Effective Maxwell's equations in general periodic microstructures

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Abstract: We study the time harmonic Maxwell equations in a meta-material consisting of perfect conductors and void space. The meta-material is assumed to be periodic with period $\eta > 0$; we study the behaviour of solutions (E^η, H^η) in the limit $\eta \to 0$ and derive an effective system. In geometries with a non-trivial topology, the limit system implies that certain components of the effective fields vanish. We identify the corresponding effective system and can predict, from topological properties of the meta-material, whether or not it permits the propagation of waves.

Key-words: Maxwell's equations, homogenization, diffraction, periodic structure, meta-material

MSC: 35Q61, 35B27, 78M40, 78A45

1 Introduction

We are interested in transmission properties of meta-materials. In this context, a meta-material is a periodic assembly of perfect conductors, and our question concerns the behaviour of electromagnetic fields when the period of the meta-material tends to zero. We fix a frequency $\omega > 0$ and investigate solutions to the time harmonic Maxwell equations. Denoting the period of the meta-material by $\eta > 0$, we study the behaviour of solutions (E^η, H^η) to the system

$$\operatorname{curl} E^\eta = \mathrm{i}\,\omega\mu_0 H^\eta \quad \text{in } \Omega\,, \tag{1.1a}$$

$$\operatorname{curl} H^\eta = -\mathrm{i}\,\omega\varepsilon_0 E^\eta \quad \text{in } \Omega \setminus \overline{\Sigma}_\eta\,, \tag{1.1b}$$

$$E^\eta = H^\eta = 0 \quad \text{in } \Sigma_\eta\,, \tag{1.1c}$$

in the limit $\eta \to 0$. In this model, we assume that the perfect conductor fills the subset $\Sigma_\eta \subset \Omega$ of the domain $\Omega \subset \mathbb{R}^3$.

In general, meta-materials for Maxwell's equations are described with two periodic material parameters ε and μ (permittivity and permeability). We study here perfectly conducting inclusions, which formally amount to setting $\varepsilon = \infty$. In this case, the electric and the magnetic field vanish in the inclusions; see (1.1c). The material parameters in the

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other equations are given by $\varepsilon_0 > 0$ and $\mu_0 > 0$, the coefficients of vacuum. Imposing (1.1a) encodes boundary conditions: the magnetic field H has a vanishing normal component and the electric field E has vanishing tangential components on the boundary $\partial\Sigma_\eta$.

We ask: Can electromagnetic waves propagate in the periodic medium? Are there components of the effective fields that necessarily vanish? What is the effective system that describes the remaining components?

Our theory yields the following results as particular applications: In a geometry with perfect conducting plates, transmission through the meta-material is possible in two directions. Instead, in a geometry with long and thin holes in the metal no transmission is possible.

1.1 Geometry and assumptions

We are interested in studying general geometries Σ_η. Nevertheless, we remain in the framework of standard periodic homogenization, i.e., the set Σ_η of inclusions is locally periodic. A microscopic structure is considered, which is given by a perfectly conducting part $\Sigma \subset Y$ in a single periodicity cell Y, where $Y := [-1/2, 1/2]^3$. We assume that the set Σ is non empty and open with a Lipschitz boundary as a subset of the 3-torus.

Our aim is to study electromagnetic waves in an open subset $\Omega \subset \mathbb{R}^3$. The meta-material is located in a second domain $R \subset\subset \Omega$. In $\Omega \setminus R$, we have relative permeability and relative permittivity equal to unity. The microscopic structure in R is defined using indices $k \in \mathbb{Z}^3$ and shifted cubes $Y_k^\eta := \eta(k + Y)$, where $\eta > 0$. By $\mathcal{K}$ we denote the index set $\mathcal{K} := \{k \in \mathbb{Z}^3 \colon Y_k^\eta \subset R\}$. We define the meta-material Σ_η by

$$\Sigma_\eta := \bigcup_{k\in\mathcal{K}} \eta(k + \Sigma) \subset R\,. \tag{1.2}$$

Even in the above periodic framework, quite general geometries and topologies can be generated. The simplest non-trivial example occurs if we study a cylinder $\Sigma \subset Y$ that connects two opposite faces of Y; see Figure 4 (b). The cylinder Σ generates a set Σ_η that is the union of disjoint long and thin fibers. In a similar way, we can generate the macroscopic geometry of large metallic plates for which length and width are of macroscopic size and the thickness is of order η; for the corresponding local geometry Σ see Figure 3 (b).

We investigate distributional solutions $(E^\eta, H^\eta) \in H^1(\Omega; \mathbb{C}^3) \times H^1(\Omega; \mathbb{C}^3)$ to (1.1). The number $\omega > 0$ denotes the frequency, and $\mu_0, \varepsilon_0 > 0$ are the permeability and the permittivity in vacuum, respectively. We assume that we are given a sequence $(E^\eta, H^\eta)_\eta$ of solutions to (1.1) that satisfies the energy-bound

$$\sup_{\eta>0} \int_\Omega \Big(|H^\eta|^2 + |E^\eta|^2 \Big) < \infty\,. \tag{1.3}$$

If (1.3) holds, by reflexivity of $L^2(\Omega; \mathbb{C}^3)$, we find two vector fields $E, H \in L^2(\Omega; \mathbb{C}^3)$ and subsequences such that $E^\eta \rightharpoonup E$ in $L^2(\Omega; \mathbb{C}^3)$ and $H^\eta \rightharpoonup H$ in $L^2(\Omega; \mathbb{C}^3)$. Due to the compactness with respect to two-scale convergence, we may additionally assume for fields $E_0, H_0 \in L^2(\Omega \times Y; \mathbb{C}^3)$ the two-scale convergence

$$E^\eta \xrightharpoonup{2} E_0 \quad \text{and} \quad H^\eta \xrightharpoonup{2} H_0\,. \tag{1.4}$$

1.2 Main results

We obtain an effective system of equations that describes the limits E and H. The effective system depends on topological properties of the microscopic geometry $\Sigma \subset Y$. We denote by $\mathcal{N}_\Sigma \subset \{1,2,3\}$ those directions for which no curve in Σ exists that connects corresponding opposite faces of Y (the notation $\mathcal{N}_\Sigma$ indicates that there is 'no loop in Σ'; for a precise definition see (2.1)). One of our result is that the number of non-trivial components of the effective electric field is given by $|\mathcal{N}_\Sigma|$ (see Proposition 3.3). Similarly, the number of non-trivial components of the effective magnetic field is given by $|\mathcal{L}_{Y\setminus\overline{\Sigma}}|$, where $\mathcal{L}_{Y\setminus\overline{\Sigma}} \subset \{1,2,3\}$ are those directions for which a curve in $Y \setminus \overline{\Sigma}$ exists that connects corresponding opposite faces of Y (the notation $\mathcal{L}_{Y\setminus\overline{\Sigma}}$ indicates that there is a 'loop in $Y \setminus \overline{\Sigma}$'; for the result see Proposition 3.8).

With the two index sets $\mathcal{N}_\Sigma, \mathcal{L}_{Y\setminus\overline{\Sigma}} \subset \{1,2,3\}$, we can formulate the effective system of Maxwell's equations. In the meta-material, there holds

$$\operatorname{curl} \hat{E} = \mathrm{i}\,\omega\mu_0\hat{\mu}\hat{H}\,, \tag{1.5a}$$

$$(\operatorname{curl} \hat{H})_k = -\mathrm{i}\,\omega\varepsilon_0(\hat{\varepsilon}\hat{E})_k \qquad \text{for every } k \in \mathcal{N}_\Sigma, \tag{1.5b}$$

$$\hat{E}_k = 0 \qquad \text{for every } k \in \{1,2,3\} \setminus \mathcal{N}_\Sigma\,, \tag{1.5c}$$

$$\hat{H}_k = 0 \qquad \text{for every } k \in \{1,2,3\} \setminus \mathcal{L}_{Y\setminus\overline{\Sigma}}\,. \tag{1.5d}$$

In this set of equations, the effective relative permittivity $\hat{\varepsilon}$ and the effective relative permeability $\hat{\mu}$ are defined through cell-problems. Our main result is the derivation of these effective equations; see Theorem 4.1 below. Essentially, the theorem states the following: Let $(E^\eta, H^\eta)_\eta$ be a bounded sequence of solutions to (1.1) satisfying (1.3), let limit fields $(\hat{E}, \hat{H})$ be defined as weak and geometric limits of $(E^\eta, H^\eta)_\eta$, and let $\hat{\varepsilon}$ and $\hat{\mu}$ be the effective coefficients defined by cell-problems (see (4.3)). In this situation, the limit $(\hat{E}, \hat{H})$ is a solution to the effective system, which coincides with (1.5) in the meta-material. Theorem 4.1 also specifies the interface conditions along the boundary ∂R of the meta-material. The result allows to determine, by checking topological properties of Σ, if the meta-material supports propagating waves. To give an example: In the case $\mathcal{N}_\Sigma = \emptyset$ (that is, Σ connects all opposite faces of Y), the electric field $\hat{E}$ necessarily vanishes identically in the meta-material and waves cannot propagate.

Of particular interest are those cases in which some components of $\hat{E}$ and/or $\hat{H}$ vanish while the other components satisfy certain blocks of Maxwell's equations. This occurs, e.g., in wire and in plate structures. Our analysis is much more general: the effect occurs when the solution spaces to the cell-problems are not three dimensional, but have a lower dimension (drop of dimension).

1.3 Literature

From the perspective of applications, our contribution is closely related to [8], which is concerned with an interesting experimental observation: Light can propagate well in a structure made of thin silver plates; even nearly perfect transmission through such a sub-wavelength structure was experimentally observed. The mathematical analysis of [8] explains the effect with a resonance phenomenon. While [8] is purely two-dimensional, the present contribution investigates which genuinly three-dimensional structures are capable of showing similar transmission properties.

From the perspective of methods, we follow other contributions more closely. We deal with the homogenization of Maxwell's equations in periodic structures. This mathematical task has already some history: The book [14] contains the homogenization of the equations in a standard setting (i.e., periodic and uniformly bounded coefficient sequences ε_η and μ_η); for this case see also [20].

The first homogenization result for Maxwell's equations in a singular periodic structure appeared in [7]: Small split rings with a large absolute value of ε_η were analyzed, and a limit system with effective permittivity $\varepsilon_0\hat{\varepsilon}$ and effective permeability $\mu_0\hat{\mu}$ was derived. The key point is that the coefficient $\hat{\mu}$ of the limit system can have a negative real part, due to resonance of the micro-structure. A similar result was obtained in [5] with a simpler local geometry; the effect of a negative $\operatorname{Re}\hat{\mu}$ is there obtained through Mie-resonance. An extension to flat rings was performed in [15]. The construction of a negative index material (with negative permittivity *and* negative permeability) was successfully achieved in [16] with the additional inclusion of long and thin wires. For a recent overview we refer to [19].

The step towards perfectly conducting materials was done in [17], in which (1.1) was also used. The result of [17] is a limit system that takes the usual form of Maxwell's equations, again with effective permittivity $\varepsilon_0\hat{\varepsilon}$, effective permeability $\mu_0\hat{\mu}$, and negative $\operatorname{Re}\hat{\mu}$. Once more, the negative coefficient is possible since the periodic structure Σ_η has a singular (torus-like) geometry.

Compared to the results described above, the work at hand takes a different perspective: We are not interested in a negative $\operatorname{Re}\hat{\mu}$, but we are interested to see whether or not certain components of the effective fields have to vanish (due to geometrical properties of the microstructure). If some components vanish, we want to extract the equations for the remaining components. The effect of vanishing components is always a result of geometries in which the substructure Σ of the periodicity cell Y touches two opposite faces of Y. We recall that such substructures also enabled the effect of a negative index in [16]. However, in all contributions mentioned above (besides from [8] and [16]) the resonant structure Σ is assumed to be compactly contained in the cell Y.

It is worth mentioning that the study of wires (as a particular example of a periodic microstructure with macroscopic dimensions) has a longer history. Bouchitté and Felbacq showed that wire structures with extreme coefficient values can lead to the effect of a negative effective permittivity; see [6, 12]. Related wire-constructions have been analyzed by Chen and Lipton; see [9, 10].

Our results concern the scattering properties of periodic media. We emphasize that, in contrast to many classical contributions, we treat only the case that the period is small compared to the wave-length of the incident wave (prescribed by the frequency ω). Also in the case that the period and the wave-length are of the same order, one can observe interesting transmission properties, e.g., due to the existence of guided modes in the periodic structure. The corresponding results are known as 'diffraction theory' or 'grating theory'. For a fundamental analysis of existence and uniqueness questions in such a diffraction problem we mention [4]; see also [18]. Regarding classical methods we mention [11], where the transmission properties of a periodically perturbed interface are studied by means of integral methods. A more recent contribution regarding a similar periodic interface is [13]. Closer to our analysis is [2], where a three-dimensional layer of a periodic material is studied (the material is periodic in two directions); see also the overview [3].

1.4 Methods and organization

We use the tool of two-scale convergence of [1] and consider the two-scale limits $E_0 = E_0(x,y)$ and $H_0 = H_0(x,y)$ of the fields E^η and H^η. By standard arguments, we obtain cell problems for $E_0(x,\cdot)$ and $H_0(x,\cdot)$. We then characterise the solution spaces of these cell problems—that is, we determine bases of the solution spaces in terms of the index sets $\mathcal{N}_\Sigma$ and $\mathcal{L}_{Y\setminus\overline{\Sigma}}$. The crucial observation is the following: if the dimension of one of the solution spaces is less than three, the standard procedure to define homogenized coefficients does no longer work. Hence the form of the effective system is not clear. However, once the homogenized coefficients are carefully defined, the derivation of the effective system is rather standard; see [16, 17]. Note that in [7, 15, 17], a full torus geometry was considered; the fact that the complement of the torus is not simply connected leads to a 4-dimensional solution space in the cell problem for H. We, however, are interested in the opposite effect: Geometries that generate solution spaces of dimension smaller than 3.

In Section 2 we introduce the notions of simple Helmholtz domains, k-loops, and the geometric average, and prove auxiliary results. The derivation of the cell problems and the characterisation of their solution spaces is carried out in Section 3. In Section 4 we prove the main result, i.e., we derive the effective system (1.5). Section 5 is devoted to the discussion of some examples of microstructures.

2 Preliminary geometric results

2.1 Periodic functions, Helmholtz-domains, and k-loops

Let Y denote the closed cube $[-1/2, 1/2]^3$ in $\mathbb{R}^3$. We define an equivalence relation $\backsim$ on Y by identifying opposite sides of the cube: $y_a \backsim y_b$ whenever $y_a - y_b \in \mathbb{Z}^3$. The quotient space $Y/\backsim$ is identified with the flat 3-torus $\mathbb{T}^3$; the canonical projection $Y \hookrightarrow \mathbb{T}^3$ is denoted by ι.

A map $u\colon \mathbb{R}^3 \to \mathbb{C}^n$ is called *Y-periodic* if $u(\cdot + l) = u(\cdot)$ for all $l \in \mathbb{Z}^3$. For $m \in \mathbb{N}\cup\{\infty\}$ and $n \in \mathbb{N}$, the function space $C^m_\sharp(Y;\mathbb{C}^n)$ denotes the restriction of Y-periodic maps $\mathbb{R}^3 \to \mathbb{C}^n$ of class C^m to Y; we may identify this function space with $C^m(\mathbb{T}^3;\mathbb{C}^n)$. Similarly, we define

$$H^m_\sharp(Y;\mathbb{C}^n) := \{u|_Y : Y \to \mathbb{C}^n : u \in H^m_{loc}(\mathbb{R}^3;\mathbb{C}^n) \text{ is } Y\text{-periodic}\},$$

which can be identified with $H^m(\mathbb{T}^3;\mathbb{C}^n)$. We note that $L^2_\sharp(Y;\mathbb{C}^n) = H^0_\sharp(Y;\mathbb{C}^n) = L^2(Y;\mathbb{C}^n)$.

For a subset $U \subset Y$ such that $\iota(U) \subset \mathbb{T}^3$ is open, we set $H^m_\sharp(U;\mathbb{C}^n) := H^m(\iota(U);\mathbb{C}^n)$. We denote by $H_\sharp(\operatorname{curl}, U)$ and $H_\sharp(\operatorname{div}, U)$ the spaces of all $L^2(\iota(U);\mathbb{C}^3)$-vector fields $u\colon U \to \mathbb{C}^3$ such that the distributional curl and the distributional divergence satisfy $\operatorname{curl} u \in L^2(\iota(U));\mathbb{C}^3)$ and $\operatorname{div} u \in L^2(\iota(U))$, respectively. For brevity, we write $C^k_\sharp(U)$, $L^2_\sharp(U)$, and $H^1_\sharp(U)$ if no confusion about the target space is possible.

Definition 2.1 (Simple Helmholtz domain)**.** *Let $U \subset Y$ be such that $\iota(U)$ is open in $\mathbb{T}^3$. We say that U is a* simple Helmholtz domain *if for every vector field $u \in L^2_\sharp(U;\mathbb{C}^3)$ with* $\operatorname{curl} u = 0$ *in U there exist a potential $\Theta \in H^1_\sharp(U)$ and a constant $c_0 \in \mathbb{C}^3$ such that $u = \nabla\Theta + c_0$ in U.*

Remark 2.2. Note that, in general, the constant c_0 is not unique. Take, for instance, $\Sigma := B_r(0)$ with $r \in (0, 1/2)$. Then $\Theta(y) := \lambda y_k$ and $c_0 := -\lambda e_k$ yields a representation of $u = 0$ for every $\lambda \in \mathbb{C}$ and $k \in \{1, 2, 3\}$. For a generalisation see Lemma 2.5.

In what follows, we consider curves $\gamma\colon [0,1] \to Y$ (not necessarily continuous in Y) such that $\iota \circ \gamma\colon [0,1] \to \mathbb{T}^3$ is continuous (see Figure 2 (a) for a subset $U = \Sigma_1 \cup \Sigma_2 \subset Y$ that admits a discontinuous path γ in U so that $\iota \circ \gamma$ is continuous). For such a curve there exists a lift $\tilde{\gamma}$; that is, a continuous curve $\tilde{\gamma}\colon [0,1] \to \mathbb{R}^3$ with $\pi \circ \tilde{\gamma} = \gamma$, where π denotes the universal covering $\mathbb{R}^3 \to \mathbb{T}^3$, $x \mapsto x \bmod \mathbb{Z}^3$.

Definition 2.3 (k-loop)**.** *Let $U \subset Y$ be non empty and such that $\iota(U) \subset \mathbb{T}^3$ is open. Let $e_1, e_2, e_3 \in \mathbb{R}^3$ be the standard basis vectors, and let $k \in \{1, 2, 3\}$. We say that a map $\gamma\colon [0,1] \to \mathbb{T}^3$ is a k-*loop in $\iota(U)$ *if the path $\gamma\colon [0,1] \to \iota(U)$ is continuous and piecewise continuously differentiable, $\gamma(1) = \gamma(0)$, and $\langle \tilde{\gamma}(1) - \tilde{\gamma}(0), e_k \rangle \neq 0$, where $\tilde{\gamma}\colon [0,1] \to \mathbb{R}^3$ is a lift of γ.*

Remark 2.4. *a)* For a lift $\tilde{\gamma}$ of the k-loop γ, we have that $\tilde{\gamma}(1) - \tilde{\gamma}(0) \in \mathbb{Z}^3$ by $\gamma(1) = \gamma(0)$. *b)* By abuse of notation, we refer to $\gamma\colon [0,1] \to Y$ as a k-loop in U when the curve $\iota \circ \gamma\colon [0,1] \to U$ is a k-loop in U.

For a subset U of Y, we introduce the following index sets:

$$\mathcal{L}_U := \{k \in \{1,2,3\} \colon \text{there is a } k\text{-loop in } U\}\,, \tag{2.1a}$$

$$\mathcal{N}_U := \{k \in \{1,2,3\} \colon \text{there is no } k\text{-loop in } U\}\,. \tag{2.1b}$$

Note that $\mathcal{L}_U \dot{\cup} \mathcal{N}_U = \{1, 2, 3\}$. We turn to the analysis of potentials defined on U.

Lemma 2.5. *Let $U \subset Y$ be non-empty and such that $\iota(U) \subset \mathbb{T}^3$ is open. Assume further that U has only finitely many connected components. Let k be an element of $\{1, 2, 3\}$. If there is no k-loop in U, then there exists a potential $\Theta_k \in H^1_\sharp(U)$ such that $\nabla \Theta_k = e_k$.*

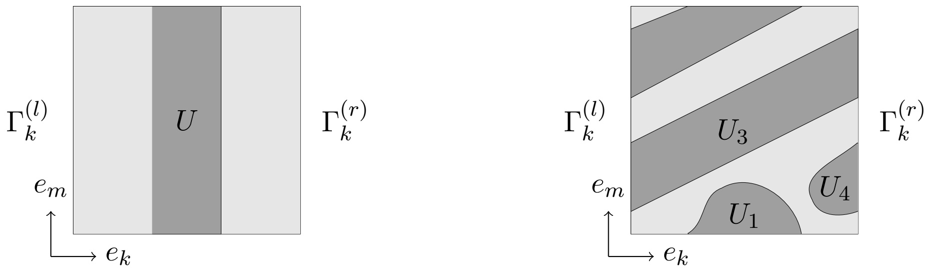
Proof. We may assume that $\iota(U)$ is connected; otherwise each connected component is treated separately. We fix $k \in \{1, 2, 3\}$, and consider the two opposite faces

$$\Gamma_k^{(l)} := \{y \in Y \colon y_k = -1/2\} \quad \text{and} \quad \Gamma_k^{(r)} := \{y \in Y \colon y_k = +1/2\}\,.$$

Idea of the proof. By assumption, U has only finitely many connected components $U_1, \dots, U_N$. Assume that none of the connected components touches the boundaries $\Gamma_k^{(l)}$ and $\Gamma_k^{(r)}$; that is, $U_i \cap \Gamma_k^{(l)} = \emptyset$ and $U_i \cap \Gamma_k^{(r)} = \emptyset$ for all $i \in \{1, \dots, N\}$ (as in Figure 1 (a)). In this case, we can define $\Theta_k \colon U \to \mathbb{C}$ by $\Theta_k(y) := y_k$ and obtain a well-defined function $\Theta_k \in H^1_\sharp(U)$.

Let us now assume that there are connected components $U_1, \dots, U_N$ such that $\iota(U_i \cup U_{i+1})$ is connected for all $i \in \{1, \dots, N-1\}$. The potential $\Theta_k \colon U_1 \cup \dots \cup U_N \to \mathbb{C}$ is defined as follows: On U_1, we set $\Theta_k(y) := y_k$. On U_2, we define $\Theta_k(y) := y_k + d_2$ for some $d_2 \in \mathbb{Z}$ such that Θ_k is continuous on $\iota(U_1 \cup U_2)$. This procedure can be continued until Θ_k is a continuous function on $U_1 \cup \dots \cup U_N$; the continuity of Θ_k is a consequence of the non-existence of a k-loop.

Rigorous proof. We denote by $\pi\colon \mathbb{R}^3 \to \mathbb{T}^3$, $x \mapsto x \bmod \mathbb{Z}^3$ the universal covering of $\mathbb{T}^3$. A lift $[0,1] \to \mathbb{R}^3$ of a continuous path $\gamma\colon [0,1] \to \iota(U)$ is denoted by $\tilde{\gamma}$; that is, $\gamma = \pi \circ \tilde{\gamma}$.



(a) (b)

Figure 1: The sketches shows the projection of the periodicity cell Y to the e_k-e_m-plane. We assume that the geometry is independent of the third coordinate. (a) The set U does not touch one of the boundaries $\Gamma_k^{(l)}$ or $\Gamma_k^{(r)}$, and hence $k \in \mathcal{N}_U$. On the other hand, $m \in \mathcal{L}_U$ since an m-loop in U exists. (b) There are connected components of $U = U_1 \cup \cdots \cup U_4$ that touch $\Gamma_k^{(l)}$ and $\Gamma_k^{(r)}$, but neither k-loops nor m-loops in U exist. That is, $k, m \in \mathcal{N}_U$.

Fix a point $x_0 \in \iota(U)$. For every $x \in \iota(U)$ choose a piecewise smooth path $\gamma\colon [0,1] \to \iota(U)$ joining $\gamma(0) = x_0$ and $\gamma(1) = x$. Let $\tilde{\gamma}\colon [0,1] \to \mathbb{R}^3$ be a lift of γ, and define $\Theta_k\colon \iota(U) \to \mathbb{C}$ by $\Theta_k(x) \coloneqq \langle \tilde{\gamma}(1) - \tilde{\gamma}(0), \mathrm{e}_k \rangle$. Note that the non-existence of a k-loop ensures that Θ_k is well defined. We further observe that the difference $\tilde{\gamma}(1) - \tilde{\gamma}(0)$ is independent of the chosen lift $\tilde{\gamma}$; indeed, for two lifts $\tilde{\gamma}$ and $\tilde{\delta}$ of γ, there is $l \in \mathbb{Z}^3$ such that $\tilde{\gamma} = \tilde{\delta} + l$. □

Corollary 2.6. *Let $U \subset Y$ be a simple Helmholtz domain. If $u \in L^2_\sharp(Y;\mathbb{C}^3)$ satisfies* $\operatorname{curl} u = 0$ *in U, then there are a potential $\Theta \in H^1_\sharp(U)$ and a constant $c_0 \in \mathbb{C}^3$ such that $u = \nabla\Theta + c_0$ in U and $\langle c_0, e_k \rangle = 0$ for every $k \in \mathcal{N}_U$.*

Proof. By the very definition of a Helmholtz domain, we find a potential $\Phi \in H^1_\sharp(U)$ and a constant $d_0 \in \mathbb{C}^3$ such that $u = \nabla\Phi + d_0$ in U. Due to Lemma 2.5, we find a potential $\Theta_k \in H^1_\sharp(U)$ such that $\nabla\Theta_k = \mathrm{e}_k$ for all $k \in \mathcal{N}_U$. Setting $\Theta \coloneqq \Phi + \sum_{k\in\mathcal{N}_U} \langle d_0, \mathrm{e}_k \rangle \Theta_k$ provides us with an element of $H^1_\sharp(U)$. Moreover,

$$\nabla\Theta = u - d_0 + \sum_{k\in\mathcal{N}_U} \langle d_0, \mathrm{e}_k \rangle \mathrm{e}_k = u - \sum_{k\in\mathcal{L}_U} \langle d_0, \mathrm{e}_k \rangle \mathrm{e}_k \,.$$

Setting $c_0 \coloneqq \sum_{k\in\mathcal{L}_U} \langle d_0, \mathrm{e}_k \rangle \mathrm{e}_k$, we find the desired representation of u. □

In the following, we need line-integrals of curl-free $L^2_\sharp(Y;\mathbb{C}^3)$-vector fields.

Lemma and Definition 2.7 (Line integral)**.** *Let $U \subset Y$ be such that $\iota(U)$ is an open subset of $\mathbb{T}^3$ with Lipschitz boundary. Assume that $\gamma\colon [0,1] \to Y$ is such that $\iota \circ \gamma$ is a k-loop in $\iota(U)$. There exists a unique linear and continuous map*

$$I_\gamma\colon \{u \in L^2_\sharp(Y;\mathbb{C}^3)\colon \operatorname{curl} u = 0 \text{ in } U\} \to \mathbb{C}\,, \quad u \mapsto I_\gamma(u)$$

such that $I_\gamma(u)$ coincides with the usual line integral

$$I_\gamma(u) = \int_\gamma u \coloneqq \int_0^1 \langle u(\gamma(t)), \dot{\gamma}(t) \rangle \, dt \,, \tag{2.2}$$

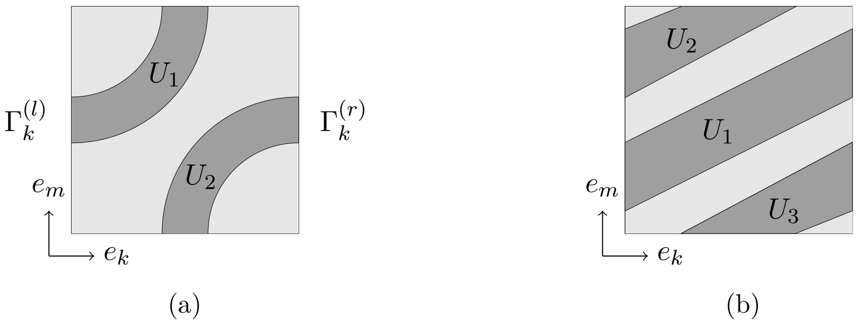


Figure 2: (a) There is a k-loop in $U = U_1 \cup U_2$ connecting $\Gamma_k^{(l)}$ and $\Gamma_k^{(r)}$ although $U = U_1 \cup U_2$ is not connected. (b) There is k-loop γ in U. Note that $\int_\gamma u = 3(\oint u)_k$ for all fields $u \in L^2_\sharp(Y;\mathbb{C}^3)$ that are curl-free in U.

for fields $u \in C^1_\sharp(Y;\mathbb{C}^3)$ that are curl free in U.

The map I_γ is called the line integral of u along γ*, and we write, by abuse of notation, $\int_\gamma u$ instead of $I_\gamma(u)$ for all $u \in L^2_\sharp(Y;\mathbb{C}^3)$ that are curl free in U.*

Idea of a proof. The space $V := \{u \in C^1_\sharp(Y;\mathbb{C}^3) \colon \operatorname{curl} u = 0 \text{ in } U\}$ is dense in $X := \{u \in L^2_\sharp(Y;\mathbb{C}^3) \colon \operatorname{curl} u = 0 \text{ in } U\}$, which can be shown with a sequence of mollifiers $(\rho_\eta)_\eta$. Defining u_η by $u_\eta := u * \rho_\eta$ provides us with a family $(u_\eta)_\eta$ in V with $u_\eta \to u$ in $L^2_\sharp(Y;\mathbb{C}^3)$. The map $\tilde{I}_\gamma \colon V \to \mathbb{C}$ defined by $\tilde{I}_\gamma(u) := \int_\gamma u$ is linear and continuous (because u is curl free in U); using density of V in X, the claim follows. □

We note the following: If γ is a k-loop in U and $\tilde{\gamma}$ is one of its lifts, then $\int_\gamma (u \circ \iota) = \int_{\tilde{\gamma}} u$ for all fields $u \in C^1_\sharp(U;\mathbb{C}^3)$. Indeed, $\dot{\tilde{\gamma}} = \dot{\gamma}$, and $u \circ \iota^{-1} \circ \gamma = u \circ \tilde{\gamma}$ for a periodic function u. Note that the line integral along γ and the line integral along $\tilde{\gamma}$ coincide.

2.2 The geometric average

The notion of a geometric average was first introduced by Bouchitté, Bourel, and Felbacq in [5]. The notion turned out to be very useful, it was extended in [17] to more general geometries. Although we focus on simple Helmholtz domains in the main part of this work, we define the geometric average for general geometries.

In the subsequent definition of a geometric average, we need three special curves $\gamma_1, \gamma_2, \gamma_3 \colon [0,1] \to Y$, which represent paths along the edges—that is, $\gamma_1(t) := (t - 1/2, -1/2, -1/2)$, $\gamma_2(t) := (-1/2, t-1/2, -1/2)$, and $\gamma_3(t) := (-1/2, -1/2, t-1/2)$. We furthermore use the index set $\mathcal{L}_U$ defined in (2.1a).

Definition 2.8 (Geometric average)**.** *Let $U \subset Y$ be non-empty and such that $\iota(U) \subset \mathbb{T}^3$ is open. Assume $u \colon U \to \mathbb{C}^3$ is an $L^2_\sharp(U;\mathbb{C}^3)$-vector field that is curl free in $\iota(U)$. We define its* geometic average *$\oint u \in \mathbb{C}^3$ as follows:*

1) *If U is a simple Helmholtz domain, then the vector field u can be written as $u = \nabla\Theta + c_0$, where $\Theta \in H^1_\sharp(U)$ and $c_0 \in \mathbb{C}^3$. In this case, for $k \in \{1,2,3\}$, we set*

$$\left(\oint u\right)_k := \begin{cases} \langle c_0, e_k\rangle & \text{for } k \in \mathcal{L}_U\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

2) If for all $k \in \{1,2,3\}$ the path γ_k along the edge is a k-loop in U, then, for $k \in \{1,2,3\}$, we set

$$\left(\oint u\right)_k := \int_{\gamma_k} u\,.$$

In later sections, we consider fields $u\colon Y \to \mathbb{C}^3$ that are curl free in $\iota(U)$ and vanish in $Y \setminus \overline{U}$, where $U \subset Y$ is non empty and proper. To define the geometric average of those fields, we restrict u to the subset U and apply the above definition.

Remark 2.9 (The geometric average is well defined). *a)* Let U be a simple Helmholtz domain. Fix $k \in \mathcal{L}_U$, and let $\gamma\colon [0,1] \to \mathbb{T}^3$ be a k-loop in $\iota(U)$. Using that $u = \nabla\Theta + c_0$ in U as well as the periodicity of Θ, we find that

$$\int_\gamma u = \int_\gamma c_0 = \langle c_0, \mathrm{e}_k\rangle\langle\tilde\gamma(1) - \tilde\gamma(0), \mathrm{e}_k\rangle\,,$$

and hence the number $\langle c_0, \mathrm{e}_k\rangle$ in *1)* is well defined (despite our observation in Remark 2.2).

b) The two definitions *1)* and *2)* coincide when both can be applied. To see this, fix an index $k \in \{1,2,3\}$. The domain U is a simple Helmholtz domain, and hence we find a potential $\Theta \in H^1_\sharp(U)$ and a constant $c_0 \in \mathbb{C}^3$ such that $u = \nabla\Theta + c_0$ in U. We may assume that $\Theta \in C^1_\sharp(U)$; otherwise we approximate by smooth functions. We then find, for the path γ_k along the edge, that

$$\int_{\gamma_k} u = \int_{\gamma_k} \nabla\Theta + \langle c_0, \mathrm{e}_k\rangle = \Theta(\gamma_k(1)) - \Theta(\gamma_k(0)) + \langle c_0, \mathrm{e}_k\rangle = \langle c_0, \mathrm{e}_k\rangle\,.$$

Remark 2.10. *a)* Let U be a simple Helmholtz domain, and let γ be a k-loop in U. We remark that $(\oint u)_k \neq \int_\gamma u$ in general. To give an example, let $\tilde\gamma$ be a lift of the k-loop γ that travels the distance 2 in the kth direction, that is, $\langle\tilde\gamma(1) - \tilde\gamma(0), \mathrm{e}_k\rangle = 2$ (as in Figure 2 (b)). In this case, $\int_\gamma u = 2(\oint u)_k$. Nevertheless, there always holds $\int_\gamma u = \lambda_k(\oint u)_k$ for $\lambda_k \in \mathbb{Z} \setminus \{0\}$.

b) There are domains for which definition *1)* can be applied, but definition *2)* cannot be used. Indeed, $U := B_r(0)$ is a simple Helmholtz domain for $r \in (0, 1/2)$. However, $\gamma_k(t) \notin U$ for all $t \in [0,1]$ and all $k \in \{1,2,3\}$.

c) On the other hand, choosing $Y \setminus U$ to be a 3-dimensional full torus $\mathbb{S}^1 \times \mathbb{D}^2 \subset\subset Y$, we find that γ_1, γ_2, and γ_3 are 1-, 2-, and 3-loops in U, respectively. The domain U, however, is not simple Helmholtz.

Properties of the geometric average. We introduce the function space

$$X := \{\varphi \in L^2_\sharp(Y;\mathbb{C}^3)\colon \varphi = 0 \text{ in } Y \setminus \overline{U} \text{ and } \operatorname{div}\varphi = 0 \text{ in } Y\}\,. \tag{2.3}$$

For a simple Helmholtz domain U and under slightly stronger assumptions on the vector field $u\colon Y \to \mathbb{C}^3$, Bouchitté, Bourel, and Felbacq, [5], define the geometric average $\oint u$ by the identity (2.4) below. We show that our definition and theirs agree when both can be applied.

Lemma 2.11. *Let $U \subset Y$ be a simple Helmholtz domain. If $u\colon Y \to \mathbb{C}^3$ is a vector field of class $L^2_\sharp(Y;\mathbb{C}^3)$ that is curl free in U, then the identity*

$$\int_Y \langle u, \varphi\rangle = \left\langle \oint u, \int_Y \varphi \right\rangle \tag{2.4}$$

holds for all $\varphi \in X$, where the function space X is defined in (2.3).

Proof. As U is a simple Helmholtz domain, by Corollary 2.6, we find a potential $\Theta \in H^1_\sharp(U)$ and a constant $c_0 \in \mathbb{C}^3$ such that $u = \nabla\Theta + c_0$ in U and $\langle c_0, \mathrm{e}_k\rangle = 0$ for all $k \in \mathcal{N}_U$. Substituting this decomposition into the left-hand side of (2.4), we find that

$$\int_Y \langle u, \varphi\rangle = \int_U \langle \nabla\Theta, \varphi\rangle + \int_U \langle c_0, \varphi\rangle = \int_U \langle \nabla\Theta, \varphi\rangle + \sum_{k\in\mathcal{L}_U} \langle c_0, \mathrm{e}_k\rangle \Big\langle \mathrm{e}_k, \int_Y \varphi \Big\rangle .$$

Note that the function space X is a subset of $H_\sharp(\operatorname{div}, Y)$. We are thus allowed to use integration by parts in the first integral on the right-hand side. Using that φ is divergence free in Y and vanishes in $Y \setminus \overline{U}$, we find that

$$\int_Y \langle u, \varphi\rangle = \sum_{k\in\mathcal{L}_U} \langle c_0, \mathrm{e}_k\rangle \Big\langle \mathrm{e}_k, \int_Y \varphi \Big\rangle = \Big\langle \oint u, \int_Y \varphi \Big\rangle ,$$

and hence the claimed equality holds. □

When we derive effective equations, we need the following property of the geometric average, which is a consequence of the Lemma 2.11.

Corollary 2.12. *Assume that $U \subset Y$ is a simple Helmholtz domain, and let $u\colon Y \to \mathbb{C}^3$ be a vector field of class $L^2_\sharp(Y;\mathbb{C}^3)$ that is curl free in U. If $E\colon Y \to \mathbb{C}^3$ is another field of class $L^2_\sharp(Y;\mathbb{C}^3)$ that is curl free and that vanishes in $Y \setminus \overline{U}$, then*

$$\int_U (u \wedge E) = \Big(\oint u \Big) \wedge \Big(\int_Y E \Big) . \tag{2.5}$$

Proof. Fix $k \in \{1,2,3\}$. Defining the field $\varphi\colon Y \to \mathbb{C}^3$ by $\varphi := E \wedge \mathrm{e}_k$ provides us with an element of X; indeed, φ is of class $L^2_\sharp(Y;\mathbb{C}^3)$ and vanishes in $Y \setminus \overline{U}$. Moreover, for every $\phi \in C^\infty_c(Y)$, we calculate that

$$\int_Y \langle \varphi, \nabla\phi\rangle = -\int_Y \langle E, \nabla\phi \wedge \mathrm{e}_k\rangle = -\int_Y \langle E, \operatorname{curl}(\phi\, \mathrm{e}_k)\rangle = 0.$$

That is, φ is divergence free in Y. We can thus apply Lemma 2.11 and obtain that

$$\int_U \langle u \wedge E, \mathrm{e}_k\rangle = \int_Y \langle u, \varphi\rangle = \Big\langle \oint u, \int_Y (E \wedge \mathrm{e}_k) \Big\rangle = \Big\langle \oint u, \Big(\int_Y E \Big) \wedge \mathrm{e}_k \Big\rangle .$$

As $k \in \{1,2,3\}$ was chosen arbitrarily, this yields (2.5). □

Remark 2.13. The statement of the corollary remains true when we replace the assumption on U to be a simple Helmholtz domain by the assumption that $\overline{U} \cap \partial Y = \emptyset$. The geometric average $\oint u$ is then given by the second part of Definition 2.8. A proof of the corollary in this situation is given in [17].

3 Cell problems and their analysis

In this section, we study sequences $(E^\eta)_\eta$ and $(H^\eta)_\eta$ of solutions to (1.1) and their two-scale limits E_0 and H_0.

3.1 Cell problem for E_0

Lemma 3.1 (Cell problem for E_0)**.** *Let $R \subset\subset \Omega \subset \mathbb{R}^3$ and $\Sigma \subset Y$ be as in Section 1.1, and let $(E^\eta, H^\eta)_\eta$ be a sequence of solutions to* (1.1) *that satisfies the energy-bound* (1.3)*. The two-scale limit $E_0 \in L^2(\Omega \times Y; \mathbb{C}^3)$ satisfies the following:*

i) For almost all $x \in R$ the field $E_0 = E_0(x, \cdot)$ is an element of $H_\sharp(\mathrm{curl}\,, Y)$ and a distributional solution to

$$\mathrm{curl}_y\; E_0 = 0 \quad \text{in } Y\,, \tag{3.1a}$$

$$\mathrm{div}_y\; E_0 = 0 \quad \text{in } Y \setminus \overline{\Sigma}\,, \tag{3.1b}$$

$$E_0 = 0 \quad \text{in } \Sigma\,. \tag{3.1c}$$

Outside the meta-material, the two-scale limit E_0 is y-independent; that is, $E_0(x, y) = E_0(x) = E(x)$ for a.e. $x \in \Omega \setminus R$ and a.e. $y \in Y$.

ii) Given $c \in \mathbb{C}^3$, there exists at most one solution $u \in L^2_\sharp(Y; \mathbb{C}^3)$ to (3.1) *with cell-average $\fint_Y u(y)\, dy = c$.*

Proof. i) The derivation of the cell problem is by now standard and can, for instance, be found in [16]. To give some ideas, fix $x \in \Omega$ and $\eta > 0$, and set $\varphi_\eta(x) := \varphi(x, x/\eta)$, where $\varphi \in C_c^\infty(\Omega; C_\sharp^\infty(Y; \mathbb{C}^3))$. Using integration by parts and the two-scale convergence of $(E^\eta)_\eta$ we obtain that

$$\lim_{\eta \to 0} \int_\Omega \langle \eta \,\mathrm{curl}\; E^\eta(x), \varphi_\eta(x) \rangle \,\mathrm{d}x = \int_\Omega \int_Y \langle E_0(x, y), \mathrm{curl}_y\; \varphi(x, y) \rangle \,\mathrm{d}y \,\mathrm{d}x\,.$$

From this and (1.1a), we deduce that E_0 is a distributional solution to (3.1a). Equation (3.1b) is a consequence of (1.1b), and (3.1c) follows from (1.1c). Due to (3.1a), there holds $E_0 \in H_\sharp(\mathrm{curl}\,, Y)$.

In $\Omega \setminus \overline{\Sigma}$, the cell problem for $E_0 = E_0(x, \cdot)$ coincides with (3.1) if we replace Σ by the empty set $\emptyset$. This problem, however, has only the constant solution.

ii) Let $u \in L^2_\sharp(Y; \mathbb{C}^3)$ be a distributional solution to (3.1) with $\fint_Y u = 0$. We claim that the field u vanishes identically in Y. Indeed, (3.1a) implies the existence of a potential $\Theta \in H^1_\sharp(Y)$ and of a constant $c_0 \in \mathbb{C}^3$ such that $u = \nabla\Theta + c_0$ in Y. We may assume that $\fint_Y \Theta = 0$. As u has vanishing average, we conclude that $c_0 = 0$. On account of (3.1b) and (3.1c), the potential Θ is a distributional solution to

$$-\Delta\Theta = 0 \quad \text{in } Y \setminus \overline{\Sigma}\,,$$

$$\Theta = d \quad \text{in } \Sigma\,,$$

for some constant $d \in \mathbb{C}$. As $\Theta \in H^1_\sharp(Y)$, the potential Θ does not jump across the boundary $\partial\Sigma$. Consequently, $\Theta = d$ in Y and thus u vanishes in Y. □

While we obtained the uniqueness result immediately, the existence statement is more involved. To investigate the solution space to the cell problem (3.1), we use the two index sets from (2.1): $\mathcal{L}_\Sigma = \{k \in \{1, 2, 3\} \colon \text{there is a } k\text{-loop in } \Sigma\}$ and $\mathcal{N}_\Sigma = \{k \in \{1, 2, 3\} \colon \text{there is no } k\text{-loop in } \Sigma\} = \{1, 2, 3\} \setminus \mathcal{L}_\Sigma$. We claim that $|\mathcal{N}_\Sigma|$ coincides with the dimension of the solution space of (3.1).

Lemma 3.2 (Connection between the E_0-problem (3.1) and $\mathcal{N}_\Sigma$)**.** *For $k \in \mathcal{N}_\Sigma$ there exists a unique solution E^k to* (3.1) *with volume average e_k. On the other hand, if $c \in \mathbb{C}^3$ is such that $c \notin \operatorname{span}\{\mathrm{e}_k : k \in \mathcal{N}_\Sigma\}$, then there is no solution E to* (3.1) *with volume average c.*

Proof. Part 1. Let k be an element of $\mathcal{N}_\Sigma$. Our aim is to show that a solution E^k exist. Due to Lemma 2.5, there exists a potential $\tilde{\Theta}_k \in H^1_\sharp(\Sigma)$ such that $\nabla\tilde{\Theta}_k = -\mathrm{e}_k$.

We extend $\tilde{\Theta}_k$ to all of Y as follows: Let $\Theta_k \in H^1_\sharp(Y)$ be the weak solution to

$$\begin{aligned} -\Delta\Theta_k &= 0 \qquad \text{in } Y \setminus \overline{\Sigma}\,,\\ \Theta_k &= \tilde{\Theta}_k \quad\ \ \text{on } \overline{\Sigma}\,. \end{aligned}$$

By setting $E^k := \nabla\Theta_k + \mathrm{e}_k$, we obtain a solution to (3.1) whose cell average is e_k.

Part 2. Let $c \in \mathbb{C}^3$ be such that $c \notin \operatorname{span}\{\mathrm{e}_k : k \in \mathcal{N}_\Sigma\}$. Assume that there is a solution E to (3.1) with $⨍_Y E = c$. By Part 1, for every $k \in \mathcal{N}_\Sigma$ there is a solution E^k to (3.1) with $⨍_Y E^k = \mathrm{e}_k$. Consider the field

$$v := E - \sum_{k\in\mathcal{N}_\Sigma} \langle c, \mathrm{e}_k\rangle E^k\,.$$

This field is a solution to (3.1) with $⨍_Y v = \sum_{k\in\mathcal{L}_\Sigma}\langle c, \mathrm{e}_k\rangle \mathrm{e}_k$. As $c \notin \operatorname{span}\{\mathrm{e}_k : k \in \mathcal{N}_{Y\setminus\overline{\Sigma}}\}$, there holds $⨍_Y v \neq 0$. We find a potential $\Phi \in H^1_\sharp(Y)$ such that $v = \nabla\Phi + \sum_{l\in\mathcal{L}_\Sigma}\langle c, \mathrm{e}_l\rangle \mathrm{e}_l$ in Y since $\operatorname{curl} v = 0$ in Y. Fix an index $k \in \mathcal{L}_\Sigma$, and let γ be a k-loop in Σ. By (3.1c), we have that $\nabla\Phi \in C^0_\sharp(\Sigma;\mathbb{C}^3)$. As $v = 0$ in Σ, we calculate, by exploiting the periodicity of Φ in Y,

$$0 = \int_\gamma v = \int_\gamma \nabla\Theta + \sum_{l\in\mathcal{L}_\Sigma}\langle c, \mathrm{e}_l\rangle \int_\gamma \mathrm{e}_l = \langle c, \mathrm{e}_k\rangle\langle\tilde{\gamma}(1) - \tilde{\gamma}(0), \mathrm{e}_k\rangle\,. \tag{3.2}$$

Note that $\langle\tilde{\gamma}(1) - \tilde{\gamma}(0), \mathrm{e}_k\rangle \neq 0$ since γ is a k-loop. Thus $\langle c, \mathrm{e}_k\rangle = 0$ for all $k \in \mathcal{L}_\Sigma$. This, however, contradicts $⨍_Y v \neq 0$. □

Thanks to this lemma, we have the following result.

Proposition 3.3 (Characterisation of the solution space of the E_0-problem)**.** *For every index $k \in \mathcal{N}_\Sigma$, let $E^k = E^k(y)$ be the solution to* (3.1) *from Lemma 3.2. Then there holds: Every solution u to* (3.1) *can be written as*

$$u = \sum_{k\in\mathcal{N}_\Sigma} \alpha_k E^k \tag{3.3}$$

with constants $\alpha_k \in \mathbb{C}$ for $k \in \mathcal{N}_\Sigma$. In particular, the dimension of the solution space coincides with $|\mathcal{N}_\Sigma|$.

Proof. We need to prove that every solution u to (3.1) can be written as in (3.3).

Let $u \in H_\sharp(\operatorname{curl}, Y)$ be an arbitrary solution to (3.1). On account of (3.1a), we find a potential $\Theta \in H^1_\sharp(Y)$ and $c_0 \in \mathbb{C}^3$ such that $u = \nabla\Theta + c_0$ in Y. For each $k \in \{1, 2, 3\}$, we set $\alpha_k := \langle c_0, \mathrm{e}_k\rangle$. Consider the field

$$v := u - \sum_{k\in\mathcal{N}_\Sigma} c_k E^k\,.$$

This field is a solution to (3.1) with

$$\fint_Y v = \sum_{l\in\mathcal{L}_\Sigma} \alpha_l \mathrm{e}_l \,. \tag{3.4}$$

By the second statement of Lemma 3.2, the coefficient $\alpha_l = 0$ for all $l \in \mathcal{L}_\Sigma$. Hence $v = 0$ in Y by the uniqueness result from Lemma 3.1. This provides (3.3). □

Remark 3.4. The previous proposition implies, in particular, that the solution space to the cell problem of E_0 is at most three dimensional. Note that no assumption (such as simple connectedness) on the domain Σ was imposed here.

3.2 Cell problem for H_0

Lemma 3.5 (Cell problem for H_0). *Let $R \subset\subset \Omega \subset \mathbb{R}^3$ and $\Sigma \subset Y$ be as in Section 1.1, and let $(E^\eta, H^\eta)_\eta$ be a sequence of solutions to* (1.1) *that satisfies the energy-bound* (1.3). *The two-scale limit $H_0 \in L^2(\Omega \times Y; \mathbb{C}^3)$ satisfies the following:*

i) For almost all $x \in R$ the field $H_0 = H_0(x, \cdot)$ is an element of $H_\sharp(\operatorname{div}, Y)$ and a distributional solution to

$$\operatorname{curl}_y H_0 = 0 \quad \text{in } Y \setminus \overline{\Sigma}\,, \tag{3.5a}$$

$$\operatorname{div}_y H_0 = 0 \quad \text{in } Y\,, \tag{3.5b}$$

$$H_0 = 0 \quad \text{in } \Sigma\,. \tag{3.5c}$$

Outside the meta-material, the two-scale limit H_0 is y-independent; that is, $H_0(x, y) = H_0(x) = H(x)$ for a.e. $x \in \Omega \setminus R$ and a.e. $y \in Y$.

ii) If $Y \setminus \overline{\Sigma}$ is a simple Helmholtz domain, then for every $c \in \mathbb{C}^3$ there is at most one solution $u \in H_\sharp(\operatorname{div}, Y)$ to (3.5) *with geometric average $\oint u = c$.*

Proof. i) H_0 is a distributional solution to (3.5). Exploiting the two-scale convergence of $(H^\eta)_\eta$ and $(E^\eta)_\eta$, we deduce (3.5a) by Maxwell's equation (1.1b). By (1.1a), each H^η is a divergence-free field in Ω, and hence $\operatorname{div}_y H_0 = 0$ in Y. On account of (1.1c), the field $H^\eta \mathbb{1}_{\Sigma_\eta}$ vanishes identically in R. Thus, $0 = H^\eta \mathbb{1}_{\Sigma_\eta} \xrightarrow{2} H_0 \mathbb{1}_\Sigma$ implies that $H_0(x, y) = 0$ for almost all $(x, y) \in R \times \Sigma$, and hence (3.5c).

Outside the meta material, the fields $H_0 = H_0(x, \cdot)$ and H coincide since the corresponding cell problem admits only constant solutions.

ii) Uniqueness. Let $Y \setminus \overline{\Sigma}$ be a simple Helmholtz domain, and let $u \in L^2_\sharp(Y; \mathbb{C}^3)$ be a solution to (3.5) with vanishing geometric average, $\oint u = 0$. We claim that u vanishes identically in Y. As u is curl free in the simple Helmholtz domain $Y \setminus \overline{\Sigma}$, we find a potential $\Theta \in H^1_\sharp(Y \setminus \overline{\Sigma})$ and a constant $c_0 \in \mathbb{C}^3$ such that $u = \nabla\Theta + c_0$ in $Y \setminus \overline{\Sigma}$. For an index $k \in \mathcal{L}_{Y\setminus\overline{\Sigma}}$, we can apply the first part of Definition 2.8 and find that $(\oint u)_k = \langle c_0, \mathrm{e}_k\rangle$. By assumption, $\oint u = 0$ and hence $\langle c_0, \mathrm{e}_k\rangle = 0$ for all $k \in \mathcal{L}_{Y\setminus\overline{\Sigma}}$. Due to Lemma 2.5, for every $k \in \mathcal{N}_{Y\setminus\overline{\Sigma}}$, we find a potential $\Theta_k \in H^1_\sharp(Y \setminus \overline{\Sigma})$ such that $\nabla\Theta_k = \mathrm{e}_k$. The function $\tilde\Theta := \Theta + \sum_{k\in\mathcal{N}_{Y\setminus\overline{\Sigma}}} \langle c_0, \mathrm{e}_k\rangle \Theta_k$ is an element of $H^1_\sharp(Y \setminus \overline{\Sigma})$. Moreover, $u = \nabla\Theta + \sum_{k\in\mathcal{N}_{Y\setminus\overline{\Sigma}}} \langle c_0, \mathrm{e}_k\rangle = \nabla\tilde\Theta$ in $Y \setminus \overline{\Sigma}$. Equations (3.5b) and (3.5c) imply

$0 = \langle u, \nu \rangle = \partial_\nu \tilde{\Theta}$ on $\partial\Sigma$, where ν is the outward unit normal vector. We conclude that $\tilde{\Theta}$ is a weak solution to

$$\begin{aligned} -\Delta \tilde{\Theta} &= 0 \quad \text{in } Y \setminus \overline{\Sigma}\,, \\ \partial_\nu \tilde{\Theta} &= 0 \quad \text{on } \partial\Sigma\,. \end{aligned}$$

Solutions to this Neumann boundary problem are constant since $Y \setminus \overline{\Sigma}$ is a domain. Hence $u = 0$ in Y. □

Remark 3.6. Note that, in contrast to the E_0-problem (see Lemma 3.1), the uniqueness statement of *ii)* is false if we do not assume that $Y \setminus \overline{\Sigma}$ is a simple Helmholtz domain. Indeed, in [17], a 3-dimensional full torus Σ is studied and a non-trivial solution with vanishing geometric average is found.

Lemma 3.7 (Connection between the H_0-problem (3.5) and $\mathcal{L}_{Y\setminus\overline{\Sigma}}$)**.** *Let $Y \setminus \overline{\Sigma}$ be a simple Helmholtz domain. If the index $k \in \{1, 2, 3\}$ is an element of $\mathcal{L}_{Y\setminus\overline{\Sigma}}$, then there exists a unique solution H^k to* (3.5) *with geometric average e_k.*

Proof. Fix $k \in \mathcal{L}_{Y\setminus\overline{\Sigma}}$ and let $\Theta_k \in H^1_\sharp(Y \setminus \overline{\Sigma})$ be a distributional solution to

$$\begin{aligned} -\Delta \Theta_k &= 0 && \text{in } Y \setminus \overline{\Sigma}\,, \\ \partial_\nu \Theta_k &= -\langle \mathrm{e}_k, \nu \rangle && \text{on } \partial\Sigma\,. \end{aligned}$$

We define $H^k \colon Y \to \mathbb{C}^3$ by

$$H^k := \begin{cases} \nabla \Theta_k + \mathrm{e}_k & \text{in } Y \setminus \overline{\Sigma}\,, \\ 0 & \text{in } \Sigma\,. \end{cases}$$

In this way, we obtain an $L^2_\sharp(Y; \mathbb{C}^3)$-vector field H^k that is a distributional solution to (3.5). As $k \in \mathcal{L}_{Y\setminus\overline{\Sigma}}$, we obtain, using the definition of the geometric average, $\oint H^k = \mathrm{e}_k$. □

Proposition 3.8 (Characterisation of the solution space to the H_0-problem)**.** *Let $Y \setminus \overline{\Sigma}$ be a simple Helmholtz domain. For every index $k \in \mathcal{L}_{Y\setminus\overline{\Sigma}}$, let $H^k = H^k(y)$ be the solution to* (3.5) *from Lemma 3.7. Then there holds: Every solution u to* (3.5) *can be written as*

$$u = \sum_{k \in \mathcal{L}_{Y\setminus\overline{\Sigma}}} \alpha_k H^k\,, \tag{3.6}$$

with constants $\alpha_k := (\oint u)_k \in \mathbb{C}$ for $k \in \mathcal{L}_{Y\setminus\overline{\Sigma}}$. In particular, the dimension of the solution space coincides with $|\mathcal{L}_{Y\setminus\overline{\Sigma}}|$.

Proof. We use the solutions H^k of Lemma 3.7. The set $\{H^k \colon k \in \mathcal{L}_{Y\setminus\overline{\Sigma}}\}$ is linearly independent since the geometric averages of the H^k are linear independent. We need to prove that every solution u to (3.5) can be written as in (3.6).

Let $u \in H_\sharp(\operatorname{div}, Y)$ be a solution to (3.5). We define

$$v := u - \sum_{k \in \mathcal{L}_{Y\setminus\overline{\Sigma}}} \left(\oint u \right)_k H^k\,.$$

The field v is also a solution to (3.5) and has the geometric average

$$\oint v = \sum_{k \in \mathcal{N}_{Y\setminus\overline{\Sigma}}} \left(\oint u \right)_k \mathrm{e}_k \,.$$

As those components of the geometric average for which there is no loop in $Y \setminus \overline{\Sigma}$ vanish, by the first part of the definition of the geometric average, the right-hand side vanishes. Due to the uniqueness statement of Lemma 3.5, v vanishes in Y. This provides (3.6). □

Remark 3.9. As a consequence of the previous proposition, we find that the solution space to the cell problem of H_0 is at most three dimensional if $Y \setminus \overline{\Sigma}$ is a simple Helmholtz domain.

Remark 3.10. Geometric intuition suggests that $|\mathcal{L}_{Y\setminus\overline{\Sigma}}| \geq |\mathcal{N}_\Sigma|$; there is, however, no obvious proof of this fact. As a consequence of this inequality, we find that the dimensions d_E and d_H of the solution spaces to the E_0-problem (3.1) and to the H_0-problem (3.5) satisfy $d_H \geq d_E$.

4 Derivation of the effective equations

Our aim in this section is to derive the effective Maxwell system. We assume that $\Omega \subset \mathbb{R}^3$, the subdomain $R \subset\subset \Omega$, and $\Sigma \subset Y$ are as in Section 1.1. We work with the two index sets $\mathcal{N}_\Sigma$ and $\mathcal{L}_{Y\setminus\overline{\Sigma}}$ defined in (2.1b) and (2.1a), respectively.

We define the matrices ε_{eff}, $\mu_{\text{eff}} \in \mathbb{R}^{3\times 3}$ by setting, for $k \in \{1,2,3\}$,

$$(\varepsilon_{\text{eff}})_{kl} := \begin{cases} \langle E^k, E^l \rangle_{L^2(Y;\mathbb{C}^3)} & \text{if } k,l \in \mathcal{N}_\Sigma \,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{4.1}$$

and

$$\mu_{\text{eff}}\, \mathrm{e}_k := \begin{cases} \oint_Y H^k & \text{if } k \in \mathcal{L}_{Y\setminus\overline{\Sigma}} \,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.2}$$

The *effective permittivity* $\hat{\varepsilon}\colon \Omega \to \mathbb{R}^{3\times 3}$ and the *effective permeability* $\hat{\mu}\colon \Omega \to \mathbb{R}^{3\times 3}$ are defined by

$$\hat{\varepsilon}(x) := \begin{cases} \varepsilon_{\text{eff}} & \text{if } x \in R\,, \\ \mathrm{id}_3 & \text{otherwise}\,, \end{cases} \qquad \hat{\mu}(x) := \begin{cases} \mu_{\text{eff}} & \text{if } x \in R\,, \\ \mathrm{id}_3 & \text{otherwise}\,, \end{cases} \tag{4.3}$$

where $\mathrm{id}_3 \in \mathbb{R}^{3\times 3}$ is the identity matrix.

Let $(E^\eta, H^\eta)_\eta$ be a sequence of solutions to (1.1) that satisfies the energy-bound (1.3); the corresponding two-scale limits are denoted by E_0, $H_0 \in L^2(\Omega \times Y; \mathbb{C}^3)$. Assume that $Y \setminus \overline{\Sigma}$ is such that we can define the geometric average. We then define the *limit fields* $\hat{E}, \hat{H}\colon \Omega \to \mathbb{C}^3$ by

$$\hat{E}(x) := \int_Y E_0(x,y)\,\mathrm{d}y \quad \text{and} \quad \hat{H}(x) := \oint H_0(x,\cdot)\,. \tag{4.4}$$

We recall that $H_0(x,\cdot)$ solves (3.5); it is therefore curl free in $Y \setminus \overline{\Sigma}$ and vanishes in Σ. The two fields $\hat{E}$ and $\hat{H}$ are of class $L^2(\Omega; \mathbb{C}^3)$.

Theorem 4.1 (Macroscopic equations). *Let $(E^\eta, H^\eta)_\eta$ be a sequence of solutions to* (1.1) *that satisfies the energy bound* (1.3)*. Assume that the geometry $\Sigma_\eta \subset R \subset \Omega$ is as in Section 1.1. Let $Y \setminus \overline{\Sigma}$ be a simple Helmholtz domain, and let $\mathcal{N}_\Sigma$ and $\mathcal{L}_{Y\setminus\overline{\Sigma}}$ be the corresponding index sets. Let the effective permittivity $\hat{\varepsilon}$ and the effective permeability $\hat{\mu}$ be defined as in* (4.3)*, and let the limit fields $(\hat{E}, \hat{H})$ be defined as in* (4.4)*. Then $\hat{E}$ and $\hat{H}$ are distributional solutions to*

$$\operatorname{curl} \hat{E} = \mathrm{i}\,\omega\mu_0\hat{\mu}\hat{H} \qquad \text{in } \Omega\,, \tag{4.5a}$$

$$\operatorname{curl} \hat{H} = -\,\mathrm{i}\,\omega\varepsilon_0\hat{\varepsilon}\hat{E} \qquad \text{in } \Omega \setminus R\,, \tag{4.5b}$$

$$(\operatorname{curl} \hat{H})_k = -\,\mathrm{i}\,\omega\varepsilon_0(\hat{\varepsilon}\hat{E})_k \qquad \text{in } \Omega\,,\ \text{for every } k \in \mathcal{N}_\Sigma\,, \tag{4.5c}$$

$$\hat{E}_k = 0 \qquad \text{in } R\,,\ \text{for every } k \in \{1,2,3\} \setminus \mathcal{N}_\Sigma\,, \tag{4.5d}$$

$$\hat{H}_k = 0 \qquad \text{in } R\,,\ \text{for every } k \in \mathcal{N}_{Y\setminus\overline{\Sigma}}\,. \tag{4.5e}$$

Proof. Thanks to the preparations of the last section, we can essentially follow [17] to derive (4.5a)–(4.5c). The remaining relations (4.5d) and (4.5e) follow from the characterisation of the solution spaces of the cell problems.

Step 1: Derivation of (4.5a)*.* The distributional limit of (1.1a) reads

$$\operatorname{curl} E = \mathrm{i}\,\omega\mu_0 H \quad \text{in } \Omega\,. \tag{4.6}$$

We recall that E and H are the weak $L^2_\sharp(Y;\mathbb{C}^3)$-limits of $(E^\eta)_\eta$ and $(H^\eta)_\eta$, respectively. By the definition of the limit $\hat{E}$ in (4.4) and the volume-average property of the two-scale limit E_0, we find that

$$\hat{E}(x) = \int_Y E_0(x,y)\,\mathrm{d}y = E(x)\,,$$

for a.e. $x \in \Omega$. Thus, $\operatorname{curl} E = \operatorname{curl} \hat{E}$.

On the other hand, using that $Y \setminus \overline{\Sigma}$ is a simple Helmholtz domain, for almost all $(x,y) \in R \times Y$, the two-scale limit H_0 can be written with coefficients $H_k(x)$ as

$$H_0(x,y) = \sum_{k\in\mathcal{L}_{Y\setminus\overline{\Sigma}}} H_k(x) H^k(y)\,, \tag{4.7}$$

by Proposition 3.8. The averaging property of the two-scale limit, the identity (4.7), and the definition of μ_{eff} imply that

$$H(x) = \int_Y H_0(x,y)\,\mathrm{d}y = \sum_{k\in\mathcal{L}_{Y\setminus\overline{\Sigma}}} H_k(x)\int_Y H^k(y)\,\mathrm{d}y = \mu_{\text{eff}} \sum_{k\in\mathcal{L}_{Y\setminus\overline{\Sigma}}} H_k(x)\mathrm{e}_k\,. \tag{4.8}$$

Using the definition of the limit field $\hat{H}$ in (4.4) and identity (4.7), we conclude from (4.8) that

$$H(x) = \mu_{\text{eff}}\hat{H}(x)\,. \tag{4.9}$$

Outside the meta-material the $L^2(Y;\mathbb{C}^3)$-weak limit H and the two-scale limit H_0 coincide due to Lemma 3.5. Moreover, $\hat{\mu}$ equals the identity, and hence (4.9) holds also in $\Omega \setminus R$. From (4.9) and (4.6) we conclude (4.5a).

Step 2: Derivation of (4.5b)*,* (4.5d) *and* (4.5e)*.* To prove (4.5b), we first observe that $\Omega \setminus R \subset \Omega \setminus \Sigma_\eta$. We can therefore take the distributional limit in (1.1b) as η tends to zero and obtain that

$$\operatorname{curl} H = -\,\mathrm{i}\,\omega\varepsilon_0 E \quad \text{in } \Omega \setminus R\,.$$

This shows (4.5b), since $\hat{H} = H$ and $\hat{E} = E$ in $\Omega \setminus R$.

By Proposition 3.3, the two-scale limit E_0 can be written with coefficients $E_k(x)$ as $E_0(x, y) = \sum_{k\in\mathcal{N}_\Sigma} E_k(x)E^k(y)$. Due to the definition of the limit field $\hat{E}$ in (4.4), we find that

$$\hat{E}(x) = \int_Y \sum_{k\in\mathcal{N}_\Sigma} E_k(x)E^k(y)\,\mathrm{d}y = \sum_{k\in\mathcal{N}_\Sigma} E_k(x)\mathrm{e}_k\,, \tag{4.10}$$

for $x \in R$. Consequently, equation (4.5d) holds. Similarly, the definition of $\hat{H}$ and Proposition 3.8 imply that

$$\hat{H}(x) = \oint \sum_{k\in\mathcal{L}_{Y\setminus\overline{\Sigma}}} H_k(x)H^k(y)\,\mathrm{d}y = \sum_{k\in\mathcal{L}_{Y\setminus\overline{\Sigma}}} H_k(x)\mathrm{e}_k$$

for almost all $x \in R$. This proves (4.5e).

Step 3: Derivation of (4.5c). We use the defining property of the two-scale convergence and appropriate oscillating test functions. For $k \in \mathcal{N}_\Sigma$ and $\theta \in C_c^\infty(\Omega;\mathbb{R})$, we set $\varphi(x, y) := \theta(x)E^k(y)$ for $x \in \Omega$ and $y \in Y$. We use $\varphi_\eta(x) := \varphi(x, x/\eta)$ for $x \in \Omega$. From the two-scale convergence of $(H^\eta)_\eta$, we obtain that

$$\begin{aligned}\lim_{\eta\to 0}\int_\Omega \langle H^\eta, \operatorname{curl}\,\varphi_\eta\rangle &= \int_\Omega\int_Y \langle H_0(x, y), \nabla\theta(x) \wedge E^k(y)\rangle\,\mathrm{d}y\,\mathrm{d}x\\ &= -\int_\Omega \Big\langle \nabla\theta(x), \int_Y H_0(x, y) \wedge E^k(y)\,\mathrm{d}y\Big\rangle\,\mathrm{d}x\,.\end{aligned}$$

Thanks to Corollary 2.12 on the geometric average, the identity

$$\int_Y H_0(x, y) \wedge E^k(y)\,\mathrm{d}y = \oint H_0(x, \cdot) \wedge \mathrm{e}_k$$

holds for almost all $x \in \Omega$. Consequently,

$$\begin{aligned}\lim_{\eta\to 0}\int_\Omega \langle H^\eta, \operatorname{curl}\,\varphi_\eta\rangle &= -\int_\Omega \Big\langle \nabla\theta(x), \oint H_0(x, \cdot) \wedge \mathrm{e}_k\Big\rangle\,\mathrm{d}x\\ &= \int_\Omega \langle \hat{H}(x), \operatorname{curl}\,\big(\theta(x)\mathrm{e}_k\big)\rangle\,\mathrm{d}x\,.\end{aligned} \tag{4.11}$$

On the other hand, H^η is a distributional solution to Maxwell's equation (1.1b). Thus, using (4.10)

$$\begin{aligned}\lim_{\eta\to 0}\int_\Omega \langle H^\eta, \operatorname{curl}\,\varphi_\eta\rangle &= -\,\mathrm{i}\,\omega\varepsilon_0 \lim_{\eta\to 0}\int_\Omega \langle E^\eta, \varphi_\eta\rangle\\ &= -\,\mathrm{i}\,\omega\varepsilon_0 \int_\Omega \Big(\int_Y \langle E_0(x, y), E^k(y)\rangle\,\mathrm{d}y\Big)\theta(x)\,\mathrm{d}x\\ &= -\,\mathrm{i}\,\omega\varepsilon_0 \sum_{l\in\mathcal{N}_\Sigma}\int_\Omega (\varepsilon_{\mathrm{eff}})_{kl}E_l(x)\theta(x)\,\mathrm{d}x\\ &= \int_\Omega \langle -\,\mathrm{i}\,\omega\varepsilon_0\varepsilon_{\mathrm{eff}}\hat{E}(x), \theta(x)\mathrm{e}_k\rangle\,\mathrm{d}x\,.\end{aligned} \tag{4.12}$$

As $\theta \in C_c^\infty(\Omega;\mathbb{R})$ was chosen arbitrarily, (4.11) and (4.12) imply that

$$(\operatorname{curl}\,\hat{H})_k = -\,\mathrm{i}\,\omega\varepsilon_0(\varepsilon_{\mathrm{eff}}\hat{E})_k \quad \text{for all } k \in \mathcal{N}_\Sigma\,.$$

This shows (4.5c) and hence the theorem is proved. □

Remark 4.2 (Well-posedness of system (4.5)). We claim that the effective Maxwell system (4.5) forms a complete set of equations. To be more precise, we show the following: Let $R \subset \Omega$ be a cuboid that is parallel to the axes and let Ω be a bounded Lipschitz domain. Assume that $\hat{\mu}$ is real and positive definite on $V \coloneqq \operatorname{span}\{\mathrm{e}_k \colon k \in \mathcal{L}_{Y\setminus\overline{\Sigma}}\}$—that is, there is a constant $\alpha > 0$ such that $\langle \hat{\mu}\xi, \xi\rangle \geq \alpha|\xi|^2$ for all $\xi \in V$. We also assume that $\hat{\varepsilon}$ is real and positive definite on $\operatorname{span}\{\mathrm{e}_k \colon k \in \mathcal{N}_\Sigma\}$, and that $\mu_0 > 0$, $\operatorname{Re}\varepsilon_0 > 0$, and $\operatorname{Im}\varepsilon_0 > 0$. Let $(\hat{E}, \hat{H})$ be a solution to (4.5) with boundary condition $\hat{H} \wedge \nu = 0$ on $\partial\Omega$, and assume that both $\hat{E}$ and $\hat{H}$ are of class H^1 in R and in $\Omega \setminus R$. We claim that $\hat{E}$ and $\hat{H}$ are trivial.

To prove that $\hat{E} = \hat{H} = 0$, we first show that the integration by parts formula

$$\int_\Omega \langle \operatorname{curl} \hat{H}, \hat{E}\rangle = \int_\Omega \langle \hat{H}, \operatorname{curl} \hat{E}\rangle \,. \tag{4.13}$$

holds. Equation (4.13) is a consequence of an integration by parts provided that the integral $\int_{\partial R} [\hat{H}](\hat{E} \wedge \nu)$ vanishes, where $[\hat{H}]$ is the jump of the field $\hat{H}$ across the boundary ∂R and ν is the outward unit normal vector on ∂R. As $\hat{E} \in H(\operatorname{curl}, \Omega)$, there holds $[\hat{E} \wedge \nu] = 0$ (i.e., the tangential components of $\hat{E}$ do not jump).

Let Γ be one face of R. To prove $\int_\Gamma [\hat{H}](\hat{E} \wedge \nu) = 0$, it suffices to show that for all $k, l \in \{1, 2, 3\}$ with $k \neq l$ and $\langle \mathrm{e}_k, \nu\rangle = \langle \mathrm{e}_l, \nu\rangle = 0$, we have that $\hat{E}_k [\hat{H}_l]_\Gamma = 0$. We obtain this relation from (4.5c) and (4.5d): Indeed, for $k \notin \mathcal{N}_\Sigma$, there holds $\hat{E}_k = 0$ because of (4.5d). On the other hand, by (4.5c), for $k \in \mathcal{N}_\Sigma$ there holds $\partial_m H_l - \partial_l H_m = \mp \mathrm{i}\, \omega\varepsilon_0 (\hat{\varepsilon}\hat{E})_k$ in the distributional sense, where $\nu = \mathrm{e}_m$. This implies that $[H_l]_\Gamma = 0$.

We now show that for almost all $x \in \Omega$

$$\langle \operatorname{curl} \hat{H}(x), \hat{E}(x)\rangle = -\mathrm{i}\, \omega\varepsilon_0 \langle \hat{\varepsilon}\hat{E}(x), \hat{E}(x)\rangle \,. \tag{4.14}$$

In $\Omega \setminus R$, this identity is a consequence of (4.5b). In R on the other hand, we conclude from (4.5d) and (4.5c) that, for $x \in R$,

$$\langle \operatorname{curl} \hat{H}(x), \hat{E}(x)\rangle = \sum_{k \in \mathcal{N}_\Sigma} (\operatorname{curl} \hat{H}(x))_k \, \overline{\hat{E}_k(x)} = -\mathrm{i}\, \omega\varepsilon_0 \sum_{k \in \mathcal{N}_\Sigma} (\varepsilon_{\mathrm{eff}} \hat{E}(x))_k \, \overline{\hat{E}_k(x)} \,.$$

Applying again (4.5d), we obtain (4.14).

From (4.5a), the integration by parts formula (4.13), and (4.14), we obtain

$$\int_\Omega \langle \hat{\mu}\hat{H}, \hat{H}\rangle = -\frac{\mathrm{i}}{\omega\mu_0} \int_\Omega \langle \operatorname{curl} \hat{E}, \hat{H}\rangle = -\frac{\mathrm{i}}{\omega\mu_0} \int_\Omega \langle \hat{E}, \operatorname{curl} \hat{H}\rangle = \frac{\varepsilon_0}{\mu_0} \langle \hat{\varepsilon}\hat{E}, \hat{E}\rangle \,.$$

As $\hat{\mu}$ and μ_0 are assumed to be real, by taking the imaginary part, we find that $\int_\Omega \operatorname{Im}\varepsilon_0 \langle \hat{\varepsilon}\hat{E}, \hat{E}\rangle = 0$, and hence $\langle \hat{\varepsilon}\hat{E}, \hat{E}\rangle = 0$ almost everywhere in Ω. This implies that $\hat{E} = 0$ in $\Omega \setminus R$ and, taking into account that $\hat{\varepsilon}$ is positive definite on $\operatorname{span}\{\mathrm{e}_k \colon k \in \mathcal{N}_\Sigma\}$, we also find $\hat{E} = 0$ in R. We can therefore conclude from (4.5a) that $\hat{H}$ vanishes in $\Omega \setminus R$ and that $\hat{H}_k = 0$ in R for all $k \in \mathcal{L}_{Y\setminus\overline{\Sigma}}$. On account of (4.5e), $\hat{H} = 0$ in Ω. This shows the uniqueness of solutions.

5 Discussion of examples

In this section, we apply Theorem 4.1 to some examples. In what follows, d_E and d_H denote the dimension of the solution space to the cell problem (3.1) of E_0 and (3.5) of H_0, respectively. Due to Propositions 3.3 and 3.8, we find that $d_E = |\mathcal{N}_\Sigma|$ and $d_H = |\mathcal{L}_{Y\setminus\overline{\Sigma}}|$.

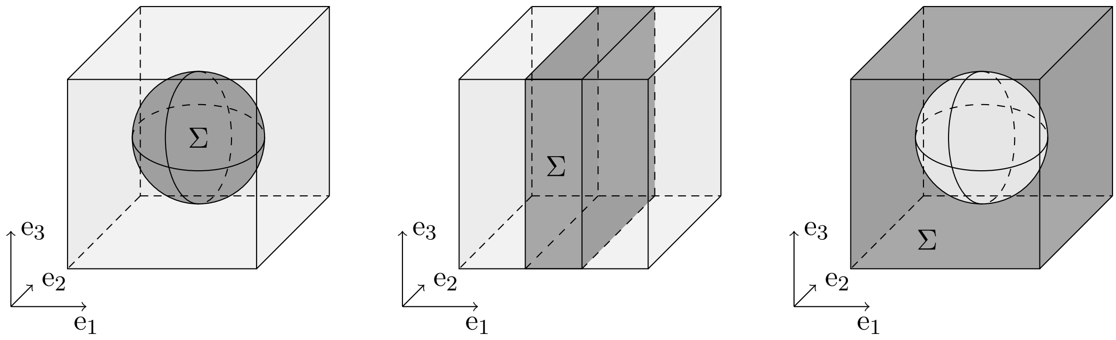


Figure 3: The periodicity cell Y is represented by the cube. (a) The metal ball of Example 5.1. (b) The metal plate of Example 5.2. (c) A sketch of the air ball (see Remark 5.3).

5.1 The metal ball

To define the metallic ball structure, we fix a number $r \in (0, 1/2)$, and set

$$\Sigma := \{y = (y_1, y_2, y_3) \in Y : y_1^2 + y_2^2 + y_3^2 < r^2\}\,. \tag{5.1}$$

A sketch of the periodicity cell is given in Figure 3 (a). We note that $Y \setminus \overline{\Sigma}$ is a simple Helmholtz domain. As each two opposite faces of Y can be connected by a loop in $Y \setminus \overline{\Sigma}$, we have that $\mathcal{L}_{Y \setminus \overline{\Sigma}} = \{1, 2, 3\}$. On the other hand, we find no loop in Σ that connects two opposite faces of Y; hence $\mathcal{N}_\Sigma = \{1, 2, 3\}$. To summarise, we have that

$\mathcal{N}_\Sigma$	d_E	$\mathcal{L}_{Y\setminus\overline{\Sigma}}$	d_H
$\{1, 2, 3\}$	3	$\{1, 2, 3\}$	3

.

The Maxwell system (4.5) is of the usual form; to be more precise, the following result holds.

Corollary 5.1 (Macroscopic equations of the metal ball)**.** *For* Σ *as in* (5.1)*, a sequence* $(E^\eta, H^\eta)_\eta$*, and limits* $\hat{E}$ *and* $\hat{H}$ *as in Theorem 4.1, the macroscopic equations read*

$$\operatorname{curl} \hat{E} = \mathrm{i}\,\omega\mu_0\hat{\mu}\hat{H} \quad \text{in } \Omega\,, \tag{5.2a}$$

$$\operatorname{curl} \hat{H} = -\,\mathrm{i}\,\omega\varepsilon_0\hat{\varepsilon}\hat{E} \quad \text{in } \Omega\,. \tag{5.2b}$$

5.2 The metal plate

To define the metal plate structure, fix a number $\gamma \in (0, 1/2)$ and set

$$\Sigma := \{y = (y_1, y_2, y_3) \in Y : y_1 \in (-\gamma, \gamma)\}\,. \tag{5.3}$$

We refer to Figure 3 (b) for a sketch of the periodicity cell Y. Observe that $Y \setminus \overline{\Sigma}$ is a simple Helmholtz domain. We obtain the table

$\mathcal{N}_\Sigma$	d_E	$\mathcal{L}_{Y\setminus\overline{\Sigma}}$	d_H
$\{1\}$	1	$\{2, 3\}$	2

.

In fact, we do not only know the dimensions of the solution spaces to (3.1) and (3.5) but also bases for these spaces. Indeed, for the volume fraction $\alpha := |Y \setminus \overline{\Sigma}|$, the field $E^1 : Y \to \mathbb{C}^3$ given by $E^1(y) := \mathrm{e}_1 \alpha^{-1} \mathbb{1}_{Y\setminus\overline{\Sigma}}(y)$ is a solution to (3.1) with $\fint_Y E^1 = \mathrm{e}_1$. On the other hand, for $k \in \{2,3\}$, the field $H^k : Y \to \mathbb{C}^3$, $H^k(y) := \mathrm{e}_k \mathbb{1}_{Y\setminus\overline{\Sigma}}(y)$ is a solution to (3.5). By the first part of the definition of the geometric average, $\oint H^k = \mathrm{e}_k$ and hence $\{H^2, H^3\}$ is a basis of the solution space to (3.5). Having the basis for the solution spaces at hand, we can compute ε_{eff} and μ_{eff} defined in (4.1) and (4.2): we have that $\varepsilon_{\text{eff}} = \alpha^{-1} \operatorname{diag}(1,0,0)$ and $\mu_{\text{eff}} = \alpha \operatorname{diag}(0,1,1)$. An application of Theorem 4.1 yields the effective equations for the metal plate.

Corollary 5.2 (Macroscopic equations for the metal plate)**.** *Let Σ be as in* (5.3) *and let $\alpha := |Y \setminus \overline{\Sigma}|$. For a sequence $(E^\eta, H^\eta)_\eta$, and limit fields $\hat{E}$ and $\hat{H}$ as in Theorem 4.1, the macroscopic equations read*

$$\begin{aligned} \operatorname{curl} \hat{E} &= \mathrm{i}\,\omega\mu_0 \hat{H} && \text{in } \Omega \setminus R\,, && \text{(5.4a)}\\ \operatorname{curl} \hat{H} &= -\mathrm{i}\,\omega\varepsilon_0 \hat{E} && \text{in } \Omega \setminus R\,, && \text{(5.4b)}\\ (\partial_3, -\partial_2)\hat{E}_1 &= \mathrm{i}\,\omega\mu_0\alpha(\hat{H}_2, \hat{H}_3) && \text{in } R\,, && \text{(5.4c)}\\ \partial_2\hat{H}_3 - \partial_3\hat{H}_2 &= -\mathrm{i}\,\omega\varepsilon_0\alpha^{-1}\hat{E}_1 && \text{in } R\,, && \text{(5.4d)}\\ \hat{E}_2 = \hat{E}_3 &= 0 && \text{in } R\,, && \text{(5.4e)}\\ \hat{H}_1 &= 0 && \text{in } R\,. && \text{(5.4f)} \end{aligned}$$

5.3 The air cylinder

To define the metallic box with a cylinder removed, we fix a number $r \in (0, 1/2)$ and set

$$\Sigma := Y \setminus \{y = (y_1, y_2, y_3) \in Y : y_1^2 + y_2^2 < r^2\}\,.$$

A sketch of the periodicity cell is given in Figure 4 (a). The air cylinder $Y \setminus \overline{\Sigma}$ is a simple Helmholtz domain. We obtain

$\mathcal{N}_\Sigma$	d_E	$\mathcal{L}_{Y\setminus\overline{\Sigma}}$	d_H
$\emptyset$	0	$\{3\}$	1

.

Once again, we do not only know the dimension of the solution space to (3.5) but also its basis. Indeed, the field $H^3 : Y \to \mathbb{C}^3$ given by $H^3(y) := \mathrm{e}_k \mathbb{1}_{Y\setminus\overline{\Sigma}}(y)$ is a solution to (3.5). We can thus compute ε_{eff} and μ_{eff}; for $\alpha := |Y \setminus \overline{\Sigma}|$, we find that $\varepsilon_{\text{eff}} = 0$ and $\mu_{\text{eff}} = \alpha \operatorname{diag}(0,0,1)$. Although the solution space to the cell problem of H_0 is not trivial, there is only the trivial solution to the effective equations in R; that is,

$$\hat{E} = \hat{H} = 0 \quad \text{in } R\,.$$

Note that $\hat{H}_3 = 0$ is a consequence of (1.1a) and $\hat{E} = 0$ in R.

Remark 5.3. Instead of the air cylinder, we can also consider the air ball (see Figure 3 (c) for a sketch) and find that there are also only the trivial solutions $\hat{E} = \hat{H} = 0$ in R.

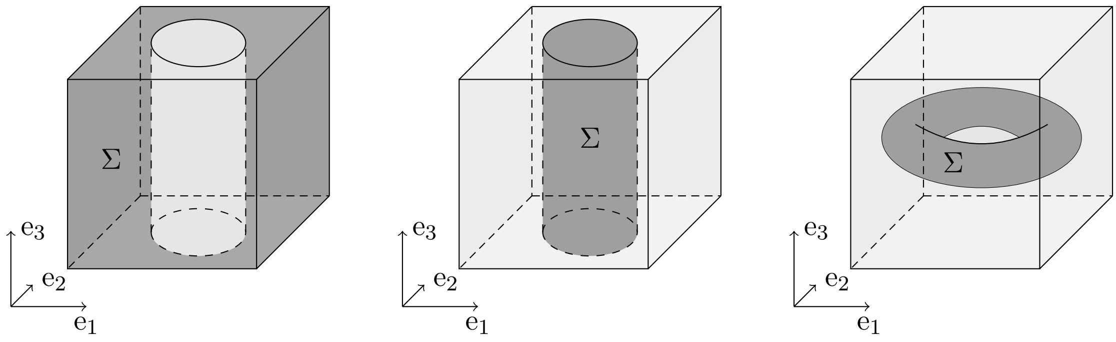


Figure 4: The periodicity cell Y is represented by the cube. (a) The air cylinder $Y \setminus \overline{\Sigma}$ of Example 5.3 (b) The metal cylinder Σ of Example 5.4. (c) Σ is a 2-dimensional full torus. In this case, $Y \setminus \overline{\Sigma}$ is not a simple Helmholtz domain.

5.4 The metal cylinder

Fix a number $r \in (0, 1/2)$. To model a metallic cylinder, Σ is defined as the set

$$\Sigma := \{y = (y_1, y_2, y_3) \in Y : y_1^2 + y_2^2 < r^2\}\,. \tag{5.5}$$

A sketch of the periodicity cell is given in Figure 4 (b).

We claim that $Y \setminus \overline{\Sigma}$ is a simple Helmholtz domain. Indeed, there are only the three standard nontrivial loops in $Y \setminus \overline{\Sigma}$; namely, γ_1, γ_2, and γ_3, which are given by $\gamma_1(t) := (t - 1/2, -1/2, -1/2)$, $\gamma_2(t) := (-1/2, t - 1/2, -1/2)$, and $\gamma_3(t) := (-1/2, -1/2, t - 1/2)$ for $t \in [0, 1]$. Thus, every $L^2_\sharp(Y; \mathbb{C}^3)$-vector field u that is curl free in $Y \setminus \overline{\Sigma}$ can be written as $u = \nabla\Theta + c_0$ for $\Theta \in H^1_\sharp(Y \setminus \overline{\Sigma})$ and $c_0 \in \mathbb{C}^3$.

We find the table

$\mathcal{N}_\Sigma$	d_E	$\mathcal{L}_{Y\setminus\overline{\Sigma}}$	d_H
$\{1, 2\}$	2	$\{1, 2, 3\}$	3

As for the metal plate, we find an interesting non-trivial limit system.

Corollary 5.4 (Macroscopic equations for the metal cylinder)**.** *For Σ as in* (5.5)*, a sequence $(E^\eta, H^\eta)_\eta$, and limit fields $\hat{E}$ and $\hat{H}$ as in Theorem 4.1, the macroscopic equations read*

$$\operatorname{curl} \hat{E} = \mathrm{i}\,\omega\mu_0\hat{\mu}\hat{H} \quad \text{in } \Omega\,, \tag{5.6a}$$

$$\operatorname{curl} \hat{H} = -\mathrm{i}\,\omega\varepsilon_0\hat{E} \quad \text{in } \Omega \setminus R\,, \tag{5.6b}$$

$$\partial_2\hat{H}_3 - \partial_3\hat{H}_2 = -\mathrm{i}\,\omega\varepsilon_0(\hat{\varepsilon}\hat{E})_1 \quad \text{in } \Omega\,, \tag{5.6c}$$

$$\partial_3\hat{H}_1 - \partial_1\hat{H}_3 = -\mathrm{i}\,\omega\varepsilon_0(\hat{\varepsilon}\hat{E})_2 \quad \text{in } \Omega\,, \tag{5.6d}$$

$$\hat{E}_3 = 0 \quad \text{in } R\,. \tag{5.6e}$$

Proof. We have that $\mathcal{L}_{Y\setminus\overline{\Sigma}} = \{1, 2, 3\}$, $\mathcal{N}_\Sigma = \{1, 2\}$, and $\hat{\varepsilon} = \mathrm{id}_3$ in $\Omega \setminus R$. Thus, (5.6a) and (5.6b) follow from (4.5a) and (4.5b), respectively. Equations (5.6c) and (5.6d) follow from (4.5c), and (5.6e) is a consequence of (4.5d). □

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