}$$

Subtracting the first term of the right-hand side, and using a Poincaré inequality for the last integral, we find

$$\int_{\Sigma_r} \rho|\nabla v|^2 \;\le\; C \int_{W_r} \rho|\nabla v|^2\,. \tag{C.5}$$

The constant $C > 1$ is independent of r, it is a consequence of the Poincaré inequality $\|v\|_{L^2(W_r)} \le C_P \|\nabla v\|_{L^2(W_r)}$ for all $v \in H^1_0(\Omega)$. Inequality (C.5) allows to calculate

$$\int_{\Sigma_{r+1}} \rho|\nabla v|^2 \;=\; \int_{\Sigma_r} \rho|\nabla v|^2 \;-\int_{W_r} \rho|\nabla v|^2 \le \left(1 - \frac{1}{C}\right) \int_{\Sigma_r} \rho|\nabla v|^2\,. \tag{C.6}$$

This implies, in particular, the exponential decay (C.2).

Step 3: Exponential decay for the Neumann problem. We follow the ideas of Step 2 and consider the solution $v \in \dot H^1(\Omega)$ of (C.3). Without loss of generality, subtracting a constant, we can assume $v \in \dot H^1_*(\Omega)$; this means that we work with the solution v that was found by solving (C.4).

For $r > R$, we consider the average $c_r := \frac{1}{|W_r|}\int_{W_r} v$ of the function v in W_r and define

$$\varphi(x) := \begin{cases} v(x) & \text{for } x_1 \leq r\,, \\ c_r + (v(x) - c_r)(r+1-x_1) & \text{for } r \leq x_1 \leq r+1\,, \\ c_r & \text{for } x_1 > r+1\,. \end{cases}$$

We note that, for $x_1 = r$, the expression in the middle coincides with v, while, for $x_1 = r+1$, the expression in the middle coincides with c_r. In particular, there holds $\varphi \in \dot{H}^1(\Omega)$.

Repeating the first calculation of Step 2 yields

$$\int_{\Sigma_r} \rho\,|\nabla v|^2 \;=\; \int_{W_r} \rho\,\nabla v \cdot \left[(\nabla \bar{v})(r+1-x_1) - (\bar{v} - \bar{c}_r)\,e_1\right].$$

The constant c_r is chosen such that $\int_{W_r}(v - c_r) = 0$. This allows to use a Poincaré inequality on W_r: There exists $C_P > 0$ with $\|\tilde{v}\|_{L^2(W_r)} \leq C_P\|\nabla\tilde{v}\|_{L^2(W_0)}$ for all $\tilde{v} \in H^1(W_r)$ with $\int_{W_r}\tilde{v} = 0$, the constant C_P is independent of r. Using this inequality for $\tilde{v} = v - c_r$, we obtain

$$\int_{\Sigma_r} \rho\,|\nabla v|^2 \;\leq\; C\int_{W_r} \rho\,|\nabla v|^2$$

with C independent of g, v, and r. We have thus obtained (C.5) also in the Neumann case and the exponential decay follows as in Step 2. □

Data Availability Statement: No datasets were generated or analysed during the current study.

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