Interface conditions for Maxwell's equations by homogenization of thin inclusions: transmission, reflection or polarization

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Abstract: We consider the time-harmonic Maxwell equations in a complex geometry. We are interested in geometries that model polarization filters or Faraday cages. We study the situation that the underlying domain contains perfectly conducting inclusions, the inclusions are distributed in a periodic fashion along a surface. The periodicity is $\eta > 0$ and the typical scale of the inclusion is η, but we allow also the presence of even smaller scales, e.g. when thin wires are analyzed. We are interested in the limit $\eta \to 0$ and in effective equations. Depending on geometric properties of the inclusions, the effective system can imply perfect transmission, perfect reflection or polarization.

Keywords: Homogenization, Maxwell's equations, Interface conditions, Polarization
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1. Introduction

The propagation of electromagnetic waves can be influenced by thin metallic structures. Probably best known is the fact that (appropriately designed) meshes of metallic wires are impenetrable for the waves of a micro-wave oven and they can protect the user. Vice versa, metallic surfaces can be made penetrable for electromagnetic waves by means of thin slits. Another effect can be the polarization of the wave. We are interested in a mathematical analysis of such effects.

We study the time-harmonic Maxwell equations in a domain that contains a thin layer of a meta-material: A perfectly conductive material is distributed in a periodic fashion along a surface. The material can be distributed in a multi-scale fashion, e.g. a wire structure with a periodicity $\eta > 0$ and a thickness of the wires of a size much smaller than η. Another example would be a plate structure with small slits in thin plates. We are interested in the limit $\eta \to 0$ and in an effective description of this situation. Our results show that the geometrical and topological properties of the inclusions are crucial for the qualitative effect of the meta-material. The effective system can show reflection or polarization. For structures with slits or

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structures that do not have certain connectivity properties, the meta-material can also be invisible in the sense that it does not appear in the effective equations.

Maxwell's equations describe electromagnetic waves with an electric field E and a magnetic field H. The equations use two coefficients, the permeability μ and the permittivity ε. Since we are interested in complex geometries, the domain of interest $\Omega_\eta \subset \Omega \subset \mathbb{R}^3$ depends on a (small) parameter $\eta > 0$. In the time-harmonic setting with a fixed frequency $\omega > 0$, the system reads

$$\operatorname{curl} E^\eta = i\omega\mu H^\eta + f_h \quad \text{in } \Omega_\eta\,, \tag{1.1a}$$

$$\operatorname{curl} H^\eta = -i\omega\varepsilon E^\eta + f_e \quad \text{in } \Omega_\eta\,, \tag{1.1b}$$

$$E^\eta \times \nu = 0 \quad \text{on } \partial\Omega_\eta\,. \tag{1.1c}$$

The last equation is a boundary condition that models a perfectly conducting material outside Ω_η, it uses the exterior normal ν of Ω_η. The index η indicates that we will analyze a fine-scale structure. The solution is a map $(E^\eta, H^\eta)\colon \Omega_\eta \ni x \mapsto (E^\eta, H^\eta)(x) \in \mathbb{C}^3 \times \mathbb{C}^3$. Since the solution depends on the domain Ω_η, it depends on η. The right hand side $f_e = f_e(x)$ of the second equation models prescribed external currents; we include $f_h = f_h(x)$ to treat a more general and more symmetric system.

We consider the macroscopic domain Ω, which is separated by Γ into an upper part Ω_+ and a lower part Ω_- as follows:

$$\Omega := (0,1)^2 \times (-1,1)\,, \qquad \Gamma := (0,1)^2 \times \{0\}\,, \tag{1.2}$$

$$\Omega_+ := (0,1)^2 \times (0,1)\,, \qquad \Omega_- := (0,1)^2 \times (-1,0)\,. \tag{1.3}$$

We assume that there exists a sequence of sets $\Sigma_\eta \subset \Omega$ which represents the perfectly conducting inclusion. The sets Σ_η concentrate at the interface $\Gamma = \{x_3 = 0\}$ in the sense that $\Sigma_\eta \subset (0,1)^2 \times (0,\eta)$. The domain Ω_η is given by $\Omega_\eta := \Omega \setminus \Sigma_\eta$. We are interested in the effective interface condition along Γ.

1.1. Asymptotic connectedness and disconnectedness of conductors.

In the simplest case, the micro-structure consists of a periodic array of parallel wires, let us assume that they are oriented in direction e_1. We will introduce a definition for "asymptotic connectedness" and will see that such parallel wires are asymptotically connected in direction e_1 and asymptotically disconnected in direction e_2. In the limit $\eta \to 0$, this micro-structure reflects electromagnetic waves that have an electric field parallel to e_1, when the electric field is orthogonal to e_1, the structure is invisible for the electromagnetic wave. Since a general wave can be understood as a superposition of the above described waves, the microstructure leads to a polarization of the general electromagnetic wave, compare Figure 1.

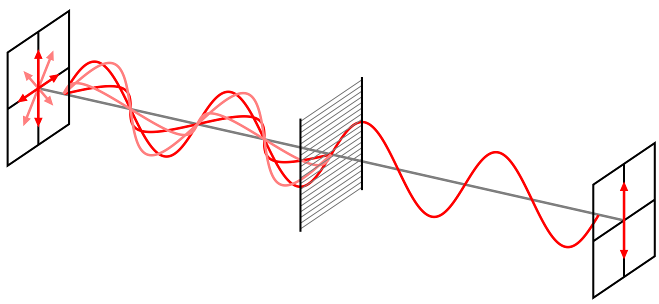


FIGURE 1. Polarization: Electromagnetic waves (represented by the electric field) interacting with a mesh of wires. Components that are orthogonal to the wires can pass the microstructure.

The limit of (1.1) depends on an asymptotic analysis of the connectedness properties. We define a notion of *asymptotic (dis-)connectedness* in certain directions for the obstacle set by means of the existence of cell functions, see Definitions 3.1 and 3.2. With these notions, the homogenization of (1.1) leads to one of the following three homogenization results: perfect transmission (inactive interface), perfect reflection or polarization.

Having used cell functions in order to define asymptotic (dis-)connectedness, it is an important task to study the connectedness properties for concrete geometrical objects. Our investigation is focused on thin parallel wires that may also have thin slits (gaps in the wire). The existence of the corresponding cell functions and, thus, the asymptotic connectedness or disconnectedness depends on the asymptotic behavior of the parameters describing the structures; we will see that the asymptotics of the number $\eta|\ln(r_\eta)|$ are decisive, where r_η is the radius of the wire, see Proposition 7.2.

To enlarge the class of domains under consideration, we introduce a monotonicity and a deformation argument. Loosely speaking: Enlarging a connected structure leaves the structure connected, making a disconnected structure smaller, leaves a disconnected structure disconnected (large and small is meant here in the sense of set inclusions, see Proposition 5.1). The deformation result provides that Lipschitz deformations of a (dis-)connected structure is again (dis-)connected, see Proposition 5.2.

1.2. **Interface conditions of the limit system.** To formulate the limit system, we use the following notation: Whenever a scalar function $u\colon \Omega \to \mathbb{C}$ possesses traces on the interface Γ, we write $u|_{\Gamma,+}$ for the trace of $u|_{\Omega_+}$, i.e. the boundary values taken from above the interface. Correspondingly, we write $u|_{\Gamma,-}$ for the trace of $u|_{\Omega_-}$, which are the boundary values from below. For the jump of such a function across Γ we write $[\![u]\!]_\Gamma := u|_{\Gamma,+} - u|_{\Gamma,-}$. We will actually use the brackets only to formulate that a function u has no jump across Γ by writing $[\![u]\!]_\Gamma = 0$.

In the case $[\![u]\!]_\Gamma = 0$, the function u has no jump across Γ and we can write $u|_\Gamma := u|_{\Gamma,+} = u|_{\Gamma,-}$. In fact, we will use traces only in this case. For vector valued functions $E\colon \Omega \to \mathbb{C}^3$ of the class $H(\mathrm{curl}, \Omega\setminus\Gamma)$, there is no continuous trace operator for E, but there are continuous trace operators for the tangential components.

Limit system. In our setting, the homogenization of (1.1) results in the following limit system for E^{hom}, H^{hom}:

$$\operatorname{curl} E^{\mathrm{hom}} = i\omega\mu H^{\mathrm{hom}} + f_h \quad \text{in } \Omega\,, \tag{1.4a}$$

$$\operatorname{curl} H^{\mathrm{hom}} = -i\omega\varepsilon E^{\mathrm{hom}} + f_e \quad \text{in } \Omega\setminus\Gamma\,, \tag{1.4b}$$

$$E^{\mathrm{hom}} \times \nu = 0 \quad \text{on } \partial\Omega\,. \tag{1.4c}$$

The system (1.4) must be complemented by interface conditions at Γ (this is a consequence of the fact that (1.4b) is not imposed on all of Ω). Depending on the asymptotic connectedness property of Σ_η, one of the interface conditions (1.5) or (1.6) or (1.7) completes the system.

We note that (1.4a) implies that the tangential components of E^{hom} have no jump across Γ:

$$[\![E_1^{\mathrm{hom}}]\!]_\Gamma = 0\,, \qquad [\![E_2^{\mathrm{hom}}]\!]_\Gamma = 0\,,$$

hence these conditions are always satisfied. By contrast, the tangential components of H^{hom} can have jumps across Γ since (1.4b) imposes an equation only on $\Omega \setminus \Gamma$.

Reflecting interface: If the obstacles are asymptotically connecting in both directions e_1 and e_2, we obtain the interface conditions

$$E_1^{\mathrm{hom}}|_\Gamma = 0\,, \qquad E_2^{\mathrm{hom}}|_\Gamma = 0\,. \tag{1.5}$$

Remark: The interface condition (1.5) means that the domains Ω_+ and Ω_- are separated. The solution $(E^{\mathrm{hom}}, H^{\mathrm{hom}})$ can equivalently be found by solving the equations in Ω_+ and Ω_- with boundary conditions $E^{\mathrm{hom}} \times \nu = 0$ on $\partial\Omega_-$ and $\partial\Omega_+$.

Inactive interface: If the obstacles are asymptotically disconnected in both directions e_1 and e_2, we obtain the interface conditions

$$[\![H_1^{\mathrm{hom}}]\!]_\Gamma = 0\,, \qquad [\![H_2^{\mathrm{hom}}]\!]_\Gamma = 0\,. \tag{1.6}$$

Remark: The relations (1.6) are satisfied if and only if (1.4b) holds in the entire domain Ω. We may therefore also say in this case: The micro-structured interface has no effect in the limit system.

Polarizing interface: Let $i \in \{1, 2\}$ and $j := 3 - i$ be fixed. If the obstacles are asymptotically connecting for the direction e_i and asymptotically disconnected for the direction e_j, we obtain the interface conditions

$$E_i^{\mathrm{hom}}|_\Gamma = 0\,, \qquad [\![H_i^{\mathrm{hom}}]\!]_\Gamma = 0\,. \tag{1.7}$$

1.3. **Outlook regarding highly conducting obstacles.** System (1.1) models perfect conductors in the inclusion Σ_η. Another interesting model is constructed by introducing a high conductivity in the inclusion. For a sequence $\gamma_\eta \to \infty$ and $\varepsilon_0 \in \mathbb{R}$, one may consider $\varepsilon_\eta = \varepsilon_0 + i\gamma_\eta \mathbf{1}_{\Sigma_\eta}$ as permittivity in Maxwell's equations. The homogenization limit can be considered also for this model. In this slightly more complex setting, one has to take into account the asymptotics of γ_η and its relation to the scaling and the geometry of Σ_η. We refrain from an analysis of this interesting setting in the present contribution. Highly conducting media were investigated in related context in [2, 4, 5, 16].

1.4. **Literature.** In the original form, Maxwell's equations form a time-dependent system, but one is very often interested in the analysis for a fixed frequency $\omega > 0$. The ansatz $u(x,t) = u(x)e^{-i\omega t}$ leads from Maxwell's equations to the time-harmonic Maxwell system (1.1). We note that the same procedure leads from the scalar wave equation to the Helmholtz equation. Both systems, Maxwell and Helmholtz, describe wave phenomena and their analysis has many similarities. Accordingly, we describe here also some work on the Helmholtz equation. We focus on works that are related to homogenization and suppress many interesting fields such as radiation conditions, periodic media or domain truncation methods. Regarding general mathematical concepts for Maxwell's equations, we refer to [15] and [18].

Homogenization in the bulk. The theory of homogenization started with the analysis of periodic systems – a periodicity in all directions. It can be either the domain that contains, e.g. small and periodically spaced inclusions, or it can be a coefficient in the equation that is periodic with a small periodicity, say $\eta > 0$. The homogenization of such system can be performed, e.g. with the method of two-scale convergence [1]. A homogenization of the Maxwell system in this classical spirit is performed in [25], error estimates have been obtained in [24], we mention [13, 14] for numerical aspects. The analysis becomes more intricate when the system is near resonance, we refer to [21] for an overview. Wires with large absolute values of the permittivity ε can lead to resonances, this was investigated in [4] for an essentially two-dimensional setting, in [2] for a setting towards three space dimensions, in [3] for a random permittivity in the wires.

Other resonances are possible for three-dimensional inclusions in the bulk, namely the famous split-ring resonators. They were first analyzed mathematically in [5], the homogenization was performed, the homogenized system can show an effective negative permeability μ. The topology is important in this resonance: For positive η, the rings are simply connected, but they are closing in the limit, which changes their topology. In particular, one has to analyze a microstructure that has some geometrical features below the η-scale. The results were transferred to perfectly conducting inclusions in [17]. The generation of a negative index material occurs when macroscopically connecting wires are additionally included, see [16]. A more general investigation of topological implications was performed in [23] for perfect conductors and in [19] for high-contrast media. One result is that if the inclusion is connected in one direction, the limiting E field must vanish in that component.

A composite medium for Maxwell's equations in which one component has a negative index was homogenized in [7]. A high-contrast homogenization for a large electric permittivity in a periodic distribution was performed in [8]. A non-trivial asymptotic geometry was analyzed in two dimensions for the Helmholtz equation in [10, 20].

We note that a special scaling of a micro-structure was used in [6], again for the Helmholtz equation: The interface has thickness of order 1, the micro-structure is η-periodic in the tangential direction, the aim is to understand the coupling between upper and lower surface of this interface; the result is the possible effect of perfect transmission due to a resonance effect. Sound absorption in this scaling was investigated in [12].

Homogenization along a surface. The above list of references indicates that very interesting effects, most of them related to resonance, can occur when small structures are distributed in the bulk (and, often, three scales are used in such constructions). The geometry of the present article is such that, along a manifold (here: a two-dimensional surface in three dimensions, compare (2.1)), a micro-structure is distributed. The fundamental task is to determine the effective description of this microstructure.

For the Helmholtz equation with fixed frequency, the result at leading order is the same as for the Poisson problem. When inclusions of scale η are distributed with typical distances η, the effective system is not very interesting: When a Dirichlet condition is imposed on the boundaries of the inclusions, the effective system contains a Dirichlet condition at the limiting interface. When a Neumann condition is imposed along the inclusion, the limiting interface does not enter the effective equation. We note that smaller inclusions can have a different effective behavior, see [9].

The results become interesting when one asks for the effects in the next order of the expansion. In the fundamental contribution [11], this was answered: The derivatives of the leading order solutions must be evaluated at the limiting interface and they are responsible for a correction term of first order in the small parameter. We mention that the results of [11] were slightly generalized and simplified in [22].

The present contribution can be seen in this context. We do not ask here for first order correction terms. On the other hand, we treat the three-dimensional situation and Maxwell's equations; we can therefore find interesting results even at leading order. Most notable, for wires: Cancellation effect in one component, invisibility of the effective surface in the other component. This can be seen like the effect of a Dirichlet condition in one component, and the effect of a Neumann condition in the other component.

1.5. **Content layout.** The manuscript is organized as follows. In Section 2, we present the geometries under consideration, formulate the weak form for the η-problem and the weak forms for the homogenized systems. We formulate the main homogenization result in Theorem 2.1. In Section 3, we introduce the notion of asymptotic (dis-)connectedness by means of the existence of cell functions. Using these cell functions, we homogenize (1.1) for generic geometries that are asymptotically (dis-)connected in Section 4. Section 5 illustrates that one can conclude connectedness properties of one inclusion from the connectedness properties of another inclusion: Deformations of the inclusion do not change the connectedness properties. Furthermore: When an inclusion is connected, every larger inclusion (in the sense of sets) is also connected. In Section 6, we investigate cylindrical geometries and show that wires are connecting in one direction and are disconnected in the other direction. The fact that the correct critical asymptotics for the radius are indeed found is shown in Section 7.

2. Main results

In this section, we describe the microscopic and the macroscopic geometry. We present the weak form of the η-problem and of the limit problem with different boundary conditions. Theorem 2.1 states the main result.

2.1. **Geometry and notations.** For a measurable set $U \subset \mathbb{R}^n$, we denote the indicator function by $\mathbf{1}_U$, i.e. $\mathbf{1}_U(x) = 1$ for $x \in U$ and $\mathbf{1}_U(x) = 0$ for $x \notin U$. In estimates, we use generic constants C that may change from one inequality to the next.

We recall the macroscopic geometry of (1.2)–(1.3): The domain is $\Omega = (0,1)^2 \times (-1,1)$, the interface $\Gamma = (0,1)^2 \times \{0\}$ separates $\Omega_+ = (0,1)^2 \times (0,1)$ and $\Omega_- = (0,1)^2 \times (-1,0)$.

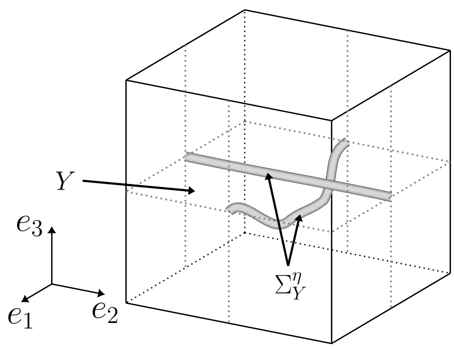
The small parameter is $\eta > 0$. We denote the three dimensional unit cell by $Y := (0,1)^3$. The microscopic geometry is prescribed by a sequence of compact obstacle shapes $\Sigma_Y^\eta \subset [0,1]^2 \times (0,1) \subset \overline{Y}$. To extend the structure periodically, we use indices $k = (k_1, k_2) \in \mathbb{Z}^2$ and set

$$\Sigma_\#^\eta := \bigcup_{k \in \mathbb{Z}^2} (k_1, k_2, 0) + \Sigma_Y^\eta \subset \mathbb{R}^3 .$$

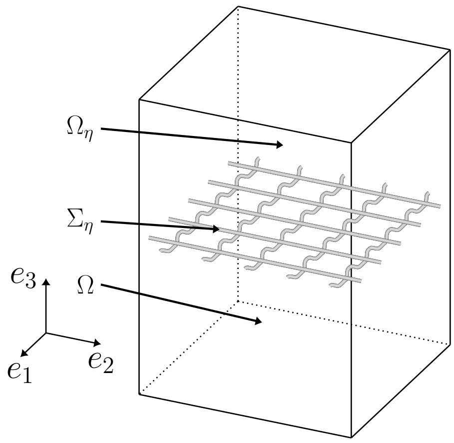
We assume that $\Sigma_\#^\eta$ has a Lipschitz boundary. A scaling of $\Sigma_\#^\eta$ with η provides the inclusion, Ω_η is the perforated macroscopic domain:

$$\Sigma_\eta := \eta\, \Sigma_\#^\eta \cap \Omega = \bigcup_{k \in \mathbb{Z}^2} \eta[(k_1, k_2, 0) + \Sigma_Y^\eta] \cap \Omega\,, \qquad \Omega_\eta := \Omega \setminus \Sigma_\eta\,. \tag{2.1}$$

For an illustration of a possible cell-geometry see Figure 2a, the resulting domain Ω_η is shown in Figure 2b.



(A) The set Σ_Y^η representing the conductor in the reference cell



(B) The domain Ω_η with obstacle set Σ_η

FIGURE 2. Geometry of the conductor

2.2. **Weak form of the η-problem.** For the analysis of Maxwell's equations we use the following function spaces: For a domain $U \subset \mathbb{R}^3$, understanding the curl of an $L^2(U)$-function in the sense of distributions, we set

$$\begin{aligned} H(\mathrm{curl}, U) &:= \{E \in L^2(U, \mathbb{C}^3) \mid \mathrm{curl}\, E \in L^2(U, \mathbb{C}^3)\}\,, \\ H_0(\mathrm{curl}, U) &:= \left\{ E \in H(\mathrm{curl}, U) \middle| \int_U E \cdot \mathrm{curl}\, \psi = \int_U \mathrm{curl}\, E \cdot \psi \ \forall \psi \in H(\mathrm{curl}, U) \right\}\,, \\ H_{\mathrm{loc}}(\mathrm{curl}, U) &:= \{E \in L^2_{\mathrm{loc}}(U, \mathbb{C}^3) \mid \mathrm{curl}\, E \in L^2_{\mathrm{loc}}(U, \mathbb{C}^3)\}\,. \end{aligned}$$

For the first two spaces, we use the norm and the scalar product that is induced by

$$\|E\|^2_{H(\mathrm{curl},U)} := \int_U \left\{ |E|^2 + |\,\mathrm{curl}\, E|^2 \right\}\,.$$

We use the following weak formulation of the η-problem (1.1): Find $(E^\eta, H^\eta) \in L^2(\Omega_\eta, \mathbb{C}^3) \times L^2(\Omega_\eta, \mathbb{C}^3)$ such that

$$\int_{\Omega_\eta} E^\eta \cdot \mathrm{curl}\, \psi = \int_{\Omega_\eta} (i\omega\mu H^\eta + f_h) \cdot \psi \qquad \text{for every } \psi \in H(\mathrm{curl}, \Omega_\eta)\,, \tag{2.2a}$$

$$\int_{\Omega_\eta} H^\eta \cdot \mathrm{curl}\, \phi = \int_{\Omega_\eta} (-i\omega\varepsilon E^\eta + f_e) \cdot \phi \qquad \text{for every } \phi \in H_0(\mathrm{curl}, \Omega_\eta)\,. \tag{2.2b}$$

We note that (2.2a) does not only imply that (1.1a) is satisfied in the sense of distributions (and, hence, also the regularity $\mathrm{curl}\, E^\eta \in L^2(\Omega_\eta)$ is satisfied), but it implies also the boundary condition, $E^\eta \in H_0(\mathrm{curl}, \Omega_\eta)$ and, hence, (1.1c) in a weak form.

We note the following facts on trivially extended solutions (one considers E^η and H^η as functions on all of Ω by setting them equal to 0 on Σ_η). The extended functions are solutions of (2.2) with integrals over all of Ω under the assumption that f_h vanishes on Σ_η and that the test functions are $\phi, \psi \in H(\mathrm{curl}, \Omega)$ with ϕ vanishing on Σ_η.

2.3. **Weak form of the limit systems.** In this section, we state the weak forms of the limit system (1.4) with one of the interface conditions (1.5), (1.6) or (1.7). The three corresponding equations have actually all the same form and the same solution space, they only differ in the spaces of test-functions. We use the macroscopic geometry given by Ω and the interface Γ of (1.2).

We introduce spaces of test-functions. We start with the largest spaces, X_- and Y_-, the subscript indicates that there are no restrictions on the functions on the interface:

$$X_- := C_0^1(\overline{\Omega}, \mathbb{C}^3)\,, \tag{2.3a}$$

$$Y_- := \left\{\psi\colon \Omega \to \mathbb{C}^3 \mid \psi|_{\Omega_\pm} \in C^1(\overline{\Omega_\pm}, \mathbb{C}^3)\right\}\,. \tag{2.3b}$$

To recall some notation: X_- consists of differentiable functions $\phi\colon \Omega \to \mathbb{C}^3$ such that $D\phi$ is continuous and bounded and such that $\phi = 0$ holds on $\partial\Omega$. The space Y_- requires the same properties on Ω_+ and on Ω_-, but the functions can have arbitrary values on Γ and all components can also have a jump across Γ.

For an index $i \in \{1, 2\}$ that indicates a tangential direction, we define subspaces in which some restriction is introduced for the i-th component of the field:

$$X_i := \{\phi \in X_- \mid \phi_i|_\Gamma = 0\}\,, \tag{2.3c}$$

$$Y_i := \{\psi \in Y_- \mid [\![\psi_i]\!]_\Gamma = 0\}\,, \tag{2.3d}$$

where $[\![\psi_i]\!]_\Gamma$ denotes the jump of ψ_i. The jump is classically defined for test-functions.

Finally, we introduce the spaces X_{12} and Y_{12} in which we impose a condition for both tangential components:

$$X_{12} := X_1 \cap X_2\,, \tag{2.3e}$$

$$Y_{12} := Y_1 \cap Y_2\,. \tag{2.3f}$$

We mention that $X_{12} = \left\{\phi \in C_0^1(\overline{\Omega}, \mathbb{C}^3) \mid \phi_1 = \phi_2 = 0 \text{ on } \Gamma\right\}$, and that for functions $\psi \in Y_{12}$ the jumps vanish, $[\![\psi_1]\!]_\Gamma = [\![\psi_2]\!]_\Gamma = 0$. The spaces are ordered: $X_{12} \subset X_i \subset X_-$ and $Y_{12} \subset Y_i \subset Y_-$.

We can now formulate the limit system in a weak sense. We say that $E^{\text{hom}}, H^{\text{hom}} \in L^2(\Omega, \mathbb{C}^3)$ is a weak solution to (1.4) with interface conditions when

$$\int_{\Omega\setminus\Gamma} E^{\text{hom}} \cdot \operatorname{curl}\psi = \int_{\Omega\setminus\Gamma} (i\omega\mu H^{\text{hom}} + f_h)\cdot\psi \qquad \text{for every } \psi \in Y\,, \tag{2.4a}$$

$$\int_{\Omega} H^{\text{hom}} \cdot \operatorname{curl}\phi = \int_{\Omega} (-i\omega\varepsilon E^{\text{hom}} + f_e)\cdot\phi \qquad \text{for every } \phi \in X\,. \tag{2.4b}$$

The interface conditions (1.5) are encoded by the choice $X = X_{12}$ and $Y = Y_-$. The interface conditions (1.6) are encoded by the choice $X = X_-$ and $Y = Y_{12}$. The interface conditions (1.7) are encoded by the choice $X = X_i$ and $Y = Y_i$. The different cases are collected in Table 1. Further comments on the weak solution concept are given in Remarks 2.5 and 2.7 below.

2.4. **Homogenization result.** The following theorem collects the homogenization results. In the homogenized Maxwell system, the inclusions are replaced by interface conditions along Γ. These conditions depend on the asymptotic connectedness and disconnectedness of the inclusions as specified in Definitions 3.1 and 3.2.

Interface	strong form		test-functions	
Reflecting	(1.5)	$E_1^{\text{hom}}\|_\Gamma = E_2^{\text{hom}}\|_\Gamma = 0$	$X = X_{12}$	$Y = Y_-$
Inactive	(1.6)	$[\![H_1^{\text{hom}}]\!]_\Gamma = [\![H_2^{\text{hom}}]\!]_\Gamma = 0$	$X = X_-$	$Y = Y_{12}$
Polarizing	(1.7)	$E_i^{\text{hom}}\|_\Gamma = [\![H_i^{\text{hom}}]\!]_\Gamma = 0$	$X = X_i$	$Y = Y_i$

Table 1. Spaces of test functions in (2.4). Regarding the third case: The index $i \in \{1, 2\}$ indicates the direction of connectedness.

Theorem 2.1 (Main result)**.** *Let the setting be that of Section 2.1: The macroscopic sets are Ω and Γ, the sets for the η-problem are given by Σ_Y^η, Ω_η and Σ_η for a sequence $\eta \to 0$. Let $(E^\eta, H^\eta) \in L^2(\Omega_\eta, \mathbb{C}^3) \times L^2(\Omega_\eta, \mathbb{C}^3)$ be solutions to (2.2). We assume that the trivial extensions of E^η and H^η to Ω have weak limits,*

$$E^\eta \rightharpoonup E^{hom}, \qquad H^\eta \rightharpoonup H^{hom} \qquad \text{weakly in } L^2(\Omega, \mathbb{C}^3).$$

Then, the equations for E^{hom} and H^{hom} are given as follows, depending on the connectedness properties of the micro-structure:

Case 1: **Reflecting** *When Σ_Y^η is asymptotically connected in both directions e_1 and e_2, then (E^{hom}, H^{hom}) solves (1.4) and (1.5) weakly, ie (2.4) holds for $X = X_{12}$ and $Y = Y_-$.*

Case 2: **Inactive** *When Σ_Y^η is asymptotically disconnected in both directions e_1 and e_2, then (E^{hom}, H^{hom}) solves (1.4) and (1.6) weakly, ie (2.4) holds for $X = X_-$ and $Y = Y_{12}$.*

Case 3: **Polarizing** *Let $i \in \{1, 2\}$. If Σ_Y^η is asymptotically connected in the direction e_i and asymptotically disconnected in the direction e_j for $j = 3 - i$, then (E^{hom}, H^{hom}) solves (1.4) and (1.7) weakly, ie (2.4) holds for $X = X_i$ and $Y = Y_i$.*

Lemma 2.2 (Trivial part of the limit system)**.** *Let the setting be that of Theorem 2.1, no assumptions on connectedness of Σ_Y^η. Then, the limit fields (E^{hom}, H^{hom}) satisfy (2.4) for $X = X_{12}$ and $Y = Y_{12}$.*

Proof. Let $\psi \in Y_{12}$ be arbitrary, we recall that this implies, in particular, that the functions ψ_1 and ψ_2 are continuous across Γ. The function ψ can therefore be interpreted (by restriction) as an element $\psi \in H(\text{curl}, \Omega_\eta)$ – jumps in the normal component do not contribute to the curl. We can therefore use ψ as a test-function in (2.2a). Passing to the limit provides (2.4a) for ψ.

Let now $\phi \in X_{12}$ be arbitrary, we recall that this implies $\phi_1 = \phi_2 = 0$ on Γ. We approximate ϕ by a sequence of functions ϕ^k; for the first two components we demand $\phi_1^k, \phi_2^k \in C_c^\infty(\Omega \setminus \Gamma, \mathbb{C}^3)$. Regarding the third component, we choose $\phi_3^k = \phi_3$ on $\{x \in \Omega \mid x_3 \notin (0, \eta)\}$ and $\phi_3^k = 0$ on $\{x \in \Omega \mid x_3 \in (0, \eta)\}$. We can find an approximation with an error $\|\phi_i^k - \phi_i\|_{H^1(\Omega)} \leq 1/k$ for $i \in \{1, 2\}$. We note that this also implies $\|\text{curl}\,\phi^k - \text{curl}\,\phi\|_{L^2(\Omega)} \to 0$ as $k \to \infty$.

Since Σ_η is contained in an η-neighborhood of Γ, there holds $\phi^k \in H_0(\text{curl}, \Omega_\eta)$ for sufficiently small $\eta > 0$. We can therefore use ϕ^k in (2.2b). Passing to the limit $\eta \to 0$, we obtain (2.4b) with ϕ^k instead of ϕ. Taking the limit $k \to \infty$, we obtain (2.4b) for ϕ. □

Remark 2.3 (Compactly contained obstacles are disconnected)**.** *Let all sets Σ_Y^η be contained in a closed set $\Sigma_Y^0 \subset Y = (0,1)^3$ that is independent of η. Then, Σ_Y^η is asymptotically disconnected in both directions e_1 and e_2. Accordingly, we are in the situation of Case 2 of Theorem 2.1: The micro-structure has no effect.*

The proof of Remark 2.3 is provided in the end of Section 6.

Theorem 2.4 (Main result for wires)**.** *In the situation of Theorem 2.1, let the inclusions be given by wires in direction $i = 1$, $\Sigma_Y^\eta = T_{r_\eta, I_\eta}$ of (6.18). We assume that the radii r_η of the wires are not too small and that the slits I_η are not too big in the sense that:*

$$\eta |\ln(r_\eta)| \to 0 \qquad \text{and} \qquad \eta^{-1} r_\eta^{-2} |I_\eta| \to 0\,.$$

Then, the limit system is given by Theorem 2.1, Case 3: E^{hom}, H^{hom} solves (1.4) and (1.7) with $i = 1$.

Proof. The wires are connecting in direction e_1 by Proposition 6.5. The wires are disconnected in direction e_2 by Proposition 6.6. Theorem 2.1 therefore yields the limit system with Case 3. □

Remark 2.5 (On the weak solution concept)**.** *The above choice of weak equations yields indeed a weak solution concept for (1.4) with interface condition. We verify this fact in the polarizing case with $i = 1$. Let (E^{hom}, H^{hom}) be a regular solution of (2.4) with $X = X_1$ and $Y = Y_1$, the solution is required to be regular enough to allow for classical integration by parts. As a first step, we use an arbitrary test-function $\phi \in C_c^\infty(\Omega \setminus \Gamma, \mathbb{C}^3)$ in (2.4b) and obtain (1.4b). Similarly, we can use an arbitrary test-function $\psi \in C_c^\infty(\Omega, \mathbb{C}^3)$ in (2.4a) and obtain (1.4a).*

Let $\phi \in X_1$ be an arbitrary test-function. We calculate with (2.4b):

$$\begin{aligned} 0 &= \int_{\Omega_+} H^{hom} \cdot \operatorname{curl} \phi + (i\omega\varepsilon E^{hom} - f_e) \cdot \phi + \int_{\Omega_-} H^{hom} \cdot \operatorname{curl} \phi + (i\omega\varepsilon E^{hom} - f_e) \cdot \phi \\ &= \int_{\Omega} (\operatorname{curl} H^{hom} + i\omega\varepsilon E^{hom} - f_e) \cdot \phi + \int_{\Gamma} [\![H^{hom}]\!]_\Gamma \cdot \phi \times e_3 = \int_{\Gamma} [\![H^{hom}]\!]_\Gamma \cdot \phi \times e_3\,. \end{aligned}$$

Since the values of ϕ_2 on Γ can be chosen arbitrarily, we can conclude from our calculation $[\![H^{hom}]\!]_\Gamma \cdot e_1 = 0$, which provides the second relation of (1.7).

Similarly, we can calculate for an arbitrary test-function $\psi \in Y_1$ with (2.4a):

$$\begin{aligned} 0 &= \int_{\Omega_+} E^{hom} \cdot \operatorname{curl} \psi - (i\omega\mu H^{hom} + f_h) \cdot \psi + \int_{\Omega_-} E^{hom} \cdot \operatorname{curl} \psi - (i\omega\mu H^{hom} + f_h) \cdot \psi \\ &= \int_{\Omega} (\operatorname{curl} E^{hom} - i\omega\mu H^{hom} + f_h) \cdot \psi + \int_{\Gamma} E^{hom} \cdot [\![\psi]\!]_\Gamma \times e_3 = \int_{\Gamma} E^{hom} \cdot [\![\psi]\!]_\Gamma \times e_3\,. \end{aligned}$$

Since the jump $[\![\psi]\!]_\Gamma$ can have an arbitrary second component, our calculation implies $E^{hom} \cdot e_1 = 0$ on Γ, and hence the first relation of (1.7).

Remark 2.6 (L^2-boundedness of the solution sequence)**.** *The statement of Theorem 2.1 assumes the boundedness of the solution sequence $(E^\eta, H^\eta) \in L^2(\Omega_\eta, \mathbb{C}^3) \times L^2(\Omega_\eta, \mathbb{C}^3)$. The derivation of this boundedness is non-trivial due to the possible presence of localized (surface) waves along the microstructure. We plan to investigate such resonance type phenomena in a future publication. The present work deals* *with the homogenization of the system, which requires the derivation of equations for weak limits.*

Remark 2.7 (Density, symmetry properties of the limit system)**.** *One can expect density of the test-functions in all cases of Table 1. More precisely:* X_{12} *is dense in the space of functions* $E \in H(\mathrm{curl}, \Omega)$ *with* $E_1|_\Gamma = E_2|_\Gamma = 0$*. The space* Y_{12} *is dense in the space of functions* $H \in H(\mathrm{curl}, \Omega)$ *with* $[\![H_1]\!]_\Gamma = [\![H_2]\!]_\Gamma = 0$*. Analogously for* X_i *and* Y_i *and* X_- *and* Y_-*. In all cases except for* X_i *and* Y_i*, these are actually classical results.*

Once the density is verified, the limit system has actually a symmetric form in solutions and test-functions. Let us formulate this statement in the polarizing case: Test functions are $\phi \in \overline{X}_i$ *and* $\psi \in \overline{Y}_i$*, the closures are taken in* $H(\mathrm{curl}, \Omega)$*. The solution is in the same space,* $E^{hom} \in \overline{X}_i$ *and* $H^{hom} \in \overline{Y}_i$*. Such a symmetric formulation is probably useful in the derivation of existence and uniqueness results.*

3. Cell functions and asymptotic (dis-)connectedness

The homogenization of (2.2) is based on an appropriate choice of test-functions. We will actually *define* connectedness properties of Σ_η with the existence of sequences of functions in a cell-geometry; the sequence of functions, if it exists, is used to construct test-functions for the original problem in Ω_η.

To motivate these cell functions, let us describe the homogenization procedure: Let $i \in \{1, 2\}$ and $j = 3 - i$. Lemma 2.2 shows (2.4a) for $Y = Y_{12}$. This space does not provide information on E^{hom} at the interface Γ. Let us assume that we want to use the larger set of test-functions $Y = Y_i$. Functions in Y_i have no jump in the i-th component, but they can have a jump in the j-th component. Thus, we cannot use them in the η-problem (2.2a). Instead, we consider a sequence of functions $\Psi_\eta^{(i)}$ that approximate a jump in the j-th component at Γ. Passing to the limit, we obtain (2.4a) for $Y = Y_i$. If such a sequence $\Psi_\eta^{(i)}$ exists, we say that Σ_Y^η is connected in the direction e_i, see Definition 3.2.

For (2.4b) the procedure is similar. Lemma 2.2 shows (2.4b) for $X = X_{12}$, let us assume that we want to use $X = X_i$. The j-component of a function in X_i does not vanish on Γ, we cannot use it directly as a test-function. But let us assume that there is a sequence of functions $\Phi_\eta^{(j)}$, vanishing in Σ_η and approximating, for $\eta \to 0$, functions that do not vanish in the j-th component at Γ. With these test-functions, we can pass to the limit and obtain (2.4b) for $X = X_i$. If such a sequence $\Phi_\eta^{(j)}$ exists, we say that Σ_Y^η is disconnected in the direction e_j, see Definition 3.1.

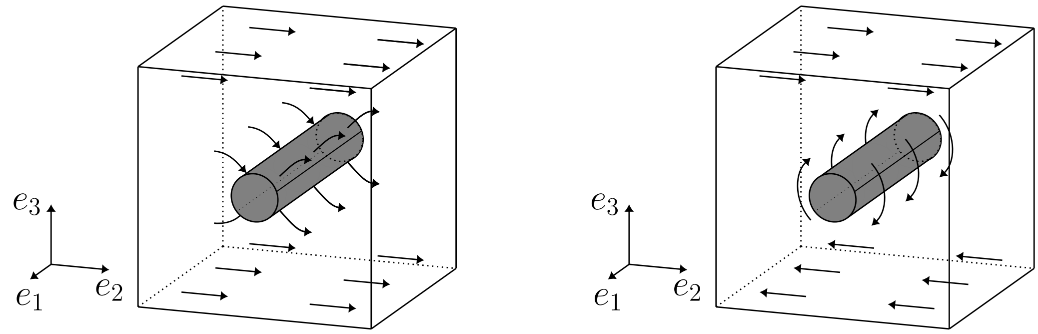
The cell functions. The cell functions are defined in the cylinder Z and we consider $Z^\eta \subset Z$ given by

$$Z := (0,1)^2 \times \mathbb{R}\,, \qquad Z^\eta := Z \setminus \Sigma_Y^\eta\,.$$

We define the function space

$$\overset{\bullet}{H}_\#(\mathrm{curl}, Z) := \{u|_Z \mid u \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3) \text{ is } e_1\text{- and } e_2\text{-periodic}\}\,.$$

We always identify a function $\overset{\bullet}{H}_\#(\mathrm{curl}, Z)$ with its periodic extension to $\mathbb{R}^3$.



(A) The cell function $\Phi_\eta^{(2)}$ shows that the wire is disconnected in direction e_2

(B) The cell function $\Psi_\eta^{(1)}$ shows that the wire is connected in direction e_1

FIGURE 3. Visualization of the cell functions in Y for $i = 1$, $j = 2$ and $\Sigma_Y^\eta = T_{0.15}^{(1)}$. The illustration in (B) does not show the cell function $\Psi_\eta^{(1)}$, but sketches $2\Psi_\eta^{(1)} - e_2$.

3.1. **E-type cell functions and asymptotic disconnectedness.** Let $j \in \{1, 2\}$, we use cell functions $\Phi_\eta^{(j)} \in \dot{H}_\#(\mathrm{curl}, Z)$ that converge to e_j for $|x_3| \to \infty$, but vanish on Σ_Y^η. Important is an asymptotic quantitative control of the curl.

Definition 3.1 (Asymptotically disconnected obstacles). *Let Σ_Y^η be a sequence of obstacles as in Section 2.1. For $j \in \{1, 2\}$, we say that the sequence Σ_Y^η is asymptotically disconnected in the direction e_j if there exists a sequence of cell functions $\Phi_\eta^{(j)} \in \dot{H}_\#(\mathrm{curl}, Z)$ of the E-type with direction e_j as follows:*

$$\eta^{\frac{1}{2}} \|\Phi_\eta^{(j)} - e_j\|_{L^2(Z)} \to 0\,, \tag{3.1a}$$

$$\Phi_\eta^{(j)}|_{\Sigma_Y^\eta} = 0\,, \tag{3.1b}$$

$$\eta^{-\frac{1}{2}} \|\,\mathrm{curl}\,\Phi_\eta^{(j)}\|_{L^2(Z)} \to 0\,. \tag{3.1c}$$

3.2. **H-type cell functions and asymptotic connectedness.** We are interested in cell functions $\Psi_\eta^{(i)} \in \dot{H}_\#(\mathrm{curl}, Z)$ that converge to a direction e_1 or e_2 for $x_3 \to \infty$, but to 0 for $x_3 \to -\infty$. Important is an asymptotic quantitative control of the curl.

Definition 3.2 (Asymptotically connecting obstacles). *Let Σ_Y^η be a sequence of obstacles as in Section 2.1. For $i \in \{1, 2\}$ and $j = 3 - i$, we say that the sequence Σ_Y^η is asymptotically connecting in the direction e_i if there exists a sequence of cell functions $\Psi_\eta^{(i)} \in \dot{H}_\#(\mathrm{curl}, Z)$ of the H-type with direction e_i as follows:*

$$\eta^{\frac{1}{2}} \|\Psi_\eta^{(i)} - e_j \mathbf{1}_{\{x_3>0\}}\|_{L^2(Z^\eta)} \to 0\,, \tag{3.2a}$$

$$\eta^{-\frac{1}{2}} \|\,\mathrm{curl}\,\Psi_\eta^{(i)}\|_{L^2(Z^\eta)} \to 0\,. \tag{3.2b}$$

Remark 3.3 (Cell functions pointing in direction e_3). *We note that* (3.1) *for $j = 3$ has the trivial solution $e_3 \mathbf{1}_{\{x_3>1\}\cup\{x_3<0\}}$. Similarly,* (3.2) *for $j = 3$ has the trivial solution $e_3 \mathbf{1}_{\{x_3>1\}}$.*

3.3. **Rescaling of the cell functions.** To employ the above introduced cell functions for homogenization, we scale them by η and consider, e.g. the function $\Phi_\eta^{(j)}(\cdot/\eta)$. The subsequent lemma collects the properties of the rescaled functions.

Lemma 3.4 (η-scaling of the cell functions)**.** *Let $\Phi_\eta^{(j)} \in \dot{H}_\#(\mathrm{curl}, Z)$ be a sequence that satisfies* (3.1). *Then, $\Phi_\eta^{(j)}(\cdot/\eta) \in H(\mathrm{curl}, \Omega)$ is a sequence with the property $\Phi_\eta^{(j)}(\cdot/\eta)|_{\Sigma_\eta} = 0$ and with the convergences*

$$\Phi_\eta^{(j)}(\cdot/\eta) \to e_j \qquad \text{in } L^2(\Omega)\,, \tag{3.3a}$$

$$\mathrm{curl}(\Phi_\eta^{(j)}(\cdot/\eta)) \to 0 \qquad \text{in } L^2(\Omega)\,. \tag{3.3b}$$

Let $\Psi_\eta^{(i)} \in \dot{H}_\#(\mathrm{curl}, Z)$ be a sequence that satisfies (3.2) *for $j = 3 - i$. Then, $\Psi_\eta^{(i)}(\cdot/\eta) \in H(\mathrm{curl}, \Omega)$ and*

$$\mathbf{1}_{\Omega_\eta}(\cdot)\Psi_\eta^{(i)}(\cdot/\eta) \to e_j \mathbf{1}_{\{x_3>0\}}(\cdot) \qquad \text{in } L^2(\Omega)\,, \tag{3.4a}$$

$$\mathbf{1}_{\Omega_\eta}(\cdot)\,\mathrm{curl}(\Psi_\eta^{(i)}(\cdot/\eta)) \to 0 \qquad \text{in } L^2(\Omega)\,. \tag{3.4b}$$

Proof. The calculations for (3.3) and (3.4) are almost identical. We present the calculations for (3.4), since these are slightly more interesting due to the presence of characteristic functions in the limit.

The set of relevant indices $k = (k_1, k_2)$ is $K_\eta := \{k \in \mathbb{Z}^2 \mid \eta(k + Z) \cap \Omega \neq \emptyset\}$. The number of elements of K_η is of the order η^{-2}; since we do not assume $\eta^{-1} \in \mathbb{N}$, we have $|K_\eta| \leq (\eta^{-1} + 1)^2$ and, for η small, $|K_\eta| \leq 2\eta^{-2}$. The convergence (3.4a) is obtained from (3.2a) with the following computation:

$$\begin{aligned}
\left\|\mathbf{1}_{\Omega_\eta}\Psi_\eta^{(i)}(\cdot/\eta) - e_j\mathbf{1}_{\{x_3>0\}}\right\|_{L^2(\Omega)}^2 &\leq \sum_{k\in K_\eta} \left\|\mathbf{1}_{\Omega_\eta}\Psi_\eta^{(i)}(\cdot/\eta) - e_j\mathbf{1}_{\{x_3>0\}}\right\|_{L^2(\eta k+\eta Z)}^2 \\
&= \eta^3 \left(\sum_{k\in K_\eta} \left\|\Psi_\eta^{(i)} - e_j\mathbf{1}_{\{x_3>0\}}\right\|_{L^2(Z^\eta)}^2 + \|e_j\|_{L^2(\Sigma_Y^\eta)}^2\right) \\
&\leq \eta^3\, 2\eta^{-2} \left(\|\Psi_\eta^{(i)} - e_j\mathbf{1}_{\{x_3>0\}}\|_{L^2(Z^\eta)}^2 + \|e_j\|_{L^2(\Sigma_Y^\eta)}^2\right) \to 0\,.
\end{aligned}$$

Using $\mathrm{curl}(\Psi_\eta^{(i)}(\cdot/\eta)) = \eta^{-1}(\mathrm{curl}\,\Psi_\eta^{(i)})(\cdot/\eta)$ and (3.2b), a similar computation gives:

$$\begin{aligned}
\|\mathbf{1}_{\Omega_\eta}(\mathrm{curl}\,\Psi_\eta^{(i)}(\cdot/\eta))\|_{L^2(\Omega)}^2 &\leq \sum_{k\in K_\eta} \left\|\eta^{-1}(\mathrm{curl}\,\Psi_\eta^{(i)})(\cdot/\eta)\right\|_{L^2(\eta k+\eta Z^\eta)}^2 \\
&= \eta^3 \sum_{k\in K_\eta} \|\eta^{-1}\,\mathrm{curl}\,\Psi_\eta^{(i)}\|_{L^2(Z^\eta)}^2 = \eta^3\, 2\eta^{-2}\|\eta^{-1}\,\mathrm{curl}\,\Psi_\eta^{(i)}\|_{L^2(Z^\eta)}^2 \\
&= \eta^{-1}\, 2\|\,\mathrm{curl}\,\Psi_\eta^{(i)}\|_{L^2(Z^\eta)}^2 \to 0\,.
\end{aligned}$$

This shows (3.4b). □

4. Homogenization of Maxwell's equations

In this section, we pass to the limit $\eta \to 0$ in (1.1) and prove Theorem 2.1. The theorem states that system (2.4) holds for test-functions $\phi \in X$ and $\psi \in Y$, with specific spaces X and Y. By Lemma 2.2, system (2.4) is satisfied for test-functions $\phi \in X_{12}$ and $\psi \in Y_{12}$.

Let us recall the principle of our notation for test-functions: The spaces X_- and Y_- have no condition on Γ, the spaces X_i and Y_i contain a condition on the i-th component on Γ, the spaces X_{12} and Y_{12} contain conditions on both tangential components on Γ. When we want to show, e.g. that an equation holds for all $\phi \in X_1$, we show that: (a) the equation holds for all $\phi \in X_{12}$, and (b) the equation holds for all functions $\phi \in C_0^1(\overline{\Omega}, \mathbb{C}\, e_2)$.

4.1. **Limits of E^η-fields at the interface Γ.** Here, we want to verify (2.4a) for certain test-functions. We can restrict ourselves to the E-equation and study the following situation: Let E^η and f^η be two sequences in $L^2(\Omega, \mathbb{C}^3)$ with

$$\int_\Omega E^\eta \cdot \operatorname{curl} \psi = \int_\Omega f^\eta \cdot \psi \qquad \text{for every } \psi \in H(\operatorname{curl}, \Omega)\,, \tag{4.1a}$$

$$E^\eta = 0 \qquad \text{on } \Sigma_\eta\,. \tag{4.1b}$$

We recall that this can be regarded as a weak formulation of the equation $\operatorname{curl} E^\eta = f^\eta$ in Ω_η and $E^\eta \in H_0(\operatorname{curl}, \Omega_\eta)$. In particular, by (4.1b), $f^\eta|_{\Sigma_\eta} = 0$. Note that (4.1a) corresponds to (2.2a) for $f^\eta := \mathbf{1}_{\Omega_\eta}(i\omega\mu H^\eta + f_h)$.

Proposition 4.1 (Interface condition for the limit E)**.** *Let Ω, Σ_Y^η, Σ_η and Γ be given as in Section 2.1, let $i \in \{1, 2\}$ and $j = 3 - i$ indicate orthogonal directions. We assume that Σ_Y^η is asymptotically connected in the direction e_i in the sense of Definition 3.2, i.e. there is a sequence $\Psi_\eta^{(i)}$ satisfying* (3.2)*. Let E^η and f^η be sequences satisfying* (4.1) *that converge weakly in $L^2(\Omega, \mathbb{C}^3)$ to limit functions E and f, respectively. Then, the limits satisfy, for both $+$ and $-$:*

$$\int_{\Omega_\pm} E \cdot \operatorname{curl} \psi = \int_{\Omega_\pm} f \cdot \psi \quad \textit{for every } \psi \in C^1(\overline{\Omega_\pm}, e_j\, \mathbb{C})\,. \tag{4.2}$$

We recall that (4.2) *encodes $E_i|_\Gamma = 0$ (for both traces, from Ω_+ and from Ω_-).*

Proof. We present the proof for Ω_+. Let $\psi \in C^1(\overline{\Omega_+}, e_j\, \mathbb{C})$ be arbitrary, we write the function as $\psi = \varphi\, e_j$ with $\varphi \in C^1(\overline{\Omega_+}, \mathbb{C})$.

We extend φ to Ω by reflection across Γ, i.e. $\varphi(x_1, x_2, x_3) = \varphi(x_1, x_2, -x_3)$, respectively, for $x \in \Omega_-$. The extended function φ is continuous across Γ and of class $H^1(\Omega)$. Let $\Psi_\eta^{(i)}$ be the cell function of (3.2). We note that the strong convergences (3.4) imply the strong $L^2(\Omega)$-convergence

$$\begin{aligned} \mathbf{1}_{\Omega_\eta}(\cdot) \operatorname{curl}\left(\varphi(\cdot)\Psi_\eta^{(i)}(\cdot/\eta)\right) &= \varphi \mathbf{1}_{\Omega_\eta}(\operatorname{curl} \Psi_\eta^{(i)}(\cdot/\eta)) + \nabla\varphi \times \Psi_\eta^{(i)}(\cdot/\eta)\mathbf{1}_{\Omega_\eta} \\ &\to \nabla\varphi \times e_j \mathbf{1}_{\{x_3>0\}} = \operatorname{curl}(\varphi e_j)\mathbf{1}_{\{x_3>0\}}\,. \end{aligned} \tag{4.3}$$

Because of $\varphi(\cdot)\Psi_\eta^{(i)}(\cdot/\eta) \in H(\operatorname{curl}, \Omega)$, we can use this test-function in (4.1a). Since E^η and f^η vanish in Ω_η, we can multiply the integrand of (4.1a) also with $\mathbf{1}_{\Omega_\eta}$. We calculate for the limit $\eta \to 0$:

$$\begin{aligned} 0 &= \int_\Omega f^\eta(x) \cdot \varphi(x)\mathbf{1}_{\Omega_\eta}(x)\Psi_\eta^{(i)}(x/\eta) - E^\eta(x) \cdot \mathbf{1}_{\Omega_\eta}(x) \operatorname{curl}\left(\varphi(x)\Psi_\eta^{(i)}(x/\eta)\right) \mathrm{d}x \\ &\to \int_\Omega f(x) \cdot \varphi(x) e_j \mathbf{1}_{\{x_3>0\}}(x) - E(x) \cdot \operatorname{curl}(\varphi(x) e_j)\mathbf{1}_{\{x_3>0\}}(x)\, \mathrm{d}x \end{aligned}$$

$$= \int_{\Omega_+} f \cdot \varphi\, e_j - E \cdot \operatorname{curl}(\varphi\, e_j) = \int_{\Omega_+} f \cdot \psi - E \cdot \operatorname{curl}(\psi)\,.$$

This was the claim. □

4.2. **Limits of H^η-fields at the interface Γ.** We turn to the verification of (2.4b) for test-functions ϕ. As in the last subsection, we consider only the relevant equation. Here, we consider sequences H^η and f^η in $L^2(\Omega, \mathbb{C}^3)$ with

$$\int_\Omega H^\eta \cdot \operatorname{curl}\phi = \int_\Omega f^\eta \cdot \phi \quad \text{for every } \phi \in H_0(\operatorname{curl}, \Omega) \text{ with } \phi|_{\Sigma_\eta} = 0\,. \tag{4.4}$$

We recall that this is a weak formulation of $\operatorname{curl} H^\eta = f^\eta$ in Ω_η. Note that (4.4) is (2.2b) for $f^\eta := \mathbf{1}_{\Omega_\eta}(-i\omega\varepsilon E^\eta + f_e)$.

Proposition 4.2 (Interface condition for the limit H)**.** *Let Ω, Σ_Y^η, Σ_η and Γ be given as in Section 2.1, let $i \in \{1, 2\}$ and $j = 3 - i$ indicate orthogonal directions. We assume that Σ_Y^η is asymptotically disconnected in the direction e_j in the sense of Definition 3.1, i.e. there is a sequence $\Phi_\eta^{(j)}$ satisfying* (3.1)*. Let H^η and f^η be sequences satisfying* (4.4)*, with $H^\eta \rightharpoonup H$ and $f^\eta \rightharpoonup f$, weakly in $L^2(\Omega, \mathbb{C}^3)$. Then, the limits satisfy*

$$\int_\Omega H \cdot \operatorname{curl}\phi = \int_\Omega f \cdot \phi \quad \textit{for every } \phi \in C_0^1(\overline{\Omega}, e_j\mathbb{C})\,. \tag{4.5}$$

We recall that (4.5) *encodes, in the weak sense, the condition $[\![H_i]\!]_\Gamma = 0$.*

Proof. Let $\phi \in C_0^1(\overline{\Omega}, e_j\mathbb{C})$ be arbitrary, we write it as $\phi = \varphi\, e_j$ for $\varphi \in C_0^1(\overline{\Omega}, \mathbb{C})$. Let $\Phi_\eta^{(j)}$ be the cell function of (3.1). The strong convergences (3.3) imply the strong $L^2(\Omega)$ convergence

$$\begin{aligned} \operatorname{curl}\left(\varphi(\cdot)\Phi_\eta^{(j)}(\cdot/\eta)\right) &= \varphi \operatorname{curl}(\Phi_\eta^{(j)}(\cdot/\eta)) + \nabla\varphi \times \Phi_\eta^{(j)}(\cdot/\eta) \\ &\to \nabla\varphi \times e_j = \operatorname{curl}(\varphi\, e_j)\,. \end{aligned} \tag{4.6}$$

Because of $\varphi(\cdot)\Phi_\eta^{(j)}(\cdot/\eta) \in H_0(\operatorname{curl}, \Omega_\eta)$, we can use this test-function in (4.4). We calculate for the limit $\eta \to 0$:

$$\begin{aligned} 0 &= \int_\Omega f^\eta(x) \cdot \varphi(x)\Phi_\eta^{(j)}(x/\eta) - H^\eta(x) \cdot \operatorname{curl}\left(\varphi(x)\Phi_\eta^{(j)}(x/\eta)\right) \mathrm{d}x \\ &\to \int_\Omega f(x) \cdot \varphi(x)e_j - H(x) \cdot \operatorname{curl}(\varphi(x)e_j)\, \mathrm{d}x = \int_\Omega f \cdot \phi - H \cdot \operatorname{curl}(\phi)\,. \end{aligned}$$

This was the claim. □

Proof of Theorem 2.1. We have to show that (2.4) is satisfied for the right choice of function spaces $Y \in \{Y_-, Y_i, Y_{12}\}$ and $X \in \{X_-, X_i, X_{12}\}$. We recall that we have already verified (2.4a) for $\phi \in X_{12}$ and $\psi \in Y_{12}$, see Lemma 2.2.

If Σ_Y^η is asymptotically connecting in the direction e_i, Proposition 4.1 shows that (2.4a) is satisfied in $\Omega_\pm$ for $\psi \in C^1(\overline{\Omega_\pm}, e_j\,\mathbb{C})$. Together with the fact that the equation holds for all $\psi \in Y_{12}$, this implies that (2.4a) holds for all $\psi \in Y_i$.

Using this observation for $i = 1$ and $i = 2$, we obtain the claim of the theorem in Case 1, where the structure is connected in both directions and the space of test-functions is $Y_- = Y_1 \cup Y_2$.

When Σ_Y^η is asymptotically disconnected in the direction e_j, then Proposition 4.2 shows that (2.4b) holds for $\phi \in C_0^1(\overline{\Omega}, e_j\mathbb{C})$. Together with the fact that the equation holds for all $\phi \in X_{12}$, this implies that (2.4a) holds for all $\phi \in X_i$. This implies the theorem in Case 2 where the structure is disconnected in both directions and the space of test-functions is $X_- = X_1 \cup X_2$.

Finally, let us consider Case 3, Σ_Y^η is asymptotically connecting in the direction e_i and asymptotically disconnected in direction e_j. As shown above, this implies that test-functions $\psi \in Y_i$ and $\phi \in X_i$ are permitted. These are the desired spaces of test-functions in Case 3. □

5. Connectedness properties of general geometric structures

The last section concluded the homogenization result. We now change our perspective and turn to the analysis of connectedness properties of geometric structures. In this section, we conclude connectedness properties of one geometric structure from the connectedness properties of another geometric structure.

The observations of this section are very valuable when they are combined with our results on wires in Section 6. The monotonicity property allows to conclude: Every structure that contains e_i-connected wires is also e_i-connected. The deformation argument allows to conclude: Every smoothly deformed e_i-connected wire is also e_i-connected.

5.1. Monotonicity property for the obstacles.

We show that obstacles Σ_Y^η are asymptotically connecting if we can find subsets that are asymptotically connecting. Vice versa, obstacles $\widetilde{\Sigma}_Y^\eta$ are asymptotically disconnected if we can find supersets that are asymptotically disconnected.

Proposition 5.1 (Monotonicity). *Let $i \in \{1, 2\}$ be a direction, let Σ_Y^η and $\widetilde{\Sigma}_Y^\eta$ be two sequences of obstacles that are ordered in the sense of an inclusion: $\widetilde{\Sigma}_Y^\eta \subset \Sigma_Y^\eta$. Then, there holds:*

(i) When $\widetilde{\Sigma}_Y^\eta$ is asymptotically connecting in direction i, then also Σ_Y^η is.

(ii) When Σ_Y^η is asymptotically disconnected in direction i, then also $\widetilde{\Sigma}_Y^\eta$ is.

Proof. We consider $\widetilde{\Sigma}_Y^\eta$ that is asymptotically connecting. By definition of this property, there exists a sequence of cell functions $\Psi_\eta^{(i)} \in \dot{H}_\#(\mathrm{curl}, Z)$ that satisfies (3.2) for an integration domain $\widetilde{Z}^\eta := Z \setminus \widetilde{\Sigma}_Y^\eta$. The same sequence of cell functions $\Psi_\eta^{(i)} \in \dot{H}_\#(\mathrm{curl}, Z)$ satisfies (3.2) for the integration domain Z^η, since this domain is smaller: $Z^\eta = Z \setminus \Sigma_Y^\eta \subset Z \setminus \widetilde{\Sigma}_Y^\eta = \widetilde{Z}^\eta$. We conclude that Σ_Y^η is asymptotically connecting.

Let Σ_Y^η be asymptotically disconnected. Then, there exists a sequence of cell functions $\Phi_\eta^{(i)} \in \dot{H}_\#(\mathrm{curl}, Z)$ that satisfies (3.1) and vanishes on Σ_Y^η. In particular, $\Phi_\eta^{(i)} \in \dot{H}_\#(\mathrm{curl}, Z)$ vanishes on $\widetilde{\Sigma}_Y^\eta \subset \Sigma_Y^\eta$. The same sequence can be used for $\widetilde{\Sigma}_Y^\eta$ and we conclude that $\widetilde{\Sigma}_Y^\eta$ is asymptotically disconnected. □

5.2. **General deformation argument.** The following transformation argument allows to extend the results for straight geometries to deformed geometries.

Proposition 5.2 (Deformation of inclusions)**.** *Let $i \in \{1,2\}$ and $\widehat{\Sigma}_Y^\eta$ be a sequence of obstacles that is asymptotically connecting in the sense of Definition 3.2 (resp. disconnected in the sense of Definition 3.1) in the direction e_i. Let the sequence of deformation maps $\varphi_\eta \colon Y \to Y$ be bi-Lipschitz with $\varphi_\eta = \mathrm{id}$ on $\{x_3 = 1\}$ and on $\{x_3 = 0\}$, and such that $\varphi_\eta - \mathrm{id}$ is e_1- and e_2-periodic. We assume for a constant $C > 0$ the uniform boundedness*

$$\|\nabla \varphi_\eta\|_{L^\infty(\mathbb{R}^n)} + \|\nabla \varphi_\eta^{-1}\|_{L^\infty(\mathbb{R}^n)} \le C\,. \tag{5.1}$$

Then, $\Sigma_Y^\eta := \varphi_\eta(\widehat{\Sigma}_Y^\eta)$ is a sequence of obstacles that is asymptotically connecting in the sense of Definition 3.2 (resp. disconnected in the sense of Definition 3.1) in the direction e_i.

Proof. We present a proof for the case of asymptotically connectedness. For asymptotically disconnected obstacles the calculations are analogous. When $\widehat{\Sigma}_Y^\eta$ is asymptotically connected in the direction e_i, there exists a sequence $\widehat{\Psi}_\eta^{(i)} \in \dot{H}_\#(\mathrm{curl}, Z)$ that satisfies (3.2) for the domain $\widehat{Z}^\eta = Z \setminus \widehat{\Sigma}_\eta$.

We use the covariant Piola transform and define $\Psi_\eta^{(i)}$ by

$$\Psi_\eta^{(i)}(\varphi_\eta(x)) = D\varphi_\eta^{-\top}(x)\widehat{\Psi}_\eta^{(i)}(x)\,,$$

where $D\varphi_\eta$ denotes the Jacobian matrix, i.e. $(D\varphi_\eta)_{l,m} = \partial_{x_m}(\varphi_\eta)_l$. The transformation is such that the curl of the new field depends only on the curl of the given field (and not on other derivatives):

$$(\mathrm{curl}\,\Psi_\eta^{(i)}) \circ \varphi_\eta = \det(D\varphi_\eta)^{-1} D\varphi_\eta\, \mathrm{curl}\,\widehat{\Psi}_\eta^{(i)}\,, \tag{5.2}$$

see [18, Corollary 3.58]. Moreover, since $\varphi_\eta - \mathrm{id}$ is e_1- and e_2-periodic, this periodicity is transferred from $\widehat{\Psi}_\eta^{(i)}$ to $\Psi_\eta^{(i)}$ and there holds $\Psi_\eta^{(i)} \in \dot{H}_\#(\mathrm{curl}, Z)$.

In order to check (3.2a) for $\Psi_\eta^{(i)}$, we have to verify, for $j = 3 - i$, that $\eta\|\Psi_\eta^{(i)} - e_j \mathbf{1}_{\{x_3>0\}}\|^2_{L^2(Z^\eta)}$ vanishes in the limit $\eta \to 0$. Since $\Psi_\eta^{(i)}$ and $\widehat{\Psi}_\eta^{(i)}$ coincide outside Y, we only have to show $\eta\|\Psi_\eta^{(i)} - e_j\|^2_{L^2(Y\setminus\Sigma_Y^\eta)} \to 0$. The uniform essential boundedness of $D\varphi_\eta$ and $(D\varphi_\eta)^{-1}$ implies $\|\det(\nabla\varphi_\eta)^{-1}\|_{L^\infty(\mathbb{R}^n)} \le C$ and we can calculate

$$\begin{aligned}
\eta\|\Psi_\eta^{(i)} - e_j\|^2_{L^2(Y\setminus\Sigma_Y^\eta)} &= \eta \int_{Y\setminus\Sigma_Y^\eta} |\Psi_\eta^{(i)} - e_j|^2 \\
&= \eta \int_{Y\setminus\varphi_\eta^{-1}(\Sigma_Y^\eta)} |\det(D\varphi_\eta)|\, |\Psi_\eta^{(i)} \circ \varphi_\eta - e_j|^2 \\
&\le \eta \int_{Y\setminus\widehat{\Sigma}_Y^\eta} |\det(D\varphi_\eta)|\, |D\varphi_\eta^{-\top}\widehat{\Psi}_\eta^{(i)}|^2 + C\,\eta \\
&\le C\,\eta \int_{Y\setminus\widehat{\Sigma}_Y^\eta} |\widehat{\Psi}_\eta^{(i)}|^2 + C\,\eta \le C\,\eta \int_{Y\setminus\widehat{\Sigma}_Y^\eta} |\widehat{\Psi}_\eta^{(i)} - e_j|^2 + C\,\eta \to 0\,,
\end{aligned}$$

where we allow that C changes from one inequality to the next. In the convergence we used that $\widehat{\Psi}_\eta^{(i)}$ satisfies (3.2a).

To show that $\Psi_\eta^{(i)}$ satisfies (3.2b), we use (5.2) and calculate, on the relevant domain $Y \setminus \Sigma_Y^\eta$,

$$\begin{aligned}
\eta^{-1}\| \operatorname{curl} \Psi_\eta^{(i)}\|^2_{L^2(Y\setminus\Sigma_Y^\eta)} &= \eta^{-1} \int\limits_{Y\setminus\Sigma_Y^\eta} |\operatorname{curl} \Psi_\eta^{(i)}|^2 \, \mathrm{d}x \\
&= \eta^{-1} \int\limits_{Y\setminus\widehat{\Sigma}_Y^\eta} |\det(D\varphi_\eta)| |(\operatorname{curl} \Psi_\eta^{(i)}) \circ \varphi_\eta|^2 \\
&= \eta^{-1} \int\limits_{Y\setminus\widehat{\Sigma}_Y^\eta} |\det(D\varphi_\eta)| |\det(D\varphi_\eta)^{-1} D\varphi_\eta \operatorname{curl} \widehat{\Psi}_\eta^{(i)}|^2 \\
&\leq C\,\eta^{-1} \int\limits_{Y\setminus\widehat{\Sigma}_Y^\eta} |\operatorname{curl} \widehat{\Psi}_\eta^{(i)}|^2 \to 0\,,
\end{aligned}$$

where we used in the convergence that that $\widehat{\Psi}_\eta^{(i)}$ satisfies (3.2b). This completes the verification of (3.2). As already indicated, the calculations for $\Phi_\eta^{(i)}$ are analogous, the proof is complete. □

6. Wire geometries

In this section we analyze inclusions that model wires. We start from a concrete set Σ_Y^η that stands for a segment of a single wire. Putting together these pieces to the set Σ_η, we obtain $O(\eta^{-1})$ wires that are periodically distributed at a distance η, they are parallel, all oriented in direction e_1.

Our result is that if the wires are not too thin, these inclusions are indeed connecting in direction e_1 and not connecting in direction e_2. The situation becomes interesting when the wire (more precisely, its generator Σ_Y^η in the periodicity cell Y) has a radius that tends to 0 as $\eta \to 0$. We find the critical scaling of the radius such that the wire remains connecting in direction e_1. We also treat a situation in which the wire is interrupted by thin gaps. We check that very thin gaps do not destroy the connectedness property in direction e_1. This shows that, indeed, our asymptotic connectedness concept is different from the classical connectedness of sets.

We recall that asymptotically connecting in direction $i = 1$ is defined by the existence of cell functions $\Psi_\eta^{(i)}$, see Definition 3.2. Being asymptotically disconnected in direction $j = 2$ is declared by the existence of cell functions $\Phi_\eta^{(j)}$, see Definition 3.1. Thus, we are interested in constructing functions $\Phi_\eta^{(1)}$ and $\Psi_\eta^{(2)}$; in this section we skip the superscript and write only Φ_η and Ψ_η. We prepare the results with Section 6.1, where we construct 2D auxiliary functions. In Section 6.2, the 3D cell functions are constructed from their 2D counterparts.

6.1. **Constructions in 2D.** We consider inclusions (wires), that are not varying in the direction e_1. In such a geometry, it is convenient to reduce the construction of the cell functions to a 2D problem. We use the following notation in two space dimensions: For a vector $a = (a_1, a_2) \in \mathbb{R}^2$, we define the rotated vector by $a^\perp :=$

$(-a_2, a_1)$. The vector $a^\perp$ is obtained by applying the rotation matrix:

$$R := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad a^\perp = Ra\,.$$

When $U \subset \mathbb{R}^2$ is a Lipschitz domain and n is the unit outer normal, the left tangential vector is $t := n^\perp$. For a function $u\colon \mathbb{R}^2 \to \mathbb{R}$, we define the rotated gradient by $\nabla^\perp u := (-\partial_2 u, \partial_1 u)$. For a vector field $v\colon \mathbb{R}^2 \to \mathbb{R}^2$, we define the two-dimensional curl of v by $\nabla^\perp \cdot v := (-\partial_2 v_1 + \partial_1 v_2) = -\operatorname{div}(Rv)$. There holds $\nabla^\perp \cdot \nabla^\perp u = \Delta u$.

The notion of a weak rotated gradient or weak rotated divergence is defined in the distributional sense.

Connection to $3D$ *calculus:* For a set $U \subset \mathbb{R}^2$, we consider the corresponding cylindrical domain in three dimensions, $\widehat{U} := \{x \in \mathbb{R}^3 \mid x_1 \in \mathbb{R}\,, (x_2, x_3) \in U\}$. When $v\colon U \to \mathbb{R}^2$ is a two-dimensional vector field we define a three-dimensional vector field $\hat{v}\colon \widehat{U} \to \mathbb{R}^3$ by

$$\hat{v}(x_1, x_2, x_3) = e_2\, v_1(x_2, x_3) + e_3\, v_2(x_2, x_3)\,. \tag{6.1}$$

The curl of the new function is

$$\operatorname{curl} \hat{v}(x_1, x_2, x_3) = e_1\, (\nabla^\perp \cdot v)(x_2, x_3)\,.$$

The above observations motivate the following construction: We start from a harmonic potential u in 2D and construct a two-dimensional field v as the rotated gradient of u, i.e. $v = \nabla^\perp u$. Then, the two-dimensional curl of v vanishes: $\nabla^\perp \cdot v = \nabla^\perp \cdot \nabla^\perp u = \Delta u = 0$. In particular, for such a two-dimensional field v, the three-dimensional curl of $\hat{v}$ vanishes, $\operatorname{curl} \hat{v} = 0$. Following this idea, we will construct cell-functions as rotated gradients.

Construction of the $2D$ *cell function* ψ. Throughout this section, the two-dimensional geometry is given by a periodicity cell $V := (0,1)^2$, the independent variables are $z = (z_1, z_2) \in V$, given is a point $z_0 \in V$ and a radius $r > 0$ such that $\overline{B_r(z_0)} \subset V$. In the following, we suppress the dependence on z_0, but we indicate the dependence on $r > 0$. Later on, r will be the radius of the wire, and the asymptotics of the wire geometry will be relevant in our analysis.

The main part of our analysis regards the verification of the fact that the wire is connected in direction e_1; this requires the construction of a function Ψ_η. The three-dimensional function Ψ_η will be obtained from a two-dimensional function ψ_r as sketched above. The two-dimensional function is constructed as

$$\psi_r(z) := \begin{cases} \nabla^\perp u_r^\psi(z) & \text{for } z \in B_r(z_0)\,, \\ \nabla^\perp v_r^\psi(z) & \text{for } z \in V \setminus B_r(z_0) =: V_r\,. \end{cases} \tag{6.2}$$

The potentials v_r^ψ and u_r^ψ will be obtained as solutions of Poisson problems. We note that potentials may have jumps across $\partial B_r(z_0)$, but we have to make sure that the tangential component of ψ_r has no jumps. We therefore have to demand that the normal derivatives of the potentials have no jump at $\partial B_r(z_0)$. Additionally, u_r^ψ must be z_1-periodic. We construct v_r^ψ in Lemma 6.1 and u_r^ψ in Lemma 6.2.

Lemma 6.1 (Potential v_r^ψ). *Let* $z_0 \in V = (0,1)^2$ *be a point,* $R > 0$ *be a radius such that* $\overline{B_R(z_0)} \subset V$. *We consider* $0 < r \le R$ *and* $V_r = V \setminus B_r(z_0)$. *There exists a* *unique solution $v_r^\psi \in H^1(V_r)$, periodic in e_1-direction, of the problem*

$$-\Delta v_r^\psi = 0 \quad \text{in } V_r\,, \tag{6.3a}$$

$$\partial_2 v_r^\psi = -1 \quad \text{on } \{z_2 = 1\}\,, \tag{6.3b}$$

$$\partial_2 v_r^\psi = 0 \quad \text{on } \{z_2 = 0\}\,, \tag{6.3c}$$

$$\nabla v_r^\psi \cdot n = \frac{1}{|\partial B_r(z_0)|} \quad \text{on } \partial B_r(z_0)\,, \tag{6.3d}$$

with the normalization

$$\int_{\partial B_R(z_0)} v_r^\psi = 0\,. \tag{6.3e}$$

Moreover, there exists a constant $C_1 = C_1(R) > 0$, independent of r, such that

$$\|\nabla v_r^\psi\|^2_{L^2(V_r)} \le C_1 |\ln(r)|\,. \tag{6.4}$$

Proof. We note that (6.3a) is compatible with the boundary conditions (6.3b)–(6.3d). We denote by X_r the subset of $H^1(V_r)$ consisting of e_1-periodic functions satisfying (6.3e). We apply the theorem of Lax–Milgram and obtain the existence and uniqueness of a solution $v_r^\psi \in X_r$ of (6.3).

To derive estimate (6.4), we reformulate (6.3) as a minimization problem. Let the energy $E\colon X_r \to \mathbb{R}$ be given by

$$E(u) := \frac{1}{2}\int_{V_r} |\nabla u|^2 - \frac{1}{\partial B_r(z_0)} \int_{\partial B_r(z_0)} u + \int_{\{z_2=1\}} u\,.$$

We choose $u = 0 \in X_r$ as a competitor and obtain $E(v_r^\psi) \le E(0) = 0$. Hence, we can estimate

$$\frac{1}{2}\int_{V_r} |\nabla v_r^\psi|^2 \le \frac{1}{\partial B_r(z_0)} \int_{\partial B_r(z_0)} v_r^\psi - \int_{\{z_2=1\}} v_r^\psi\,. \tag{6.5}$$

To derive (6.4), it therefore suffices to estimate the two integrals on the right-hand side of (6.5).

For radii $r \le s \le R$ and, we apply the theorem of Gauß on the domain $B_R(z_0) \setminus \overline{B_s(z_0)}$ and find

$$\int_{\partial B_s(z_0)} n \cdot \nabla v_r^\psi = -1\,,$$

where n denotes the outer normal of $B_s(z_0)$. We rewrite this equality as

$$\begin{aligned} -\frac{1}{2\pi s} &= \frac{1}{|\partial B_s(z_0)|} \int_{\partial B_s(z_0)} n \cdot \nabla v_r^\psi = \frac{1}{|\partial B_1(z_0)|} \int_{\partial B_1(0)} \zeta \cdot \nabla v_r^\psi(z_0 + s\zeta)\,\mathrm{d}S(\zeta) \\ &= \frac{1}{2\pi} \partial_s \int_{\partial B_1(0)} v_r^\psi(z_0 + s\zeta)\,\mathrm{d}S(\zeta) = \partial_s \left(\frac{1}{|\partial B_s(z_0)|} \int_{\partial B_s(z_0)} v_r^\psi \right). \end{aligned}$$

Because of (6.3e), we obtain from the fundamental theorem of calculus

$$\begin{aligned}\frac{1}{|\partial B_r(z_0)|}\int_{\partial B_r(z_0)} v_r^\psi &= -\int_r^R \partial_s\left(\frac{1}{|\partial B_s(z_0)|}\int_{\partial B_s(z_0)} v_r^\psi\right)\mathrm{d}s \\ &= \int_r^R \frac{1}{2\pi s}\,\mathrm{d}s = \frac{1}{2\pi}(\ln(R)-\ln(r))\,.\end{aligned} \tag{6.6}$$

This provides a bound for the second integral of the energy $E(u)$ for $u = v_r^\psi$.

The last integral of E can be estimated with the trace theorem on V_R, with a Poincaré inequality and the Young inequality as

$$-\int_{\{z_2=1\}} v_r^\psi \le C_T\|v_r^\psi\|_{H^1(V_R)} \le C_P\|\nabla v_r^\psi\|_{L^2(V_R)} \le C_Y + \frac{1}{4}\|\nabla v_r^\psi\|^2_{L^2(V_R)}\,. \tag{6.7}$$

Inserting (6.6) and (6.7) in (6.5) yields

$$\frac{1}{4}\|\nabla v_r^\psi\|^2_{L^2(V_r)} \le \frac{1}{2\pi}(\ln(R)-\ln(r)) + C_Y \le 4C_1|\ln(r)|\,,$$

where the constant C_1 depends on R. This shows (6.4). □

The following lemma provides the scalar potential u_r^ψ on $B_r(z_0)$.

Lemma 6.2 (Potential u_r^ψ)**.** *Let $r > 0$ and $z_0 \in \mathbb{R}^2$. There exists a solution $u_r^\psi \in H^1(B_r(z_0))$ of the problem*

$$-\Delta u_r^\psi = \frac{1}{|B_r(z_0)|} \qquad \text{in } B_r(z_0)\,, \tag{6.8a}$$

$$\nabla u_r^\psi\cdot n = -\frac{1}{|\partial B_r(z_0)|} \qquad \text{on } \partial B_r(z_0) \tag{6.8b}$$

and constants C_2 and C_3 such that

$$\|\nabla u_r^\psi\|^2_{L^2(B_r(z_0))} = C_2\,, \qquad \|\Delta u_r^\psi\|^2_{L^2(B_r(z_0))} = C_3 r^{-2}\,. \tag{6.9}$$

Proof. We use $v_1(z_1,z_2) := -(4\pi)^{-1}(z_1^2+z_2^2)$ and $u_r^\psi(z) := v_1((z-z_0)/r)$ for $r>0$. An explicit calculation shows that $-\Delta v_1 = \pi^{-1}$ in $B_1(0)$ and $\nabla v_1\cdot n = -(2\pi)^{-1}$ on $\partial B_1(0)$. With the chain rule, we obtain $-\Delta u_r^\psi = \pi^{-1}r^{-2}$ in $B_r(0)$ and $\nabla u_r^\psi \cdot n = -(2\pi)^{-1}r^{-1}$ on $\partial B_r(0)$. Thus, u_r^ψ solves (6.8). In the same manner, the scaling by r leads to (6.9), an explicit computation shows $C_2 = (8\pi)^{-1}$ and $C_3 = \pi^{-1}$. □

We combine the 2D potentials v_r^ψ and u_r^ψ to define the 2D cell functions ψ_r via (6.2). Let us calculate the norms of the relevant quantities.

Lemma 6.3 (2D vector field ψ_r)**.** *Let $z_0 \in V = (0,1)^2$ be a point, $R > 0$ be a radius such that $\overline{B_R(z_0)} \subset V$. We consider $0 < r < R$ and $V_r = V \setminus B_r(z_0)$. There exists $\psi_r \in L^2(V,\mathbb{R}^2)$ with $\nabla^\perp\cdot\psi_r \in L^2(V,\mathbb{R}^2)$, periodic in the sense that its e_1-periodic extension $\widetilde{\psi}_r$ satisfies $\nabla^\perp\cdot\widetilde{\psi}_r \in L^2_{\mathrm{loc}}(\mathbb{R}\times(0,1))$, with the upper and lower boundary conditions*

$$\psi_r|_{\{z_2=0\}}\cdot e_1 = 0\,, \qquad \psi_r|_{\{z_2=1\}}\cdot e_1 = 1\,. \tag{6.10}$$

Furthermore, for constants C_1, C_2, C_3:

$$\|\psi_r\|^2_{L^2(V_r)} \le C_1|\ln(r)|\,, \qquad \|\psi_r\|^2_{L^2(B_r(z_0))} = C_2\,, \tag{6.11}$$

$$\|\nabla^\perp \cdot \psi_r\|^2_{L^2(V_r)} = 0\,, \qquad \|\nabla^\perp \cdot \psi_r\|^2_{L^2(B_r(z_0))} = C_3 r^{-2}\,. \tag{6.12}$$

The constants C_1, C_2, C_3 depend on R, but are independent of r, they are the constants of Lemmas 6.1 and 6.2.

Proof. Let v_r^ψ and u_r^ψ be given by Lemma 6.1 and Lemma 6.2, respectively. We define $\psi_r := \mathbf{1}_{B_r(z_0)}\nabla^\perp u_r^\psi + \mathbf{1}_{V_r}\nabla^\perp v_r^\psi$.

The first estimate of (6.11) is a direct consequence of the estimate (6.4) for ∇v_r^ψ, see Lemma 6.1. The second estimate of (6.11) is a direct consequence of (6.9) in Lemma 6.2. Since $\Delta v_r^\psi = 0$, we get the first equality in (6.12). The second equality follows from $\|\Delta u_r^\psi\|^2_{L^2(B_r(z_0))} = C_3 r^{-2}$ of (6.12).

The boundary conditions (6.10) follow from (6.3b) and (6.3c).

Regarding e_1-periodicity of ψ_r, we recall that v_r^ψ is the periodic solution of a Poisson problem. This does not only imply $v_r^\psi(1, z_2) = v_r^\psi(0, z_2)$ for every $z_2 \in (0,1)$, but also $\partial_1 v_r^\psi(1, z_2) = \partial_1 v_r^\psi(0, z_2)$ for every $z_2 \in (0,1)$, both conditions are understood in the sense of traces. This shows that ψ_r is e_1-periodic in the sense as described in the lemma.

We finally note that, by (6.3d) and (6.8b) the normal components of ∇v_r^ψ and ∇u_r^ψ coincide at $\partial B_r(z_0)$. This implies that tangential components of ψ_r do not jump at $\partial B_r(z_0)$, hence, $\nabla^\perp \cdot \psi_r \in L^2(V, \mathbb{R}^2)$. □

Construction of the 2D cell function ϕ. The construction of ϕ in two dimensions is much simpler than the construction of ψ_r, since it is independent of the radius. In the subsequent lemma, the set $U \subset (0,1)^2$ is contained in the complement of the cross section of the conductor, one may think of $U = V \setminus \overline{B_R(z_0)}$.

Lemma 6.4 (2D vector field ϕ)**.** *Let $U \subset V := (0,1)^2$ be a connected Lipschitz domain such that $\partial U \supset (0,1) \times \{0,1\}$. There exists $\phi \in L^2(V)$ with $\nabla^\perp \cdot \phi \in L^2(V)$, e_1-periodic in the sense that its e_1-periodic extension $\widetilde{\phi}$ satisfies $\nabla^\perp \cdot \widetilde{\phi} \in L^2_{\mathrm{loc}}(\mathbb{R} \times (0,1))$, satisfying*

$$\phi|_{\{z_2=0\}} \cdot e_1 = 1\,, \qquad \phi|_{\{z_2=1\}} \cdot e_1 = 1\,, \tag{6.13}$$

$$\phi|_{\{V\setminus U\}} = 0\,, \tag{6.14}$$

$$\nabla^\perp \cdot \phi = 0 \quad \textit{in } V\,. \tag{6.15}$$

Proof. We consider the system

$$-\Delta u^\phi = 0 \qquad \text{in } U\,, \tag{6.16a}$$

$$e_2 \cdot \nabla u^\phi = -1 \qquad \text{on } \{z_2 = 0\} \cup \{z_2 = 1\}\,, \tag{6.16b}$$

$$n \cdot \nabla u^\phi = 0 \qquad \text{on } \partial U \setminus \partial V\,, \tag{6.16c}$$

with periodicity conditions in e_1-direction. The Lemma of Lax–Milgram provides a unique (up to additive constants) solution $u^\phi \in H^1(U)$ of (6.16).

We define $\phi = \mathbf{1}_U \nabla^\perp u^\phi$ on V. There holds $\nabla^\perp \cdot \phi = \mathbf{1}_U \Delta u^\phi = 0$ in U. Due to (6.16c), the tangential component of ϕ vanishes at $\partial U \setminus \partial V$. This implies (6.15) in all of V. Condition (6.13) on the boundary values follows from (6.16b). The e_1 periodicity of u^ϕ is inherited by ϕ. □

6.2. **Construction of cell functions for 3D wires.** In this section, we use the 2D functions of the last subsection to construct the 3D cell functions $\Phi_\eta^{(j)}$ and $\Psi_\eta^{(i)}$ for wires. These cell functions show that the wire is asymptotically connecting in the direction e_i and asymptotically disconnected in direction e_j for $j = 3 - i$.

To simplify notation, we consider only $i = 1$. Wires are defined with a radius r_η and a center $z_0 \in V = (0,1)^2$ in the two dimensional cross section. With the two-dimensional ball $B_{r_\eta}(z_0) \subset V$, we define the wire as

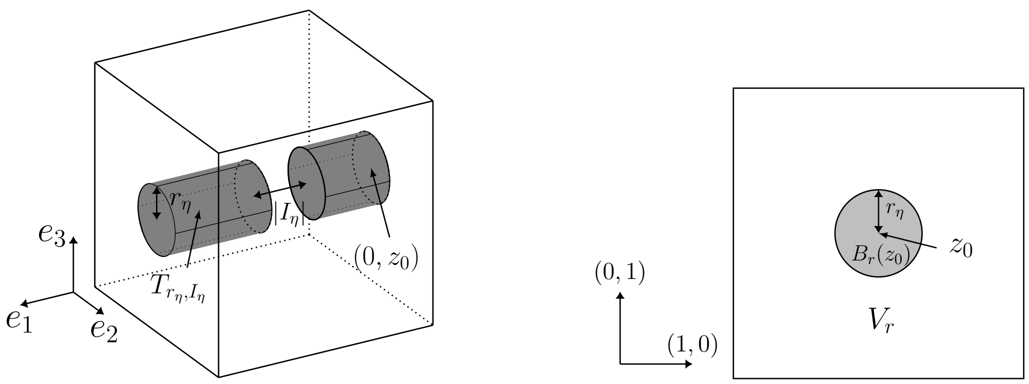
$$T_{r_\eta} := (0,1) \times B_{r_\eta}(z_0)\,. \tag{6.17}$$

The asymptotic connectedness of the wire T_{r_η} will depend on the asymptotic behavior of the radius r_η.

Additionally, we want to study the effect of gaps in the wire. Let $I_\eta \subset (0,1)$ be an open set of finitely many intervals representing gaps in the wires. We define

$$T_{r_\eta, I_\eta} := ((0,1) \setminus I_\eta) \times B_{r_\eta}(z_0)\,. \tag{6.18}$$

Note that T_{r_η, I_η} is also defined for $I_\eta = \emptyset$; in this case, no gaps are studied, $T_{r_\eta} = T_{r_\eta,\emptyset}$. The geometries are visualized in Figure 4.



(A) The set T_{r_η, I_η} in the reference cell Y (B) The cross section of T_{r_η, I_η} in $(0,1)^2$

FIGURE 4. Geometries for the construction of ϕ_η. The vector $(1,0) \in \mathbb{R}^2$ in the 2D picture corresponds with $e_2 = (0,1,0)$ in the 3D picture and $(0,1) \in \mathbb{R}^2$ with $e_3 = (0,0,1) \in \mathbb{R}^3$. When performing calculations in the 2D context, we also write e_1 for $(1,0) \in \mathbb{R}^2$ and e_2 for $(0,1) \in \mathbb{R}^2$.

For the wires T_{r_η, I_η} of (6.18) we obtain the following results:

- Proposition 6.5: If $\eta|\ln(r_\eta)| \to 0$ and $\eta^{-1} r_\eta^{-2} |I_\eta| \to 0$, then T_{r_η, I_η} is asymptotically connecting in the direction e_1.
- Proposition 6.6: Wires are asymptotically disconnected in direction e_2.
- Proposition 7.2: If $\eta|\ln(r_\eta)| \to \infty$, then $T_{r_\eta} = T_{r_\eta,\emptyset}$ is asymptotically disconnected in the direction e_1.

In the following, we construct the sequences of 3D cell functions from their 2D counterparts, exploiting that, apart from the gaps, the geometry is independent of

x_1. Starting from $\psi_{r_\eta}\colon V = (0,1)^2 \to \mathbb{R}^2$ given by Lemma 6.3, we define Ψ_η as

$$\Psi_\eta(x_1,x_2,x_3) := \begin{pmatrix} 0 \\ \psi_{r_\eta,1} \\ \psi_{r_\eta,2} \end{pmatrix} (x_2,x_3)\, \mathbf{1}_{\{0<x_3<1\}} + e_2\, \mathbf{1}_{\{x_3>1\}} \,. \tag{6.19}$$

The properties of ψ_{r_η} on $\{z_2 = 1\}$ and $\{z_2 = 0\}$ imply that the tangential components of the 3D functions have no jumps. The curl is therefore given by

$$\operatorname{curl} \Psi_\eta(x_1,x_2,x_3) = \left(\nabla^\perp \cdot \psi_{r_\eta}\right)(x_2,x_3)\, e_1\, \mathbf{1}_{\{0<x_3<1\}} \,. \tag{6.20}$$

Proposition 6.5 (Cell function Ψ_η and connectedness of wires in direction e_1)**.** *Let $z_0 \in V = (0,1)^2$ be a point, $R > 0$ be a radius such that $\overline{B_R(z_0)} \subset V$. For a sequence of radii $0 < r_\eta \leq R$ and a sequence of open sets $I_\eta \subset (0,1)$, we consider T_{r_η,I_η} of* (6.18) *and assume*

$$\eta |\ln(r_\eta)| \to 0 \qquad \text{and} \qquad \eta^{-1} r_\eta^{-2} |I_\eta| \to 0 \,. \tag{6.21}$$

Then, the inclusion $\Sigma_Y^\eta = T_{r_\eta,I_\eta}$ is asymptotically connecting in direction e_1, i.e. there exists a sequence $\Psi_\eta \in \dot{H}_\sharp(\operatorname{curl}, Z)$ satisfying (3.2) *for $j = 2$.*

Proof. Let $\psi_{r_\eta} \in H_\sharp(\nabla^\perp\cdot, V)$ be given by Lemma 6.3 and Ψ_η by (6.19). Since ψ_{r_η} is periodic in the first argument, Ψ_η is e_2-periodic (since Ψ_η is independent of x_1, it is also e_1-periodic). The curl of Ψ_η is essentially given by $\nabla^\perp \cdot \psi_{r_\eta}$, see (6.20). This will allow to derive estimates (3.2a) and (3.2b).

The estimate (3.2a) compares the cell function Ψ_η with the characteristic function $e_2 \mathbf{1}_{\{x_3>0\}}$ on the set Z^η. Since Ψ_η is identical to e_2 for $x_3 > 1$ and vanishes for $x_3 < 0$, the only contributions to the $L^2(Z^\eta)$-norm come from $Y \setminus T_{r_\eta,I_\eta}$, which we decompose into the two sets $Y \setminus T_{r_\eta}$ and $T_{r_\eta} \setminus T_{r_\eta,I_\eta} = I_\eta \times B_{r_\eta}(z_0)$. With a triangle inequality and Young's inequality, we find

$$\begin{aligned} \eta \|\Psi_\eta - e_2 \mathbf{1}_{\{x_3>0\}}\|^2_{L^2(Z^\eta)} &= \eta \|\Psi_\eta - e_2\|^2_{L^2(Y\setminus T_{r_\eta})} + \eta \|\Psi_\eta - e_2\|^2_{L^2(I_\eta \times B_{r_\eta}(z_0))} \\ &\leq 2\eta \|\psi_{r_\eta}\|^2_{L^2(V_{r_\eta})} + 2\eta |I_\eta| \|\psi_{r_\eta}\|^2_{L^2(B_{r_\eta}(z_0))} + 2\eta \\ &\leq 2\eta\, C_1 |\ln(r_\eta)| + 2\eta |I_\eta| C_2 + 2\eta \to 0 \,, \end{aligned}$$

where we used (6.11) of Lemma 6.3 and the asymptotics (6.21) in the last line. This shows that Ψ_η satisfies (3.2a).

To show (3.2b), we proceed similarly and calculate

$$\begin{aligned} \eta^{-1} \|\operatorname{curl} \Psi_\eta\|^2_{L^2(Z^\eta)} &= \eta^{-1} \|\operatorname{curl} \Psi_\eta\|^2_{L^2(Y\setminus T_{r_\eta})} + \eta^{-1} \|\operatorname{curl} \Psi_\eta\|^2_{L^2(I_\eta \times B_{r_\eta}(z_0))} \\ &= \eta^{-1} \|\nabla^\perp \cdot \psi_{r_\eta}\|^2_{L^2(V_{r_\eta})} + \eta^{-1} |I_\eta| \|\nabla^\perp \cdot \psi_{r_\eta}\|^2_{L^2(B_{r_\eta}(z_0))} \\ &\leq \eta^{-1} |I_\eta| C_3 r_\eta^{-2} \to 0 \,, \end{aligned}$$

where we used estimate (6.12) of Lemma 6.3 and the asymptotics (6.21) in the last line. This shows that Ψ_η satisfies (3.2b). ☐

The cell function Φ_η is constructed similar as Ψ_η in (6.19). For Φ_η, the estimates are actually much simpler than for Ψ_η, since we can choose $\Phi_\eta = \Phi$ independent of

η. Starting from a vector-field on $\phi\colon U \to \mathbb{C}^2$ on $U \subset V = (0,1)^2$, we define

(6.22)

$$\Phi_\eta(x_1,x_2,x_3) := \Phi(x_1,x_2,x_3) = \begin{pmatrix} 0 \\ \phi_1 \\ \phi_2 \end{pmatrix} (x_2,x_3)\, \mathbf{1}_{\{0<x_3<1\}} + e_2\, \mathbf{1}_{\{x_3>1 \text{ or } x_3<0\}}\,.$$

Choosing ϕ of Lemma 6.4 for $U = V_R$, we can exploit the properties of ϕ on $\{z_2 = 1\}$ and $\{z_2 = 0\}$. They imply that tangential components of Φ_η have no jump across $x_3 = 0$ or $x_3 = 1$. In particular, the curl is given by

$$\operatorname{curl} \Phi_\eta(x_1,x_2,x_3) = \left(\nabla^\perp \cdot \phi\right)(x_2,x_3)\, e_1\, \mathbf{1}_{\{0<x_3<1\}}\,. \tag{6.23}$$

Proposition 6.6 (Cell function Φ_η and disconnectedness of wires in direction e_2)**.** *Let $z_0 \in V = (0,1)^2$ be a point, $R > 0$ be a radius such that $\overline{B_R(z_0)} \subset V$. For a sequence of radii $0 < r_\eta \le R$ and a sequence of open sets $I_\eta \subset (0,1)$, we consider T_{r_η,I_η} of* (6.18)*. The inclusion $\Sigma_Y^\eta = T_{r_\eta,I_\eta}$ is always asymptotically disconnected in direction e_2, i.e. there exists a sequence $\Phi_\eta \in \overset{\bullet}{H}_\#(\operatorname{curl}, Z)$ satisfying* (3.1) *for $j = 2$. In fact, one can choose $\Phi_\eta = \Phi$ independent of η.*

Proof. We use ϕ of Lemma 6.4 for $U = V_R$ and $\Phi_\eta = \Phi$ by (6.22). Since ϕ is periodic in the first argument, Φ is e_2-periodic (since Φ is independent of x_1 it is also e_1-periodic). The curl of Φ is essentially given by $\nabla^\perp \cdot \phi$, see (6.23). Using $\nabla^\perp \cdot \phi = 0$ of (6.15), we obtain $\operatorname{curl} \Phi = 0$. This implies (3.1c).

The estimate (3.1a) compares the cell function Φ with the constant e_2 on the set Z. Since Φ is identical to e_2 for $x_3 \notin (0,1)$, the only contributions to the $L^2(Z)$-norm come from Y, where the L^2-norm of Φ is finite, hence

$$\eta^{1/2}\|\Phi - e_2\|_{L^2(Z)} = \eta^{1/2}\|\Phi - e_2\|_{L^2(Y)} \to 0\,.$$

This shows that $\Phi_\eta = \Phi$ satisfies (3.1a). Moreover, since ϕ vanishes on $B_R(z_0) = V \setminus U$, see (6.14), we have $\Phi|_{T_{R,\emptyset}} = 0$ and, thus, in particular, $\Phi|_{T_{r_\eta,I_\eta}} = 0$, which verifies (3.1b). □

Proposition (6.6) and its proof hold also for more general geometries:

Remark 6.7 (Disconnectedness of general cylinders in direction e_2)**.** *Let all sets Σ_Y^η be contained in a set $\Sigma_Y^0 = (0,1) \times B$ for a closed set $B \subset V = (0,1)^2$ that is independent of η. Then, Σ_Y^η is asymptotically disconnected in directions e_2.*

Proof. We note that $V \setminus B$ has one connected component $U \subset V \setminus B$ for which $\partial U \supset \partial V$ holds. Arguing as in the proof of Proposition 6.6 using the set U instead of V_R, we obtain that $(0,1) \times (V \setminus \overline{U})$ is asymptotically disconnected in the direction e_2. For Σ_Y^0 and Σ_Y^η, the asymptotically disconnectedness in direction e_2 follows from the monotonicity result of Proposition 5.1. □

Proof of Remark 2.3. For the closed set $\Sigma_Y^0 \subset Y = (0,1)^3$ one can find a closed set $B_1 \in (0,1)^2$ such that $\Sigma_Y^0 \subset (0,1) \times B_1$. Choosing the larger set $(0,1) \times B_1$ as new Σ_Y^0, Remark 6.7 shows the asymptotic disconnectedness in direction e_2. Moreover, there exists a set B_2 such that $\Sigma_Y^0 \subset \{y \in Y \mid (y_1,y_3) \in B_2\}$. By changing the role of x_1 and x_2, we obtain the asymptotic disconnectedness in direction e_1. □

7. Criticality of the radius asymptotics

In this section we show that, when the radii are substantially smaller than in Proposition 6.5, the cylindrical obstacles are disconnected even in direction e_1. This is true even if there are no gaps.

Lemma 7.1 (2D scalar field ϕ_r)**.** *Let $z_0 \in V = (0,1)^2$ be a point, $R > 0$ be a radius such that $\overline{B_R(z_0)} \subset V$. We consider $0 < r \le R^2$ and $V_r = V \setminus B_r(z_0)$. There exists $\phi_r \in H^1(V, \mathbb{R})$ and a constant C_4, which does not depend on r, such that*

$$\phi_r|_{B_r(z_0)} \equiv 0\,, \qquad \phi_r|_{V\setminus B_R(z_0)} \equiv 1\,, \tag{7.1}$$

$$\phi_r(x) \in [0,1] \qquad \textit{for a.e. } x \in V\,, \tag{7.2}$$

$$\|\nabla \phi_r\|^2_{L^2(V)} \le C_4 |\ln(r)|^{-1}\,. \tag{7.3}$$

Proof. We define ϕ_r on $B_R(z_0) \setminus B_r(z_0)$ as the minimizer of the Dirichlet energy $\frac{1}{2}\|\nabla\phi_r\|^2_{L^2(B_R(z_0)\setminus B_r(z_0))}$ subject to the boundary values $\phi_r = 0$ on $\partial B_r(z_0)$ and $\phi_r = 1$ on $\partial B_R(z_0)$. This minimizer is given by a logarithm in the radius, more precisely, with $|x| = (x_1^2 + x_2^2)^{1/2}$:

$$\phi_r(x) = \widetilde{\phi}_r(|x - z_0|) \qquad \text{for} \quad \widetilde{\phi}_r(\tau) = \frac{\ln(\tau) - \ln(r)}{\ln(R) - \ln(r)}.$$

We extend ϕ_r by 0 in $B_r(z_0)$ and by 1 in $V \setminus B_R(z_0)$ such that (7.1) and (7.2) are satisfied. Elementary calculus yields

$$\begin{aligned}\|\nabla\phi_r\|^2_{L^2(V)} &= \int_{B_R(z_0)\setminus B_r(z_0)} |\nabla\phi_r|^2 = 2\pi \int_r^R |\partial_\tau \widetilde{\phi}_r|^2\, \tau\, d\tau \\ &= 2\pi(\ln(R) - \ln(r))^{-2} \int_r^R \frac{1}{\tau}\, d\tau = 2\pi(\ln(R) - \ln(r))^{-1}\,.\end{aligned}$$

This provides (7.3). □

As in Section 6, we construct a sequence of 3D cell functions from the 2D counterpart. Here, for a sequence of radii r_η we use the 2D scalar field ϕ_{r_η} from Lemma 7.1 to construct a field Φ_η that is aligned with e_1:

$$\Phi_\eta(x_1, x_2, x_3) := e_1\, \phi_{r_\eta}(x_2, x_3)\, \mathbf{1}_{\{0<x_3<1\}} + e_1\, \mathbf{1}_{\{x_3>1 \text{ or } x_3<0\}}\,. \tag{7.4}$$

The properties of ϕ_{r_η} on $\{z_2 = 1\}$ and $\{z_2 = 0\}$ imply that the tangential components of the 3D function Φ_η have no jumps. The curl is therefore given by

$$\operatorname{curl} \Phi_\eta(x_1, x_2, x_3) = \begin{pmatrix} 0 \\ \partial_2\phi_{r_\eta} \\ -\partial_1\phi_{r_\eta}\end{pmatrix}(x_2, x_3)\, \mathbf{1}_{\{0<x_3<1\}}\,. \tag{7.5}$$

Proposition 7.2 (Cell function Φ_η and disconnectedness in direction e_1 for too thin wires)**.** *Let $z_0 \in V = (0,1)^2$ be a point and $R > 0$ with $\overline{B_R(z_0)} \subset V$. For a sequence of radii $0 < r_\eta \le R^2$, we consider T_{r_η} defined in* (6.17)*. Let the inclusions be wires $\Sigma^\eta_Y = T_{r_\eta}$ with exponentially small radius in the sense that*

$$\eta |\ln(r_\eta)| \to \infty\,. \tag{7.6}$$

Then, Σ^η_Y is asymptotically disconnected in direction e_1 in the sense of Definition 3.1: There exists a sequence $\Phi_\eta \in \dot{H}_\#(\operatorname{curl}, Z)$ satisfying (3.1) *with $j = 1$.*

Proof. We consider $\phi_{r_\eta} \in H^1(V)$ of Lemma 6.4 and use Φ_η of (7.4). The function ϕ_{r_η} is periodic in its first argument, therefore, Φ_η is e_2-periodic (it is independent of x_1, hence also e_1-periodic). The curl of Φ_η is given by (7.5) and we have $\Phi_\eta \in \dot{H}_\#(\mathrm{curl}, Z)$. It remains to show (3.1).

By construction, $\phi_{r_\eta}|_{B_{r_\eta}(z_0)} = 0$. This yields that Φ_η vanishes in T_{r_η}, thus, (3.1b) holds. Regarding (3.1a), we calculate

$$\eta^{1/2} \|\Phi_\eta - e_1\|_{L^2(Z)} = \eta^{1/2} \|e_1(\phi_{r_\eta} - 1)\|_{L^2(V)} \leq \eta^{1/2} \to 0\,,$$

where we used $\phi_{r_\eta}(x) \in [0, 1]$, which was observed in (7.2). We have thus obtained (3.1a).

In order to show (3.1c), we use the curl of Φ_η that was calculated in (7.5). The estimate (7.3) on $\nabla\phi_{r_\eta}$ allows to calculate

$$\eta^{-1} \|\,\mathrm{curl}\, \Phi_\eta\|^2_{L^2(Z)} = \eta^{-1} \|\nabla^\perp \phi_{r_\eta}\|^2_{L^2(V)} \leq \eta^{-1} C_4 |\ln(r_\eta)|^{-1} \to 0\,,$$

where we used (7.6) in the convergence. This verifies (3.1c). □

References

[1] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992.

[2] G. Bouchitté, C. Bourel, and D. Felbacq. Homogenization of the 3D Maxwell system near resonances and artificial magnetism. *C. R. Math. Acad. Sci. Paris*, 347(9-10):571–576, 2009.

[3] G. Bouchitté, C. Bourel, and L. Manca. Resonant effects in random dielectric structures. *ESAIM Control Optim. Calc. Var.*, 21(1):217–246, 2015.

[4] G. Bouchitté and D. Felbacq. Homogenization of a wire photonic crystal: the case of small volume fraction. *SIAM J. Appl. Math.*, 66(6):2061–2084, 2006.

[5] G. Bouchitté and B. Schweizer. Homogenization of Maxwell's equations in a split ring geometry. *Multiscale Model Simul*, 8(3):717–750, 2010.

[6] G. Bouchitté and B. Schweizer. Plasmonic waves allow perfect transmission through subwavelength metallic gratings. *Netw. Heterog. Media*, 8(4):857–878, 2013.

[7] R. Bunoiu, L. Chesnel, K. Ramdani, and M. Rihani. Homogenization of Maxwell's equations and related scalar problems with sign-changing coefficients. *Ann. Fac. Sci. Toulouse Math. (6)*, 30(5):1075–1119, 2021.

[8] K. Cherednichenko and S. Cooper. Homogenization of the system of high-contrast Maxwell equations. *Mathematika*, 61(2):475–500, 2015.

[9] D. Cioranescu and F. Murat. A strange term coming from nowhere. In *Topics in the mathematical modelling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 45–93. Birkhäuser Boston, Boston, MA, 1997.

[10] B. Delourme, S. Fliss, P. Joly, and E. Vasilevskaya. Trapped modes in thin and infinite ladder like domains. Part 1: Existence results. *Asymptot. Anal.*, 103(3):103–134, 2017.

[11] B. Delourme, H. Haddar, and P. Joly. Approximate models for wave propagation across thin periodic interfaces. *J. Math. Pures Appl. (9)*, 98(1):28–71, 2012.

[12] P. Donato, A. Lamacz, and B. Schweizer. Sound absorption by perforated walls along boundaries. *Applicable Analysis. An International Journal*, 101(13):4397–4411, 2022.

[13] D. Gallistl, P. Henning, and B. Verfürth. Numerical homogenization of **H**(curl)-problems. *SIAM J. Numer. Anal.*, 56(3):1570–1596, 2018.

[14] P. Henning, M. Ohlberger, and B. Verfürth. A new heterogeneous multiscale method for time-harmonic Maxwell's equations. *SIAM J. Numer. Anal.*, 54(6):3493–3522, 2016.

[15] A. Kirsch and F. Hettlich. *The mathematical theory of time-harmonic Maxwell's equations*, volume 190 of *Applied Mathematical Sciences*. Springer, Cham, 2015. Expansion-, integral-, and variational methods.

[16] A. Lamacz and B. Schweizer. A negative index meta-material for Maxwell's equations. *SIAM J. Math. Anal.*, 48(6):4155–4174, 2016.

[17] R. Lipton and B. Schweizer. Effective Maxwell's equations for perfectly conducting split ring resonators. *Arch. Ration. Mech. Anal.*, 229(3):1197–1221, 2018.

[18] P. Monk. *Finite element methods for Maxwell's equations.* Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.

[19] M. Ohlberger, B. Schweizer, M. Urban, and B. Verführt. Mathematical analysis of transmission properties of electromagnetic meta-materials. *Networks and Heterogeneous Media*, 15(1):29–56, 2020.

[20] B. Schweizer. The low-frequency spectrum of small Helmholtz resonators. *Proc. A.*, 471(2174):20140339, 18, 2015.

[21] B. Schweizer. Resonance meets homogenization: construction of meta-materials with astonishing properties. *Jahresber. Dtsch. Math.-Ver.*, 119(1):31–51, 2017.

[22] B. Schweizer. Effective Helmholtz problem in a domain with a Neumann sieve perforation. *J. Math. Pures Appl. (9)*, 142:1–22, 2020.

[23] B. Schweizer and M. Urban. Effective Maxwell's equations in general periodic microstructures. *Applicable Analysis. An International Journal*, 97(13):2210–2230, 2018.

[24] T. A. Suslina. Homogenization of the stationary Maxwell system with periodic coefficients in a bounded domain. *Arch. Ration. Mech. Anal.*, 234(2):453–507, 2019.

[25] N. Wellander and G. Kristensson. Homogenization of the Maxwell equations at fixed frequency. *SIAM J. Appl. Math.*, 64(1):170–195, 2003.