Transmission conditions for the Helmholtz-equation in perforated domains

Christina Dörlemann, Martin Heida, Ben Schweizer*

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Abstract

We study the Helmholtz equation in a perforated domain Ω_ε. The domain Ω_ε is obtained from an open set $\Omega \subset \mathbb{R}^3$ by removing small obstacles of typical size $\varepsilon > 0$, the obstacles are located along a 2-dimensional manifold $\Gamma_0 \subset \Omega$. We derive effective transmission conditions across Γ_0 that characterize solutions in the limit $\varepsilon \to 0$. We obtain that, to leading order $O(\varepsilon^0)$, the perforation is invisible. On the other hand, at order $O(\varepsilon^1)$, inhomogeneous jump conditions for the pressure and the flux appear. The form of the jump conditions is derived.

Keywords: Helmholtz equation, perforated domain, transmission conditions, Neumann sieve, acoustics

MSC: 35B27, 74Q05

1 Introduction

Our aim is to study the acoustic properties of complex domains. Assuming that acoustic waves are described by the linear wave equation, the acoustic properties of a domain Ω_ε are determined by the Helmholtz equation

$$-\Delta p^\varepsilon = \omega^2 p^\varepsilon + f \qquad \text{in } \Omega_\varepsilon, \tag{1.1}$$

where ω is the frequency of waves and f is a right hand side that models sound sources in the domain $\Omega_\varepsilon \subset \mathbb{R}^3$. Equation (1.1) is accompanied by a boundary condition on $\partial\Omega_\varepsilon$.

We use a small parameter $\varepsilon > 0$ and write Ω_ε for the domain, since we assume that the domain contains structures of typical size ε. More specifically, we investigate a perforated domain: We investigate three-dimensional domains that contain many obstacles (the number of obstacles is of order ε^{-2}) with the small diameter $\varepsilon > 0$, we denote the single obstacle by Σ_k^ε, where $k \in \mathbb{Z}^2$ is an index to number the obstacles.

*Technische Universität Dortmund, Fakultät für Mathematik, Vogelpothsweg 87, D-44227 Dortmund, Germany. ben.schweizer@tu-dortmund.de

We assume that the obstacles are periodically distributed along a 2-dimensional hyperplane $\Gamma_0 \subset \mathbb{R}^3$. The domain Ω_ε is obtained from an ε-independent domain $\Omega \subset \mathbb{R}^3$ by removing the obstacles, $\Omega_\varepsilon = \Omega \setminus \bigcup_k \Sigma_k^\varepsilon$. Every point in the open set $\Omega \setminus \Gamma_0$ does not touch any obstacle for sufficiently small $\varepsilon > 0$ (compare Figure 1).

We ask for the effective influence of the perforations along Γ_0. A rigorous description can be obtained by the analysis of solution sequences p^ε to (1.1) in the sense of homogenization. Denoting a weak limit of the solution sequence p^ε by p, we ask for the system of equations that determines p. We will show rigorously that the limit p is characterized by the Helmholtz equation in the domain Ω, hence the effect of the perforation gets lost at leading order, see (1.6).

At first glance, this result seems to be counter-intuitive: One might expect some influence of the perforation, some jump conditions for the pressure function across Γ_0 and/or some jump conditions for the velocities $-\nabla p$ across Γ_0. On the other hand, using analytical knowledge, our first result cannot be much of a surprise: The solution sequence is bounded in $H^1(\Omega_\varepsilon)$, a subsequence is converging weakly in this space, the space $H^1(\Omega_\varepsilon)$ admits to evaluate traces, the continuity of the trace operator implies that the limit function cannot have a jump of traces. Hence we expect no jumps of the limit function p across Γ_0, we write this as $[p] = 0$. Similar arguments can be used for the flux: If the flux into the obstacles vanishes on the ε-level ($\partial_n p^\varepsilon = 0$ on $\partial\Sigma_k^\varepsilon$), then, effectively, no source can appear along Γ_0. We therefore expect $[\partial_\nu p] = 0$ along Γ_0, where ν denotes a normal vector on Γ_0. The two conditions are established rigorously in Theorem 1.1.

A deeper insight can be gained by studying first order effects: The intuition (and some rule-of-thumb equations of the more physical literature) can be confirmed if one considers effects of order ε, or, in more technical terms, if one analyzes the weighted difference $v^\varepsilon := (p^\varepsilon - p)/\varepsilon$. Our result in Theorem 1.2 provides the form of the system for a weak limit v of the sequence v^ε: The function v solves the Helmholtz equation on the domain $\Omega \setminus \Gamma_0$, and the functions v and ∇v satisfy jump conditions across Γ_0. These jump conditions contain the pressure function p and its derivatives as specified in (1.10): The jump $[v]$ of v is proportional to the slope $\partial_\nu p$ of p along Γ_0. The jump $[\partial_\nu v]$ of the velocity corrector is proportional to the second derivative $\partial_\nu^2 p$ of p along Γ_0. Our limit system includes a correction coefficient α (introduced in assumption (1.9)), which is unfortunately not yet characterized. We hope that we can derive a cell problem for this parameter in a later work.

1.1 Comparison with the literature

The Helmholtz equation (1.1) describes the distribution of sound waves of a fixed frequency ω in a prescribed geometry. The geometry studied here is of much interest in applications, for example in the design of sound absorbing structures. If a wall with holes is used to separate two chambers, this wall can have a decisive effect on the distribution of sound waves.

The acoustic properties of a perforation. With this applications in mind, many contributions from a very practical point of view are available. An effective descrip-

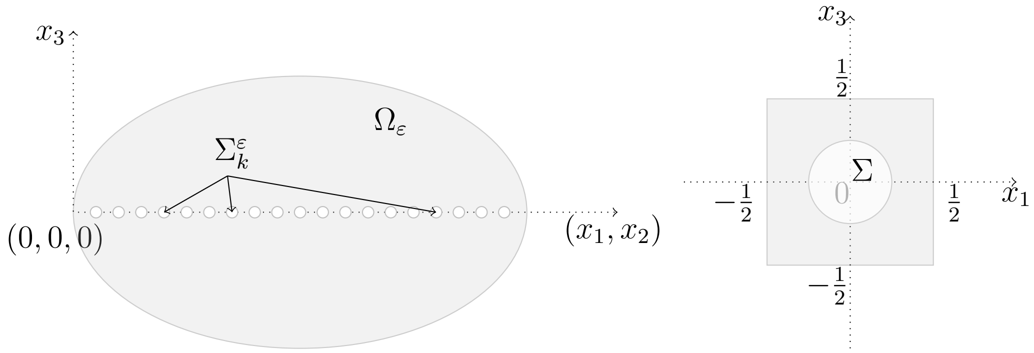

Figure 1: Left: The domain Ω_ε with many small obstacles Σ_k^ε. Right: Each obstacle is a scaled and shifted copy of a standard obstacle $\Sigma \subset \mathbb{R}^3$.

tion of the perforation that is used in the literature can be written as

$$\partial_\nu p^+ = \partial_\nu p^- = -i\frac{\omega\rho}{Z}(p^+ - p^-)\,. \tag{1.2}$$

In this formula, ρ denotes the density, ω the frequency, ν a normal vector on Γ_0, pointing into the domain Ω_+, and Z is a complex number, the *transmission impedance*, a parameter that characterizes the effective behavior of the obstacles (we cite from equation (2) of [13], where reference is given to [7]).

Let us compare the empirical formula (1.2) with our findings. As a first observation, we note that in both, in (1.2) and in our results, the normal component of the pressure gradient has no jump. The second equation in (1.2) seems to contradict our finding that also the effective pressure function p has no jump. But we may as well compare the pressure difference $p^+ - p^-$ with the jump of the first order corrector, scaled with ε, i.e.: $p^+ - p^-$ behaves like $\varepsilon[v]$. If we do so, we may also say that (1.2) is consistent with our formula $\alpha\,[v] = |\Sigma|\,\partial_\nu p$ from (1.10), where α is a correction factor that is close to 1 for small obstacles. The comparison provides a formula for the transmission impedance: $Z = -i\omega\rho\varepsilon|\Sigma|/\alpha$.

We note that a more mathematical treatment of a related problem has been performed in [13]. In that work, the authors obtain a non-trivial effective transmission condition for the pressure p. Their formula (29) can be written as

$$-i\omega D_\beta \partial_\beta p + \omega^2 F g_0 = \frac{-i\omega}{\varepsilon_0}(p^+ - p^-)\,.$$

In this formula, D_β and F are effective coefficients and are given by cell problems, ∂_β denotes derivatives in direction β, g_0 is a "fictitious acoustic transverse velocity", which re-appears in their second transmission condition, ε_0 is a thickness parameter. Also this formula can be compared to our result: After a division by $-i\omega$, the gradient of p on the left hand side is set in relation with $(p^+ - p^-)/\varepsilon_0$ on the right hand side. Neglecting g_0, the comparison suggests for the normal direction $\beta = \nu$ that $D_\beta = |\Sigma|/\alpha$.

Transmission conditions for perforated domains. Transmission problems have been studied also in many other contributions, see e.g. [1, 3, 9, 11]. The case with homogeneous Neumann boundary conditions at the inclusions was treated in [9]. The authors show that the perforation is invisible in the limit problem and provide rates of convergence. In [3], the problem is investigated with the periodic unfolding method. In [1], a coating of the inclusions with an absorbing material is introduced; this coating can lead to losses and to more complex impedance parameters Z. The geometry of our problem has been studied also in [11], where equations are formulated in the inclusions. Using an appropriate scaling with factors ε^{-1}, the authors obtain a non-trivial effective problem (jump conditions appear also at lowest order, whereas a jump condition appears in our setting only in the first order term). Related works are [12], where the problem is further analyzed, and [10], where an oscillatory (on small scales) *boundary* instead of an *interface* is studied. The transmission problem where the interface consists not of holes but of small inclusions of a second material is studied in [5], where also the asymptotic expansion of solutions is derived.

There are equations where order-1 effects are introduced by the perforation (even without an ε^{-1} boundary condition). An example is the Stokes flow in a perforated geometry, see [6, 14]. But even for the Helmholtz equation with a fixed frequency ω, order-1 effects are possible, namely in a Helmholtz resonator geometry. For a mathematical study of the Helmholtz resonator we refer to [15]. We emphasize that the lowest order effect of [15] is only possible by introducing *three* scales: The macroscopic scale (order 1, size of Ω), the microscopic scale ε (size of the resonator), and a sub-micro-scale which is small compared to ε (the diameter of a channel connecting the interior of the resonator to the exterior).

That effects of leading order can be created by small structures is also known from a related equation, namely the time homogeneous Maxwell equation (of which the Helmholtz equation is a special case): Using split-ring microscopic geometries, the effective behavior of solutions to Maxwell equations can be changed dramatically: Negative index materials with negative index of refraction can occur as homogenized materials, see [2, 8]. We note that in these works, again, three scales are used: Each microscopic element of size ε contains a substructure of a size that is small compared to ε (in this case: the diameter of the slit in the ring).

1.2 Mathematical setting and results

Let $\Omega \subset \mathbb{R}^3$ be a domain with Lipschitz boundary, containing the origin. We use the unit cell $Y := \left[-\frac{1}{2}, \frac{1}{2}\right)^2 \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ and the obstacle shape $\Sigma \subset Y$. We assume that Σ is a domain with Lipschitz boundary, which is strictly contained in Y, i.e. $\overline{\Sigma} \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^3$. To construct the obstacles in the complex geometry, we scale and shift the set Σ: We use $k \in \mathbb{Z}^2$ to label the different obstacles and set

$$Y_k^\varepsilon := \varepsilon \left(Y + (k_1, k_2, 0)\right) , \quad \Sigma_k^\varepsilon := \varepsilon \left(\Sigma + (k_1, k_2, 0)\right) \text{ for } k = (k_1, k_2) \in \mathbb{Z}^2 . \tag{1.3}$$

The indices of cells inside Ω are $I_\varepsilon := \{k \in \mathbb{Z}^2 | \, Y_k^\varepsilon \subset \Omega\}$. The number of elements of I_ε is of order ε^{-2}. We denote by $\Sigma^\varepsilon := \bigcup_{k \in I_\varepsilon} \Sigma_k^\varepsilon$ the union of all obstacles in Ω and define the perforated domain by setting $\Omega_\varepsilon := \Omega \setminus \Sigma^\varepsilon$.

We denote by n the outer normal of Ω_ε on $\partial\Omega_\varepsilon$. The perforation Σ^ε is located along the submanifold $\Gamma_0 := (\mathbb{R}^2 \times \{0\}) \cap \Omega$. The submanifold Γ_0 separates the domain Ω into two subdomains:

$$\Omega_+ := \left[\mathbb{R}^2 \times (0,\infty)\right] \cap \Omega \;\text{ and }\; \Omega_- := \left[\mathbb{R}^2 \times (-\infty,0)\right] \cap \Omega,$$

leading to the disjoint decomposition $\Omega = \Omega_+ \cup \Gamma_0 \cup \Omega_-$.

Our analysis concerns the following Helmholtz equation on Ω_ε:

$$\begin{aligned} -\Delta p^\varepsilon &= \omega^2 p^\varepsilon + f && \text{in } \Omega_\varepsilon, \\ \partial_n p^\varepsilon &= 0 && \text{on } \partial\Sigma^\varepsilon, \\ p^\varepsilon &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

In this equation, $f \in L^2(\Omega)$ is a given source term and the frequency $\omega > 0$ is a fixed parameter. The natural space of solutions of (1.4) is

$$\mathcal{H}_\varepsilon := \left\{ u \in H^1(\Omega_\varepsilon) \,|\, u|_{\partial\Omega} = 0 \right\} .$$

The weak formulation of (1.4) is: find $p^\varepsilon \in \mathcal{H}_\varepsilon$ such that

$$\int_{\Omega_\varepsilon} \nabla p^\varepsilon \cdot \nabla \varphi = \int_{\Omega_\varepsilon} \omega^2 p^\varepsilon \varphi + \int_{\Omega_\varepsilon} f\varphi \qquad \forall \varphi \in \mathcal{H}_\varepsilon \,. \tag{1.5}$$

We assume that ω^2 is not an eigenvalue of the operator $-\Delta$ to Dirichlet conditions on $\partial\Omega$, i.e. $\omega^2 \notin \sigma(-\Delta)$. In what follows, we denote by $\mathcal{P}_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Omega)$ the extension operator that continues every function by 0 to all of Ω.

Our first result characterizes limits p of solution sequences p^ε. We obtain that the perforation is invisible in the limit $\varepsilon \to 0$.

Theorem 1.1 (Limit behavior of solutions). *Let $f \in L^2(\Omega)$ be a source function and let $p^\varepsilon \in \mathcal{H}_\varepsilon$ be a sequence of weak solutions to* (1.4). *We assume $\omega^2 \notin \sigma(-\Delta)$.*

Effective system. *The norms $\|\mathcal{P}_\varepsilon p^\varepsilon\|_{L^2(\Omega)}$ and $\|\mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)}$ are bounded. There exists $p \in H^1_0(\Omega)$ such that $\mathcal{P}_\varepsilon p^\varepsilon \to p$ strongly in $L^2(\Omega)$ and $\mathcal{P}_\varepsilon \nabla p^\varepsilon \rightharpoonup \nabla p$ weakly in $L^2(\Omega)$. The limit p is the unique weak solution of*

$$-\Delta p = \omega^2 p + f \quad \text{in } \Omega \,. \tag{1.6}$$

Rate of convergence. *If f has the regularity $H^1 \cap C^0$ in an open neighborhood of Γ_0 and if $\partial\Omega$ is of class C^3 in a neighborhood of $\bar{\Gamma}_0 \cap \partial\Omega$, then there exists a constant $C = C(f) > 0$, independent of $\varepsilon > 0$, such that*

$$\|p - \mathcal{P}_\varepsilon p^\varepsilon\|_{L^2(\Omega)} + \|\nabla p - \mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)} + \|\Delta p - \mathcal{P}_\varepsilon \Delta p^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{1/2} \,. \tag{1.7}$$

In order to see the effect of the interface, we have to study the first order behavior of solutions. We consider solutions p^ε to the ε-problem (1.4) and the limit function $p \in H^1(\Omega)$ of Theorem 1.1. We define v^ε as the variation of order ε,

$$v^\varepsilon := \frac{p^\varepsilon - p}{\varepsilon} \;\text{ on }\; \Omega_\varepsilon \,. \tag{1.8}$$

In order to formulate our result, we will use some notation that is explained in more detail in Section 2: For $v \in W^{1,1}(\Omega \setminus \Gamma_0)$ with $\Delta v \in L^1(\Omega \setminus \Gamma_0)$, we denote by $[v]$ and $[\partial_\nu v]$ the jump of v and of its normal derivatives across Γ_0. In our setting, the normal vector is $\nu = e_3$. We denote by $\mathcal{H}^2$ the two dimensional Hausdorff measure.

Our second theorem provides a corrector result, i.e. formulas for limits of v^ε. This second theorem has the weakness that the a priori bounds on v^ε and the existence of a limit function v must be assumed. Furthermore, a characterization of the limit of gradients must be assumed. Both assumptions are collected in (1.9), for more comments on this assumption see Remark 1.5 below.

Theorem 1.2 (First order behavior). *Let the situation be as in Theorem 1.1, in particular, let $p^\varepsilon \in \mathcal{H}_\varepsilon$ is a sequence of weak solutions to* (1.4) *and let p be the solution of the effective system* (1.6). *Let the corrector v^ε be defined by* (1.8), $v^\varepsilon = (p^\varepsilon - p)/\varepsilon$. *Let f be of class $H^1 \cap C^0$ in an open neighborhood of Γ_0, and let $\partial\Omega$ be of class C^3. We assume that for some factor $\alpha \in C^0(\Gamma_0, \mathbb{R})$ the sequence v^ε satisfies the following: There exists a limit function $v \in W^{1,1}(\Omega \setminus \Gamma_0)$ such that, as $\varepsilon \to 0$,*

$$\mathcal{P}_\varepsilon v^\varepsilon \rightharpoonup v \text{ weakly in } L^1(\Omega) \quad \text{and} \quad \mathcal{P}_\varepsilon \nabla v^\varepsilon \rightharpoonup \nabla v + \alpha\,[v]\,\nu \mathcal{H}^2\lfloor_{\Gamma_0}\,, \tag{1.9}$$

the latter weakly in the sense of measures on Ω. Under these assumptions, the limit v is determined by the following system of equations:

$$\begin{aligned} -\Delta v &= \omega^2 v && \text{in } \Omega \setminus \Gamma_0\,,\\ \alpha\,[v] &= |\Sigma|\,\partial_\nu p && \text{on } \Gamma_0\,,\\ [\partial_\nu v] &= -\,|\Sigma|\,\partial_\nu^2 p && \text{on } \Gamma_0\,. \end{aligned} \tag{1.10}$$

Let us provide some further remarks on the two theorems.

Remark 1.3 (Well-defined expressions in the limit system of Theorem 1.2). The regularity property $v \in W^{1,1}(\Omega \setminus \Gamma_0)$ implies that the jump $[v]$ is well-defined on Γ_0 in the sense of traces in $L^1(\Gamma_0)$. Moreover, relation $(1.10)_1$ implies that Δv is an L^1-function on both sides of Γ_0, hence the jump $[\partial_\nu v]$ is well-defined on Γ_0 as a distribution. Since f is of class $H^1(\Omega)$, the solution p of the Helmholtz equation in Ω is of class $H^3(\Omega)$. This implies that the right hand side of $(1.10)_{2,3}$ is well defined in the sense of traces.

Remark 1.4 (Rate of convergence). Let us try to depict the microscopic situation in the vicinity of one obstacle. The function p is a smooth function: the gradient ∇p is essentially constant in an ε-neighborhood of the single obstacle. In contrast to that, the function p^ε sees the obstacle: the gradient ∇p^ε always has a vanishing normal component at the boundary of the obstacles due to the homogeneous Neumann condition. This implies that the values of ∇p^ε have variations of order 1 in the vicinity of the obstacle. In turn, the gradients ∇p^ε and ∇p necessarily differ by the order 1 in the neighborhood of the perforation.

This picture helps to develop an idea about the rates of convergence that can be expected. If we calculate the L^2-norm of the difference of the gradients, already an ε-layer around the obstacles (with volume of order ε) induces a contribution of order $\|\nabla p - \nabla p^\varepsilon\|_{L^2(\Omega_\varepsilon)} \gtrsim (1^2 \cdot \varepsilon)^{1/2} = \varepsilon^{1/2}$. This order of convergence is consistent with (1.7). In particular, we can expect that the rate $\varepsilon^{1/2}$ is the optimal rate of convergence for gradients in the L^2-norm.

On the other hand, the situation changes if we consider the L^1-norm. The contribution of an ε-layer around the obstacles is now $\|\nabla p - \nabla p^\varepsilon\|_{L^1(\Omega_\varepsilon)} \gtrsim (1^1 \cdot \varepsilon)^{1/1} = \varepsilon$. We can therefore hope that this error is of order $O(\varepsilon)$ when the L^1-metric is used. This order of convergence is consistent with our assumption (1.9): If ∇v^ε is bounded in L^1, we can select a subsequence which converges in the sense of measures.

Remark 1.5 (On assumption (1.9)). The assumption essentially contains two points: (i) The boundedness of the sequence v^ε in $W^{1,1}$. (ii) The characterization of the factor α (possibly with a cell-problem).

Let us assume that the boundedness of v^ε in the space $W^{1,1}(\Omega_\varepsilon)$ can be shown, i.e. $\|v^\varepsilon\|_{W^{1,1}(\Omega_\varepsilon)} \leq C$. This estimate implies that the trivial extension of the gradient is bounded in L^1, $w^\varepsilon := \mathcal{P}_\varepsilon \nabla v^\varepsilon$ satisfies $\|w^\varepsilon\|_{L^1(\Omega)} \leq C$. This boundedness implies that we can select a subsequence $\varepsilon \to 0$ and a limit measure $\mu \in \mathcal{M}(\Omega)$ with $\mathcal{P}_\varepsilon \nabla v^\varepsilon = w^\varepsilon \rightharpoonup \mu$ in the sense of measures as $\varepsilon \to 0$. We can restrict all the further considerations to this subsequence.

For an arbitrary subdomain $\tilde{\Omega} \subset\subset \Omega \setminus \Gamma_0$ we can exploit the fact that the embedding $W^{1,1}(\tilde{\Omega}) \subset L^1(\tilde{\Omega})$ is compact. This implies that v^ε converges on $\tilde{\Omega}$ strongly in L^1 to a function v. Since a family of sets $\tilde{\Omega}$ can be chosen to cover all of $\Omega \setminus \Gamma_0$ (and we can continue to take subsequences), the limit function v is defined on $\Omega \setminus \Gamma_0$. Furthermore, since the bound $\|v^\varepsilon\|_{W^{1,1}(\tilde{\Omega})} \leq C$ is satisfied independent of $\tilde{\Omega}$, the limit function satisfies $v \in W^{1,1}(\Omega \setminus \Gamma_0) \subset L^1(\Omega)$. This implies the first part of assumption (1.9).

Again restricting ourselfs to an arbitrary subset $\tilde{\Omega} \subset\subset \Omega \setminus \Gamma_0$, we note that $\nabla v^\varepsilon \to \nabla v$ holds in the sense of distributions on $\tilde{\Omega}$ since $v^\varepsilon \to v$ holds in L^1. On the other hand, we have $\nabla v^\varepsilon \to \mu$ in the sense of measures on $\tilde{\Omega}$. This implies the characterization $\mu = \nabla v$ on $\Omega \setminus \Gamma_0$. At this point we have verified for the second part of (1.9) that $\mathcal{P}_\varepsilon \nabla v^\varepsilon \rightharpoonup \mu$ with $\mu = \nabla v$ on $\Omega \setminus \Gamma_0$.

In order to verify (1.9), it remains to characterize the singular part of μ as $\alpha\, [v]\, \nu \mathcal{H}^2$. Let us assume that v^ε can be extended across the obstacles to a function $\tilde{v^\varepsilon}$ with $\|\tilde{v^\varepsilon}\|_{W^{1,1}(\Omega)} \leq C$. Such an extension is known to exist in L^2-based Sobolev spaces (see (2.3)), but we are not aware of a reference in L^1 (nevertheless, we expect the result to be true). Since $W^{1,1}(\Omega)$ is a subset of the space of functions with bounded variation, $W^{1,1}(\Omega) \subset BV(\Omega)$ with continuous embedding, the sequence $\tilde{v^\varepsilon}$ is a bounded sequence in $BV(\Omega)$. Compactness in $BV(\Omega)$ implies that there is a subsequence and a limit function $\tilde{v} \in BV(\Omega)$ such that $\tilde{v^\varepsilon} \rightharpoonup \tilde{v}$ in $BV(\Omega)$. In particular, we have $\tilde{v^\varepsilon} \to \tilde{v}$ in $L^1(\Omega)$, hence $\tilde{v} = v$. Furthermore, the gradients $\nabla \tilde{v^\varepsilon}$ converge in the sense of measures to the measure-valued gradient of v, which is $\nabla \tilde{v^\varepsilon} \rightharpoonup \nabla v|_{\Omega \setminus \Gamma_0} + [v]\, \nu \mathcal{H}^2$ in the sense of measures. This fact seems to suggest that the singular part of the measure μ is given by $[v]\, \nu \mathcal{H}^2$, but we have to take into account the error $\mathcal{P}_\varepsilon \nabla v^\varepsilon - \nabla \tilde{v^\varepsilon} \neq 0$.

We have introduced the volume factor $\alpha > 0$ in order to capture the corresponding error, i.e. the measure valued limit of $\nabla \tilde{v^\varepsilon}|_{\Sigma^\varepsilon}$ on the obstacles Σ^ε. We note that we can expect this contribution to be small for small obstacles, we therefore expect $\alpha < 1$ to be close to 1 for small obstacles.

Methods of proof. Astonishingly, our proofs do not use any of the typical homogenization tools, such as two-scale convergence, periodic unfolding, or compensated compactness (while in [3] periodic unfolding is used). This seems to be a special feature of the transmission problem (sometimes also called the "sieve-problem"): The behavior of the solution is very regular except for a lower dimensional manifold.

The only homogenization tool that we use is the extension operator $\tilde{\mathcal{P}}_\varepsilon$, which extends $H^1(\Omega_\varepsilon)$-functions to functions of the same class in all of Ω. Otherwise, only elementary calculations are performed (integration by parts, cut-off functions, dominated convergence). An interesting method of proof is used in the derivation of a priori estimates and convergence rates: We argue by contradiction and exploit compactness arguments, similar to the more intricate reasoning in [2] or [8].

2 Notation and preliminaries

For $Q \subset \mathbb{R}^3$, we write $L^2(Q)$ for the space of square integrable functions over Q and $H^k(Q) = W^{k,2}(Q)$ for the Bessel-potential spaces. We further denote $H^k_0(Q)$ the closure of $C^k_c(Q)$ in $H^k(Q)$. For a measurable domain $Q \subset \mathbb{R}^3$ of finite measure and $g \in L^1(Q)$, we write $\fint_Q g := |Q|^{-1} \int_Q g$ for the average of g over Q.

With $\Omega \subset \mathbb{R}^3$ and Γ_0 as in the introduction, we note that the hypersurface Γ_0 cuts Ω into the two subdomains $\Omega_\pm := \{x \in \Omega \mid \pm x_3 > 0\}$. For $p \in W^{1,1}(\Omega \setminus \Gamma_0)$, we denote by $p^\pm$ the trace of $p|_{\Omega_\pm}$ on Γ_0, respectively. Furthermore, if $\Delta p \in L^1(\Omega \setminus \Gamma_0)$, we use

$$\partial^\pm_\nu p := \nabla p^\pm \cdot \nu \,,$$

where $\nu = e_3$ is the outer normal of Ω_- on Γ_0. The jumps of p and ∇p are introduced as

$$\begin{aligned} [p] &:= p^+ - p^- \,, \\ [\partial_\nu p] &:= \partial^+_\nu p - \partial^-_\nu p \,. \end{aligned}$$

Note that $p \in H^1(\Omega \setminus \Gamma_0)$ together with $[p] = 0$ is equivalent to $p \in H^1(\Omega)$. This leads to the following observation:

Remark 2.1. Let $p \in H^1(\Omega \setminus \Gamma_0)$ and $f \in L^2(\Omega)$. The partial differential equation

$$-\Delta p = \omega^2 p + f \text{ in } \Omega \tag{2.1}$$

is equivalent to the system

$$\begin{aligned} -\Delta p &= \omega^2 p + f && \text{in } \Omega \setminus \Gamma_0 \,, \\ [p] &= 0 && \text{on } \Gamma_0 \,, \\ [\partial_\nu p] &= 0 && \text{on } \Gamma_0 \,. \end{aligned} \tag{2.2}$$

Both equations (2.1) and $(2.2)_1$ are understood in the sense of distributions or, equivalently, in the weak sense. We emphasize that $(2.2)_1$ guarantees $\Delta p \in L^2(\Omega_\pm)$, hence $[\partial_\nu p]$ is well defined.

In the proofs of our main theorems, we are dealing with sequences $p^\varepsilon \in \mathcal{H}_\varepsilon = \{u \in H^1(\Omega_\varepsilon) \,|\, u|_{\partial\Omega} = 0\}$. Since these functions are defined on Ω_ε and not on Ω, we need suitable extension operators. The most elementary operator is the extension by 0, which we denote as $\mathcal{P}_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Omega)$. Furthermore, it is well known, that there exists a family of extension operators $\tilde{\mathcal{P}}_\varepsilon : H^1(\Omega_\varepsilon) \to H^1(\Omega)$, such that

$$\left\| \tilde{\mathcal{P}}_\varepsilon p^\varepsilon \right\|_{H^1(\Omega)} \leq C \left\| p^\varepsilon \right\|_{H^1(\Omega_\varepsilon)} \tag{2.3}$$

for some $C > 0$ independent of ε ([4], Chapter 1). Essentially, $\tilde{\mathcal{P}}_\varepsilon$ is defined by using in each obstacle the harmonic extension of the boundary values.

The subsequent elementary lemma will turn out to be useful in the proofs. Note that the assumptions of the lemma are not yet checked for solution sequences p^ε.

Lemma 2.2 (A compactness criterion in perforated domains)**.** *Let $p^\varepsilon \in H^1(\Omega_\varepsilon)$ satisfy the a priori estimate $\|\mathcal{P}_\varepsilon p^\varepsilon\|_{L^2(\Omega)} + \|\mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)} \leq C$ for every $\varepsilon > 0$. Then, there exists $p \in H^1(\Omega)$ and a subsequence $\varepsilon \to 0$ such that $\mathcal{P}_\varepsilon p^\varepsilon \to p$ strongly in $L^2(\Omega)$, $\mathcal{P}_\varepsilon \nabla p^\varepsilon \rightharpoonup \nabla p$ weakly in $L^2(\Omega)$ and $\tilde{\mathcal{P}}_\varepsilon p^\varepsilon \rightharpoonup p$ weakly in $H^1(\Omega)$. Furthermore, if $p^\varepsilon|_{\partial\Omega} = 0$ holds for every $\varepsilon > 0$, then also $p|_{\partial\Omega} = 0$.*

Proof. In what follows, we successively pass to subsequences of p^ε, keeping the notation p^ε for each subsequence. Since $\|\mathcal{P}_\varepsilon p^\varepsilon\|_{L^2(\Omega)} + \|\mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)} \leq C$, upon changing the constant, there also holds $\left\| \tilde{\mathcal{P}}_\varepsilon p^\varepsilon \right\|_{H^1(\Omega)} \leq C$. Thus, there is $p \in H^1(\Omega)$ such that $\tilde{\mathcal{P}}_\varepsilon p^\varepsilon \rightharpoonup p$ weakly in $H^1(\Omega)$ and $\tilde{\mathcal{P}}_\varepsilon p^\varepsilon \to p$ strongly in $L^2(\Omega)$. By the trace theorem, the condition $p^\varepsilon|_{\partial\Omega} = 0$ for all $\varepsilon > 0$ implies $p|_{\partial\Omega} = 0$.

For $\delta > 0$ let $\phi_\delta \in L^\infty(\mathbb{R})$ be the indicator function $\phi_\delta(z) = 1$ for $|z| < \delta$ and $\phi_\delta(z) = 0$ for $|z| \geq \delta$. We set $\varphi_\delta : \Omega \to \mathbb{R}$, $\varphi_\delta(x) := \phi_\delta(x_3)$ and obtain for $\varepsilon < \delta$:

$$\begin{aligned} \limsup_{\varepsilon \to 0} \int_\Omega \left| \mathcal{P}_\varepsilon p^\varepsilon - \tilde{\mathcal{P}}_\varepsilon p^\varepsilon \right|^2 &= \limsup_{\varepsilon \to 0} \int_{\Sigma^\varepsilon} \left| \tilde{\mathcal{P}}_\varepsilon p^\varepsilon \right|^2 \leq \limsup_{\varepsilon \to 0} \int_\Omega \left| \tilde{\mathcal{P}}_\varepsilon p^\varepsilon \right|^2 \varphi_\delta^2 \\ &= \limsup_{\varepsilon \to 0} \left\| \varphi_\delta \tilde{\mathcal{P}}_\varepsilon p^\varepsilon \right\|_{L^2(\Omega)}^2 = \|\varphi_\delta p\|_{L^2(\Omega)}^2 . \end{aligned}$$

The last limit follows from the strong convergence $\varphi_\delta \tilde{\mathcal{P}}_\varepsilon p^\varepsilon \to \varphi_\delta p$ in $L^2(\Omega)$. Since $\delta > 0$ was arbitrary, the right hand side is arbitrarily small. We conclude that $\mathcal{P}_\varepsilon p^\varepsilon \to p$ converges strongly in $L^2(\Omega)$.

Similarly, we obtain for every $\psi \in L^2(\Omega; \mathbb{R}^3)$:

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_\Omega \mathcal{P}_\varepsilon \nabla p^\varepsilon \cdot \psi &= \lim_{\varepsilon \to 0} \int_\Omega \mathcal{P}_\varepsilon \nabla p^\varepsilon \cdot \psi(1 - \varphi_\delta) + \lim_{\varepsilon \to 0} \int_\Omega \mathcal{P}_\varepsilon \nabla p^\varepsilon \cdot \psi \varphi_\delta \\ &= \int_\Omega \nabla p \cdot \psi(1 - \varphi_\delta) + \lim_{\varepsilon \to 0} \int_\Omega \mathcal{P}_\varepsilon \nabla p^\varepsilon \cdot \psi \varphi_\delta . \end{aligned}$$

Since $\limsup_{\varepsilon \to 0} \left| \int_\Omega \mathcal{P}_\varepsilon \nabla p^\varepsilon \cdot \psi \varphi_\delta \right| \leq \limsup_{\varepsilon \to 0} \|\mathcal{P}_\varepsilon \nabla p^\varepsilon\|_{L^2(\Omega)} \|\psi \varphi_\delta\|_{L^2(\Omega)} \to 0$ as $\delta \to 0$, we obtain $\mathcal{P}_\varepsilon \nabla p^\varepsilon \rightharpoonup \nabla p$ weakly in $L^2(\Omega)$. □

3 Limit of p^ε

Proof of Theorem 1.1. We will prove Theorem 1.1 in three steps: In Step 1, we prove the homogenization result under the assumption that $\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ is bounded. In Step 2, we use Step 1 to prove boundedness of $\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ by a contradiction argument. In Step 3, we prove the convergence rates (1.7).

Step 1: Limit behavior of p^ε. We assume here that $\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ is bounded. We use p^ε as a test function in (1.5) and obtain

$$\|\nabla p^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le \|p^\varepsilon\|_{L^2(\Omega_\varepsilon)} \left(\omega^2 \|p^\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\right) , \tag{3.1}$$

which implies boundedness of $\|\nabla p^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}$.

From the estimates for $\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ and $\|\nabla p^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}$ and Lemma 2.2, we conclude the existence of $p \in H^1_0(\Omega)$ such that $\mathcal{P}_\varepsilon p^\varepsilon \to p$ strongly in $L^2(\Omega)$ and $\mathcal{P}_\varepsilon \nabla p^\varepsilon \rightharpoonup \nabla p$ weakly in $L^2(\Omega)$ along a subsequence. We choose a test function $\varphi \in \mathcal{C}^\infty_c(\Omega)$, and obtain from (1.5) and Lemma 2.2

$$\begin{aligned}\int_\Omega \nabla p \cdot \nabla\varphi &= \lim_{\varepsilon\to 0} \int_\Omega \mathcal{P}_\varepsilon \nabla p^\varepsilon \cdot \nabla\varphi \\ &= \lim_{\varepsilon\to 0} \int_\Omega \omega^2 \mathcal{P}_\varepsilon p^\varepsilon \varphi + \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f\varphi = \int_\Omega \omega^2 p\,\varphi + \int_\Omega f\varphi\,.\end{aligned} \tag{3.2}$$

This provides (1.6) and hence the homogenization result under the assumption of boundedness. We note that the above calculations also hold if in (1.5), f is replaced by a sequence $(f_\varepsilon)_{\varepsilon>0}$ with $f_\varepsilon \to f$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0$.

Step 2: $L^2(\Omega)$-boundedness of p^ε. Let us assume for a contradiction argument that the sequence $\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ is not bounded. For every $\varepsilon > 0$, we define rescaled quantities by setting

$$\tilde p^\varepsilon := \frac{p^\varepsilon}{\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}} \text{ in } \Omega_\varepsilon \text{ and } \tilde f^\varepsilon := \frac{f}{\|p^\varepsilon\|_{L^2(\Omega_\varepsilon)}} \text{ in } \Omega\,. \tag{3.3}$$

We achieve $\|\tilde p^\varepsilon\|_{L^2(\Omega_\varepsilon)} = 1$ for every $\varepsilon > 0$ and $\left\|\tilde f^\varepsilon\right\|_{L^2(\Omega)} \to 0$ for $\varepsilon \to 0$. Since p^ε solves (1.4), we conclude that $\tilde p^\varepsilon$ solves

$$\begin{aligned} -\Delta\tilde p^\varepsilon &= \omega^2\tilde p^\varepsilon + \tilde f^\varepsilon && \text{in } \Omega_\varepsilon, \\ \partial_n \tilde p^\varepsilon &= 0 && \text{on } \partial\Sigma^\varepsilon\,. \end{aligned} \tag{3.4}$$

Since $\|\tilde p^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ is bounded, we can apply Step 1 and obtain the existence of $\tilde p \in H^1_0(\Omega)$ such that $\mathcal{P}_\varepsilon \tilde p^\varepsilon \to \tilde p$ strongly in $L^2(\Omega)$ and $\mathcal{P}_\varepsilon \nabla\tilde p^\varepsilon \rightharpoonup \nabla\tilde p$ weakly in $L^2(\Omega)$, where $\tilde p$ solves

$$-\Delta\tilde p = \omega^2\tilde p \text{ in } \Omega. \tag{3.5}$$

Since $\tilde p \in H^1_0(\Omega)$ solves (3.5) and ω^2 is not an eigenvalue of $-\Delta$ on Ω, we conclude $\tilde p = 0$. We obtain the desired contradiction between the strong convergence $\mathcal{P}_\varepsilon \tilde p^\varepsilon \to 0$ in $L^2(\Omega)$ and $\|\mathcal{P}_\varepsilon \tilde p^\varepsilon\|_{L^2(\Omega)} = 1$ for every $\varepsilon > 0$.

Step 3: Rate of convergence. It remains to prove (1.7). For a contradiction argument, let us assume $\varepsilon^{-1/2} \|\mathcal{P}_\varepsilon p^\varepsilon - p\|_{L^2(\Omega)} \to \infty$, which also implies $G_\varepsilon := \varepsilon^{-1/2} \left\|\tilde{\mathcal{P}}_\varepsilon p^\varepsilon - p\right\|_{L^2(\Omega)} \to \infty$ by the uniform boundedness of p in Σ^ε. We study the sequence of functions $w^\varepsilon := G_\varepsilon^{-1}\varepsilon^{-1/2}(\tilde{\mathcal{P}}_\varepsilon p^\varepsilon - p)$ with $\|w^\varepsilon\|_{L^2(\Omega)} = 1$, satisfying

$$\begin{aligned} -\Delta w^\varepsilon &= \omega^2 w^\varepsilon && \text{in } \Omega_\varepsilon, \\ \partial_n w^\varepsilon &= -G_\varepsilon^{-1}\varepsilon^{-\frac{1}{2}}\partial_n p && \text{on } \partial\Sigma^\varepsilon, \\ w^\varepsilon &= 0 && \text{on } \partial\Omega, \end{aligned}$$

with the weak formulation

$$\int_{\Omega_\varepsilon} \nabla w^\varepsilon \cdot \nabla\varphi = -\int_{\partial\Omega_\varepsilon} G_\varepsilon^{-1}\varepsilon^{-\frac{1}{2}}\partial_n p\,\varphi\,d\mathcal{H}^2 + \int_{\Omega_\varepsilon} \omega^2 w^\varepsilon\varphi \qquad \forall\varphi \in H_0^1(\Omega)\,. \tag{3.6}$$

Due to our assumptions on Ω and f, the functions Δp and ∇p are of class C^0 and bounded in an open neighborhood of Γ_0. This allows to estimate the boundary integral as

$$\begin{aligned} \left|\int_{\partial\Omega_\varepsilon} \varepsilon^{-\frac{1}{2}}\partial_n p\,\varphi\,d\mathcal{H}^2\right| &= \left|\sum_{k\in I_\varepsilon}\int_{\partial\Sigma_k^\varepsilon} \varepsilon^{-\frac{1}{2}}\partial_n p\,\varphi\,d\mathcal{H}^2\right| = \left|\sum_{k\in I_\varepsilon}\int_{\Sigma_k^\varepsilon} \varepsilon^{-\frac{1}{2}}\left(-\Delta p\varphi - \nabla p\cdot\nabla\varphi\right)\right| \\ &\le \varepsilon^{-\frac{1}{2}}\,\||\Delta p| + |\nabla p|\|_{L^2(\Sigma^\varepsilon)}\cdot\||\varphi| + |\nabla\varphi|\|_{L^2(\Sigma^\varepsilon)} \le C\,\|\varphi\|_{H^1(\Sigma^\varepsilon)}\,. \end{aligned} \tag{3.7}$$

Using $\varphi = w^\varepsilon$ as a test function in (3.6), exploiting $\|\nabla w^\varepsilon\|_{L^2(\Sigma^\varepsilon)} \le C\,\|\nabla w^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ from (2.3), we obtain

$$\int_{\Omega_\varepsilon} |\nabla w^\varepsilon|^2 \le CG_\varepsilon^{-1}\,\|w^\varepsilon\|_{H^1(\Omega_\varepsilon)} + \omega^2\,\|w^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2\,, \tag{3.8}$$

and thus the boundedness of w^ε in $H^1(\Omega_\varepsilon)$. From the construction of w^ε and Lemma 2.2 we conclude that, for a limit function $w \in H_0^1(\Omega)$ and a subsequence, there holds $\mathcal{P}_\varepsilon\left(w^\varepsilon|_{\Omega_\varepsilon}\right) \to w$ strongly in $L^2(\Omega)$ and $\mathcal{P}_\varepsilon\left(\nabla w^\varepsilon|_{\Omega_\varepsilon}\right) \rightharpoonup \nabla w$ weakly in $L^2(\Omega)$ and $w^\varepsilon \to w$ strongly in $L^2(\Omega)$.

Since $G_\varepsilon^{-1} \to 0$ as $\varepsilon \to 0$, (3.6) yields the following limit equation for w:

$$\int_\Omega \nabla w\cdot\nabla\varphi = \int_\Omega \omega^2 w\varphi \qquad \forall\varphi \in H_0^1(\Omega)\,.$$

Since ω^2 is not an eigenvalue of $-\Delta$, we find $w = 0$. We obtain the desired contradiction, since the strong convergence of w^ε to 0 contradicts the normalization $\|w^\varepsilon\|_{L^2(\Omega)} = 1$.

With this contradiction to the assumption $G_\varepsilon \to \infty$, we have $\|p - \mathcal{P}_\varepsilon p^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}}$. Estimate (3.8) is valid in general and provides the estimate with improved regularity: boundedness of ∇w^ε in $L^2(\Omega_\varepsilon)$ and thus $\|\nabla p - \mathcal{P}_\varepsilon\nabla p^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}}$. The estimate $\|\Delta p - \mathcal{P}_\varepsilon\Delta p^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{\frac{1}{2}}$ follows from the Helmholtz equations $(1.4)_1$ and (1.6). □

4 First order behavior

Proof of Theorem 1.2. We prove the theorem in three steps. In Step 1, we reduce the proof of the statement to the convergence behavior of a boundary integral. In Step 2, we prove the convergence of this boundary integral. In Step 3, we show that the weak limit problem is equivalent to the distributional formulation of (1.10).

Step 1: Reduction to one boundary integral. Our aim is to analyze the first order corrector function $v^\varepsilon := \varepsilon^{-1}(p^\varepsilon - p)$. The function v^ε solves the following Helmholtz equation:

$$\begin{aligned} -\Delta v^\varepsilon &= \omega^2 v^\varepsilon && \text{in } \Omega_\varepsilon\,, \\ \partial_n v^\varepsilon &= -\frac{1}{\varepsilon}\partial_n p && \text{on } \partial\Sigma^\varepsilon\,, \\ v^\varepsilon &= 0 && \text{on } \partial\Omega\,. \end{aligned} \tag{4.1}$$

System (4.1) has the following weak formulation: $v^\varepsilon \in H^1(\Omega_\varepsilon)$ satisfies $v^\varepsilon|_{\partial\Omega} = 0$ and

$$\int_{\Omega_\varepsilon} \nabla v^\varepsilon \cdot \nabla\varphi = -\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon}\partial_n p\,\varphi\, d\mathcal{H}^2 + \int_{\Omega_\varepsilon} \omega^2 v^\varepsilon \varphi \qquad \forall\varphi \in H^1_0(\Omega)\,. \tag{4.2}$$

Our aim is to analyze the limit $\varepsilon \to 0$ in relation (4.2). On the right hand side, we use assumption (1.9), which contains the weak L^1-convergence of $\mathcal{P}_\varepsilon v^\varepsilon \rightharpoonup v$ and hence the convergence of the bulk integral. Also on the left hand side, we use assumption (1.9), but now the measure convergence $\mathcal{P}_\varepsilon \nabla v^\varepsilon \rightharpoonup \nabla v + \alpha\,[v]\,\nu\mathcal{H}^2\lfloor_{\Gamma_0}$. For smooth test-functions φ we obtain

$$\int_\Omega \nabla v\cdot\nabla\varphi + \int_{\Gamma_0} \alpha[v]\nu\cdot\nabla\varphi\, d\mathcal{H}^2 = -\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon}\partial_n p\,\varphi\; d\mathcal{H}^2 + \int_\Omega \omega^2 v\varphi \quad \forall\varphi \in C^\infty_c(\Omega)\,. \tag{4.3}$$

The main step of the proof is therefore to determine the limit of the boundary integral. We will derive in Step 2

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon}\partial_n p\,\varphi\, d\mathcal{H}^2 = -\,|\Sigma| \int_{\Gamma_0} \left(\partial^2_\nu p\,\varphi + \partial_\nu p\,\partial_\nu\varphi\right)\, d\mathcal{H}^2 \quad \forall\varphi \in C^\infty_c(\Omega)\,. \tag{4.4}$$

Inserting the characterization (4.4) in (4.3) will provide the limit system for v.

Step 2: Proof of (4.4)**.** Let $\varphi \in C^\infty_c(\Omega)$ be a test function. We consider the contribution of the boundary integral for every obstacle; we recall that the obstacles are numbered with indices $k \in I_\varepsilon$. The contribution of the boundary integral in the cell with number k is denoted by $F^\varepsilon(k)$, i.e.: For every $k \in I_\varepsilon$, we set

$$F^\varepsilon(k) := \varepsilon^{-2}\int_{\partial\Sigma^\varepsilon_k} \frac{1}{\varepsilon}\partial_n p\,\varphi\, d\mathcal{H}^2\,.$$

An integration by parts can be used to evaluate $F^\varepsilon(k)$ as

$$F^\varepsilon(k) = \varepsilon^{-2}\int_{\Sigma^\varepsilon_k} \frac{-1}{\varepsilon}\left(\Delta p\,\varphi + \nabla p\cdot\nabla\varphi\right) = -\,|\Sigma| \fint_{\Sigma^\varepsilon_k} \left(\varphi\Delta p + \nabla p\cdot\nabla\varphi\right)\,,$$

where we have used that n is the inner normal of Σ_k^ε and that the measure of obstacle k is $|\Sigma_k^\varepsilon| = \varepsilon^3 |\Sigma|$ for every $k \in I_\varepsilon$.

Our next aim is to construct out of the sequence $(F^\varepsilon(k))_{k\in I_\varepsilon}$ a function that lives on the interface Γ_0. To this end, let $\varepsilon > 0$ be fixed and let $y \in \Gamma_0$ be any point on the interface. We define the index $k(y,\varepsilon) \in \mathbb{Z}^2$ to be that index such that $y \in Y^\varepsilon_{k(y,\varepsilon)}$. This index is well-defined for almost every $y \in \Gamma_0$.

The elliptic equation $-\Delta p = \omega^2 p + f$ and our regularity assumptions imply that the functions ∇p and Δp are of class C^0 in a neighborhood of Γ_0. This allows to calculate, for almost every point $y \in \Gamma_0$, the limit of the above functions:

$$
\begin{aligned}
F(y) &:= \lim_{\varepsilon\to 0} F^\varepsilon(k(y,\varepsilon)) \\
&= -\lim_{\varepsilon\to 0} |\Sigma| \fint_{\Sigma^\varepsilon_{k(y,\varepsilon)}} (\Delta p\, \varphi + \nabla p \cdot \nabla \varphi) = -|\Sigma| \left(\Delta p\, \varphi + \nabla p \cdot \nabla\varphi\right)(y) \,.
\end{aligned} \tag{4.5}
$$

We now want to conclude from this point-wise convergence a convergence for integrals, more precisely, the convergence $\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon}\partial_n p(y)\, \varphi(y) d\mathcal{H}^2(y) \to \int_{\Gamma_0} F(y) d\mathcal{H}^2(y)$ as $\varepsilon \to 0$. Since the interface area in the single cell is $|Y_k^\varepsilon \cap \Gamma_0|_{\mathcal{H}^2} = \varepsilon^2$ for every $k \in I_\varepsilon$, we obtain

$$
\begin{aligned}
\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon}\partial_n p\, \varphi\, d\mathcal{H}^2 &= \sum_{k\in I_\varepsilon} \int_{\partial\Sigma_k^\varepsilon} \frac{1}{\varepsilon}\partial_n p\, \varphi\, d\mathcal{H}^2 \\
&= \sum_{k\in I_\varepsilon} F^\varepsilon(k)\, |Y_k^\varepsilon \cap \Gamma_0|_{\mathcal{H}^2} = \int_{\Gamma_0} F^\varepsilon(k(y,\varepsilon))\, d\mathcal{H}^2(y) \,.
\end{aligned} \tag{4.6}
$$

By definition of F in $(4.5)_1$, we have the pointwise convergence $F^\varepsilon(k(y,\varepsilon)) \to F(y)$. Since ∇p and Δp are bounded in a neighborhood of Γ_0, the family $F^\varepsilon(k)$ is uniformly bounded. We can therefore apply Lebesgue's dominated convergence theorem and obtain, in the limit $\varepsilon \to 0$,

$$
\int_{\partial\Omega_\varepsilon} \frac{1}{\varepsilon}\partial_n p\, \varphi\, d\mathcal{H}^2 \to \int_{\Gamma_0} F\, d\mathcal{H}^2 = -|\Sigma| \int_{\Gamma_0} \left(\Delta p\, \varphi + \nabla p\, \cdot \nabla\varphi\right)\, d\mathcal{H}^2 \,. \tag{4.7}
$$

Since $\varphi|_{\Gamma_0} \in C_c^\infty(\Gamma_0)$, we may integrate by parts in the last expression with respect to the tangential coordinates x_1 and x_2, with vanishing boundary integrals. We obtain

$$
\int_{\Gamma_0} F\, d\mathcal{H}^2 = -\int_{\Gamma_0} |\Sigma| \left(\partial_3^2 p\, \varphi + \partial_3 p\, \partial_3\varphi\right)\, d\mathcal{H}^2 \,. \tag{4.8}
$$

Because of $e_3 = \nu$, we have thus obtained (4.4).

Step 3: The limit equations. It remains to insert (4.4) into (4.3), which provides

$$
\int_\Omega \nabla v \cdot \nabla\varphi + \int_{\Gamma_0} \alpha[v]\partial_\nu\varphi\, d\mathcal{H}^2 = \int_{\Gamma_0} |\Sigma| \left(\partial_\nu^2 p\, \varphi + \partial_\nu p\, \partial_\nu\varphi\right)\, d\mathcal{H}^2 + \int_\Omega \omega^2 v\varphi \,.
$$

for every $\varphi \in C_c^\infty(\Omega)$. This relation is the weak formulation of (1.10), since a formal integration by parts yields

$$-\int_\Omega \Delta v\, \varphi - \int_{\Gamma_0} [\partial_\nu v]\varphi \, d\mathcal{H}^2 + \int_{\Gamma_0} \alpha[v]\partial_\nu \varphi \, d\mathcal{H}^2 \\ = \int_{\Gamma_0} |\Sigma| \left(\partial_\nu^2 p\, \varphi + \partial_\nu p\, \partial_\nu \varphi \right) \, d\mathcal{H}^2 + \int_\Omega \omega^2 v \varphi$$

for every smooth φ. Comparing the factors of φ in the bulk provides $-\Delta v = \omega^2 v$ (the equation thus holds rigorously in the sense of distributions in $\Omega \setminus \Gamma_0$). Comparing the factors of $\partial_\nu \varphi$ in boundary integrals provides $(1.10)_2$. Comparing the factors of φ in boundary integrals provides $(1.10)_3$. □

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References

[1] A. S. Bonnet-Ben Dhia, D. Drissi, and N. Gmati. Mathematical analysis of the acoustic diffraction by a muffler containing perforated ducts. *Math. Models Methods Appl. Sci.*, 15(7):1059–1090, 2005.

[2] G. Bouchitté and B. Schweizer. Homogenization of Maxwell's equations in a split ring geometry. *Multiscale Model. Simul.*, 8(3):717–750, 2010.

[3] D. Cioranescu, A. Damlamian, G. Griso, and D. Onofrei. The periodic unfolding method for perforated domains and Neumann sieve models. *J. Math. Pures Appl. (9)*, 89(3):248–277, 2008.

[4] D. Cioranescu and J. Saint Jean Paulin. *Homogenization of reticulated structures*, volume 136 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999.

[5] X. Claeys and B. Delourme. High order asymptotics for wave propagation across thin periodic interfaces. *Asymptot. Anal.*, 83(1-2):35–82, 2013.

[6] C. Conca. Étude d'un fluide traversant une paroi perforée. I. Comportement limite près de la paroi. *J. Math. Pures Appl. (9)*, 66(1):1–43, 1987.

[7] R. Kirby and A. Cummings. The impedance of perforated plates subjected to grazing gas flow and backed by porous media. *Journal of Sound and Vibration*, 217(4):619–636, 1998.

[8] A. Lamacz and B. Schweizer. Effective Maxwell equations in a geometry with flat rings of arbitrary shape. *SIAM J. Math. Anal.*, 45(3):1460–1494, 2013.

[9] M. Lobo, O. A. Oleinik, M. E. Perez, and T. A. Shaposhnikova. On homogenization of solutions of boundary value problems in domains, perforated along manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 25(3-4):611–629 (1998), 1997. Dedicated to Ennio De Giorgi.

[10] N. Neuss, M. Neuss-Radu, and A. Mikelić. Effective laws for the Poisson equation on domains with curved oscillating boundaries. *Appl. Anal.*, 85(5):479–502, 2006.

[11] M. Neuss-Radu and W. Jäger. Effective transmission conditions for reaction-diffusion processes in domains separated by an interface. *SIAM J. Math. Anal.*, 39(3):687–720 (electronic), 2007.

[12] M. Neuss-Radu, S. Ludwig, and W. Jäger. Multiscale analysis and simulation of a reaction-diffusion problem with transmission conditions. *Nonlinear Anal. Real World Appl.*, 11(6):4572–4585, 2010.

[13] E. Rohan and V. Lukeš. Homogenization of the acoustic transmission through a perforated layer. *J. Comput. Appl. Math.*, 234(6):1876–1885, 2010.

[14] J. Sanchez-Hubert and E. Sánchez-Palencia. Acoustic fluid flow through holes and permeability of perforated walls. *J. Math. Anal. Appl.*, 87(2):427–453, 1982.

[15] B. Schweizer. The low-frequency spectrum of small Helmholtz resonators. *Proc. A.*, 471(2174):20140339, 18, 2015.