A radiation box domain truncation scheme for the wave equation

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Abstract: We consider the wave equation in an unbounded domain and are interested in domain truncation methods. Our aim is to develop a numerical scheme that allows calculations for truncated waveguide geometries with periodic coefficient functions. The scheme is constructed with radiation boxes that are attached to the artificially introduced boundaries. A Dirichlet-to-Neumann operator N is calculated in these radiation boxes. Efficiency of the scheme is obtained by calculating N not with an iteration, but with a single run through the time interval. We observe speed-up factors of up to 20 in comparison to calculations without domain truncation.

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1. Introduction

Domain truncation methods are needed to calculate finite element solutions of partial differential equations on unbounded domains. We are interested here in the wave equation

$$\partial_t^2 \tilde{u} - \nabla \cdot (a\nabla)\tilde{u} = f\,, \tag{1.1}$$

posed on $\tilde{\Omega} \times (0,T)$, where $T > 0$ is a time horizon, $a = a(x)$ a positive coefficient field $a : \tilde{\Omega} \to \mathbb{R}$, and $\tilde{\Omega} \subset \mathbb{R}^n$ an unbounded domain. The aim is to truncate the domain and to replace $\tilde{\Omega}$ by a bounded subset $\Omega \subset \tilde{\Omega}$. The new unknown is a function $u : \Omega \times (0,T) \to \mathbb{R}$ on the smaller domain and we seek for a system of equations for u such that u agrees with $\tilde{u}$ on the bounded subset, $\tilde{u}|_{\Omega\times(0,T)} = u$. The new system of equations will use the wave equation on Ω, we demand $\partial_t^2 u - \nabla \cdot (a\nabla)u = f$ on $\Omega \times (0,T)$. The interest is to introduce some sort of boundary conditions for u such that the original solution $\tilde{u}$ is recovered on Ω.

Setting of the problem. Let us introduce a specific geometry. We consider a half waveguide geometry that extends unbounded in positive x_1-direction, and truncate the domain at $x_1 = 0$. The proposed method is very general and can also be applied,

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e.g., when the waveguide is unbounded in multiple directions, or when the geometry is locally perturbed. To simplify the notation, we consider here only $\tilde{\Omega} := (-L, \infty) \times \Gamma$ and the truncated domain $\Omega := (-L, 0) \times \Gamma$. The cross section $\Gamma \subset \mathbb{R}^{n-1}$ is a fixed bounded Lipschitz domain. The truncation introduces an artificial boundary, in our setting given by $\Gamma_0 = \{0\} \times \Gamma \subset \partial\Omega$. Our method applies also in the one-dimensional case, $n = 1$, in this case one should set $\Gamma := \{0\}$ and identify functions on Γ with real numbers. The length parameter $L > 0$ and the time parameter $T > 0$ are kept fixed throughout this work.

We are given a coefficient matrix $a : \tilde{\Omega} \to \mathbb{R}^{n\times n}$, $a = a(x)$ with $x \in \mathbb{R}^n$. We assume that a satisfies bounds $0 < \lambda \leq a(x) \leq \Lambda < \infty$ and that it is 1-periodic in x_1-direction for $x_1 > 0$, $a(x + e_1) = a(x)$ for every $x \in \tilde{\Omega}$ with $x_1 > 0$. For notational simplicity, we complement the wave equation (1.1) with trivial initial conditions $u_0 = u_1 = 0$ on $\tilde{\Omega}$ and with homogeneous Dirichlet boundary conditions on $\partial\tilde{\Omega}$. For $n > 1$ we note that $\partial\tilde{\Omega} = \{-L\} \times \Gamma \cup (-L, \infty) \times \partial\Gamma$. We always assume that f is compactly supported in $\Omega \times (0, T)$.

Main results. We discretize the time interval with $0 = t_0 < ... < t_m = T$ and use a finite element approximation $X_{\tilde{\Omega}} \subset H^1_0(\tilde{\Omega})$. After this discretization, in every time step, a finite dimensional, coercive, symmetric system must be solved.

Our aim is to perform calculations on a truncated domain. To this end, let $X_\Omega = \{u|_\Omega \,|\, u \in X_{\tilde{\Omega}}\}$ be the space of functions on the smaller domain Ω. The main result of this work is to introduce linear operators $N_k : X_\Omega \to X'_\Omega$ for $k \in \{1, ..., m-1\}$. These operators encode approximate transparent boundary conditions on $\Gamma_0 = \partial\Omega \cap \tilde{\Omega}$. The exact transparent boundary conditions are encoded with an operator family $\tilde{N} = (\tilde{N}_1, ..., \tilde{N}_{m-1})$, which is defined with the help of the unbounded domain $R := (0, \infty) \times \Gamma$, see (3.10). Our result is that the operator $N = (N_1, ..., N_{m-1})$, which is easy to calculate, is a good approximation of $\tilde{N}$, see Theorem 4.1.

The relevance of our result is closely related to the algorithmic aspect of how to calculate the approximate operators $N_k : X_\Omega \to X'_\Omega$. We use a boundary box $B := (0, 1) \times \Gamma$ and solve wave equations on B. For algorithmic complexity, it is desirable that we do not use an iteration to determine N. Instead, N is calculated by solving, for every discretization point on the left boundary, only one wave equation on B. The operators N can be calculated in a pre-processing step. Once calculated, we have an approximate transparent boundary condition in a periodic medium at our disposal. Since N is independent of f and of the coefficients a in Ω, the operators N can be used also for different right hand sides f, for different coefficients, and for different geometries (as long as the periodic geometry for $x_1 > 0$ is kept fixed).

With N at hand we seek, for each time step $j < m$, we seek a solution $u^{j+1} \in X_\Omega$ of

$$\int_\Omega \frac{u^{j+1}}{\Delta t^2}\, w + \theta \int_\Omega \nabla u^{j+1} \cdot a \nabla w + \langle N_1(u^{j+1}), w\rangle = \langle b_j, w\rangle \qquad \forall w \in X_\Omega\,, \tag{1.2}$$

compare (3.12). In this equation θ and $\Delta t = T/m$ are positive real numbers. We show that the left hand side, as a function of u^{j+1} and w, defines a coercive bilinear form on X_Ω, see Proposition 3.5. This makes (1.2) a linear problem that is easy to solve, analytically with the Lax-Milgram theorem. The right hand side b_j is calculated from three contributions: 1.) The right hand side f. 2.) The explicit terms in the time discretization scheme. 3.) The explicit contributions of N, namely $\sum_{i=0}^{j-2} N_{j-i}(u^{i+2})$.

1.1. **An informal description of Dirichlet-to-Neumann operators.** The principal idea of the truncation scheme is very direct and can easily be explained, at least on an informal level. Such a description is the aim of this subsection.

Let $\tilde{u}$ be the solution of (1.1) on $\tilde{\Omega}$. We write in the following $\Box\tilde{u} = f$ on $\tilde{\Omega}$, using the wave operator $\Box := \partial_t^2 - \nabla \cdot (a\nabla)$ for the x-dependent coefficients a. Our aim is to do calculations on the bounded domain Ω and to consider the wave equation $\Box u = f$ on Ω. This wave equation has to be complemented with a boundary condition on Γ_0. In the following we write $g := u|_{\Gamma_0}$ for the trace of a function u that is defined on Ω.

We define an operator N formally as follows: Given a function $g : \Gamma_0 \times (0,T)$, solve the wave equation $\Box v = 0$ on the unbounded domain $R := (0,\infty) \times \Gamma$. The equation is solved with vanishing initial data and with the boundary condition $v|_{\Gamma_0} = g$. The operator N is defined as the map

$$N : g \mapsto e_1 \cdot (a\nabla v)|_{\Gamma_0} . \tag{1.3}$$

Recall that g is defined on the space-time domain $\Gamma_0 \times (0,T)$ and v is also evaluated as a function on $\Gamma_0 \times (0,T)$. We emphasize that equation (1.3) is only a formal definition since we do not see how to choose appropriate function spaces for such a map N.

Let us nevertheless continue with our formal description. Let us assume that we can solve

$$\Box u = f \text{ on } \Omega \times (0,T) , \tag{1.4}$$

$$e_1 \cdot (a\nabla u) = N(u|_{\Gamma_0\times(0,T)}) \text{ on } \Gamma_0 \times (0,T) . \tag{1.5}$$

We claim that the solution u satisfies the desired property $\tilde{u}|_\Omega = u$. Indeed, let us consider the concatenated function $\tilde{v} := u$ in $\Omega \times (0,T)$ and $\tilde{v} := v$ in $R \times (0,T)$, where v is the solution of the wave equation on R as in the definition of N. Then, by construction, $\tilde{v}$ solves the wave equation on Ω and also on R. Furthermore, $\tilde{v}$ has no jump across Γ_0 (in the sense that traces from both sides coincide), since we demand $v|_{\Gamma_0} = g = u|_{\Gamma_0}$. Finally, the fluxes coincide on Γ_0 because of $e_1 \cdot (a\nabla u) = N(u|_{\Gamma_0\times(0,T)}) = e_1 \cdot (a\nabla v)|_{\Gamma_0}$. By uniqueness of solutions of the wave equation, this shows $\tilde{v} = \tilde{u}$ and hence the claim, $\tilde{u}|_\Omega = u$.

Let us continue our formal presentation in order to present the idea of performing calculations only in a radiation boundary box.

We define a radiation boundary box by truncating the exterior domain R; we set $B := (0,1) \times \Gamma \subset \tilde{\Omega}$ and denote the right boundary by $\Gamma_1 := \{1\} \times \Gamma$. Given an operator N_{right} that maps boundary data $g_1 : \Gamma_1 \to \mathbb{R}$ to fluxes $e_1 \cdot (a\nabla v)|_{\Gamma_1}$, we can define an operator N_{left} that maps boundary data $g_0 : \Gamma_0 \to \mathbb{R}$ to fluxes $e_1 \cdot (a\nabla v)|_{\Gamma_0}$ by solving the following problem on B:

$$\Box u = 0 \text{ on } B \times (0,T) , \tag{1.6}$$

$$u = g_0 \text{ on } \Gamma_0 \times (0,T) , \tag{1.7}$$

$$e_1 \cdot (a\nabla u) = N_{\text{right}}(u|_{\Gamma_1\times(0,T)}) \text{ on } \Gamma_1 \times (0,T) , \tag{1.8}$$

and setting $N_{\text{left}}(g_0) := e_1 \cdot (a\nabla u)|_{\Gamma_0}$. This definition has – formally – the following property: Assume that N_{right} is the desired operator N for the whole space, i.e., it maps boundary data g_1 onto the flux of the corresponding Dirichlet solution on the unbounded domain $(1,\infty) \times \Gamma$. Then N_{left} is (also) the desired operator N. This follows since, in that case, problem (1.6)–(1.8) is equivalent to the problem on the domain R.

A natural idea to use the above description in a numerical method is to iterate the operators N: Given N_{right}, the above equations define an operator N_{left}. This operator can also be used at the position $x_1 = 1$ and used as a new operator N_{right}. A fixed point N of this iteration is an exact Dirichlet-to-Neumann operator.

One-swipe-calculation of N. Finally, we can present an idea of algorithmic importance, we call it the *one-swipe calculation* of the operator N. The idea is based on the fact that the wave equation has a finite speed of propagation.

For a time parameter $\tau > 0$ we can consider operators N_{left} and N_{right} on the time interval $(0, \tau)$: An operator N_{right} maps a function on $\Gamma_1 \times (0, \tau)$ to a function on the same domain. By finite speed of propagation, there is a time parameter $\tau > 0$ such that, given *any* operator N_{right} (e.g.: $N_{\text{right}} \equiv 0$) on $(0, \tau)$, the solution to problem (1.6)–(1.8) coincides with the solution $\tilde{v}$ on the whole space $R \times (0, \tau)$. Loosely speaking, the wave that is induced by g did not reach the right boundary in the time window $(0, \tau)$. The result is that any choice of N_{right} leads to a correct calculation of N_{left}.

We therefore set $N_{\text{right}} \equiv 0$ on $(0, \tau)$ and calculate N_{left} on $(0, \tau)$. As argued above, this operator coincides with the desired operator N on $(0, \tau)$. We now continue the scheme and calculate N on the interval $(0, 2\tau)$. The boundary data $u|_{\Gamma_1}$ are vanishing on $(0, \tau)$ by choice of τ; these boundary data have no effect for fluxes at later times and can be neglected. The boundary data on Γ_1 in the interval $(\tau, 2\tau)$ is important, but their effect can be calculated with the previously calculated operator N on $(0, \tau)$, using an appropriate time shift by τ.

For two numbers $0 < \tau < T < \infty$, it is sufficient to perform the above definition finitely many times in order to construct the operator N. This shows that, in the case of the wave equation, we do not have to perform an iteration of the type $N_{\text{right}} \mapsto N_{\text{left}}$. Instead, we can calculate the desired operator N in one single swipe through the time interval $(0, T)$.

The goal of this article is to make the above ideas precise and to cast them in a mathematically rigorous framework. We have the feeling that a rigorous framework cannot be constructed in the time continuous setting and we will therefore study only a time discrete setting in the following.

Difficulty in the choice of function spaces. Let us describe why the natural choice of function spaces does not allow to make the above ideas rigorous in the time continuous setting. Natural function spaces for solutions of the wave equation are suggested by the energies that are associated to the wave equation, the potential energy $\frac{1}{2} \int_\Omega \nabla u \cdot a \nabla u$ and the kinetic energy $\frac{1}{2} \int_\Omega |\partial_t u|^2$. Estimates for solutions u of the wave equation are found by multiplication of the wave equation with $\partial_t u$. They yield estimates of the type

$$u \in L^\infty(0, T; H^1(R)) \quad \text{and} \quad \partial_t u \in L^\infty(0, T; L^2(R)) \,. \tag{1.9}$$

Let us investigate the homogeneous wave equation $\Box v = 0$ with vanishing initial data on R, with the Dirichlet boundary condition $v|_{\Gamma_0} = U|_{\Gamma_0}$, where U has the regularity of (1.9) in R. Testing $\Box v = 0$ with $\partial_t v$ yields

$$\partial_t \frac{1}{2} \int_R \left\{ |\partial_t v|^2 + \nabla v \cdot a \nabla v \right\} = - \int_{\Gamma_0} e_1 \cdot a \nabla v \, \partial_t U \,.$$

In order to estimate the right hand side by the norms of v that appear on the left hand side, we must re-write the term as a volume integral,

$$-\int_{\Gamma_0} e_1 \cdot a\nabla v\, \partial_t U = \int_R \nabla \cdot (a\nabla v\, \partial_t U) = \int_R \nabla \cdot (a\nabla v)\, \partial_t U + \int_R a\nabla v \cdot \partial_t \nabla U\,.$$

We do not see a way how to write the right hand side in such a way that only first order derivatives (in either x or t) of U and v appear.

These considerations suggest that the operator N cannot be defined rigorously in energy spaces.

1.2. **Literature.** If an equation on an unbounded domain has to be solved numerically, it is necessary to truncate the domain and to impose a boundary condition on the artificial boundary. For the description of the problem it is convenient to introduce a Dirichlet-to-Neumann operator (DtN operator), which allows to re-write the original problem equivalently as a problem on a bounded domain. The important task is then to either calculate the DtN operator explicitly or to approximate it. In the case of homogeneous media and time dependent problems, one can use spherical artificial boundaries and spherical harmonics [13], or more general domains and integral operators [8]. For numerical purposes more interesting is the approximation of the DtN operator by local operators; this was also studied extensively, starting with the seminal paper [7], for a modern treatment and further references we mention [12], and, for waveguides, [16].

In the case of heterogeneous media, one can describe the DtN operator with cell problems. This was done in [9] and [18] in the time-harmonic case, its numerical implementation was the subject of [15]. A slightly different approach, still for domain truncation of heterogeneous media in the time-harmonic case, was formulated in [4] and suggests the use of radiation boxes instead of DtN maps. Regarding general analytical aspects of this problem we mention the important paper [10].

Regarding the time dependent case in heterogeneous media, the corresponding task was investigated by Coatleven in [2], which is closely related to the work at hand. In [2], after a discretization in time, the scheme for the single time step is investigated. Since, in each time step, an elliptic problem has to be solved, one can use ideas of the time-harmonic studies to treat the single time step. Indeed, quite similar to our approach, a problem in a boundary box $\mathcal{C}$ (our B) is investigated, see equations (3.4) and (3.5) in [2]. The equations define operators that map boundary data to solutions and that allow to describe the DtN operator. The resulting numerical scheme for the time dependent problem uses, like our scheme, a boundary operator of discrete convolution type in time. For later contributions we mention [17] and [1].

Let us compare our scheme to that of [2]. To start with, the scheme in [2] is formulated for a larger class of equations, parabolic equations are also included. This makes a difference since we exploit the finite speed of propagation of the wave equation in order to construct our scheme. The finite speed of propagation allows us to describe the desired approximate DtN operator not as a solution of a fixed point map in the space of operators, but directly by solving forward problems. We quantify the error and find that it is small when the finite speed of propagation is well approximated. The iteration of operators is avoided essentially by a simplification step that we call the one-swipe calculation: The operator that was calculated until time step j is exploited in time step $j+1$ in order not to have to iterate.

Regarding general numerical schemes for the wave equation we mention [11] and [3] for constructions of finite element schemes, [14] for the stability investigations in terms of the parameter θ, [5] and [6] for more recent results on improved schemes.

2. Time discrete setting

The construction of the operator N can be formalized in the time discrete setting. We fix a number $m \in \mathbb{N}$ and choose points $0 = t_0 < t_1 < ... < t_m = T$ to discretize the time interval $[0, T]$. For simplicity of notation, we assume here that the points are chosen equidistant, i.e., $t_j = j\,\Delta t$, $j = 0, 1, ..., m$, with $\Delta t = T/m$.

A time dependent solution u is a tuple of the form $u = (u^0, ..., u^m)$. The discretization of the second time derivative in t_j will always be performed with the expression $(u^{j+1} - 2u^j + u^{j-1})/(\Delta t)^2$. We discretize the right hand side f with a sequence $(f_1, f_2, ..., f_{m-1})$, an adequate choice is given below. The time discrete version of $\Box u = f$ with a weight $\theta \in (0, 1)$ reads:

$$\int_{\tilde\Omega} \frac{u^{j+1} - 2u^j + u^{j-1}}{\Delta t^2}\, w + \int_{\tilde\Omega} \nabla(\theta u^{j+1} + (1-2\theta)u^j + \theta u^{j-1}) \cdot a\nabla w = \int_{\tilde\Omega} f_j\, w \tag{2.1}$$

for all $1 \le j \le m$ and all w. We have chosen the symmetric weights θ, $(1-2\theta)$, and θ for the three consecutive time steps. The two subsequent remarks give further details on the choice of the weights.

Remark 2.1 (The weights for the space time finite element discretization)**.** *We can identify a tuple $(u^0, ..., u^m)$ with a function u that satisfies $u(t_j) = u^j$ for all j and that is linearly affine in all time intervals (t_j, t_{j+1}). In this setting, we may write the wave equation in space and time as*

$$\int_0^T \int_{\tilde\Omega} \{-\partial_t u\, \partial_t \varphi + \nabla u \cdot a\nabla\varphi\} = \int_0^T \int_{\tilde\Omega} f\,\varphi \qquad \forall \varphi\,. \tag{2.2}$$

We evaluate both sides for piecewise affine (in time) functions u and φ, assuming $\varphi(t_j) = w$ and $\varphi(t_i) = 0$ for all $i \neq j$:

$$\begin{aligned}
&\int_{t_j}^{t_{j+1}} \int_{\tilde\Omega} \frac{u^{j+1} - u^j}{\Delta t}\, \frac{w}{\Delta t} - \int_{t_{j-1}}^{t_j} \int_{\tilde\Omega} \frac{u^j - u^{j-1}}{\Delta t}\, \frac{w}{\Delta t} \\
&\qquad + \Delta t \int_0^1 \int_{\tilde\Omega} \nabla((1-s)u^j + su^{j+1}) \cdot a(1-s)\nabla w \\
&\qquad + \Delta t \int_0^1 \int_{\tilde\Omega} \nabla(su^j + (1-s)u^{j-1}) \cdot as\nabla w \\
&= \Delta t \int_0^1 \int_{\tilde\Omega} f((1-s)t_j + st_{j+1})(1-s)w \;+\; f((1-s)t_j + st_{j+1})sw\,.
\end{aligned}$$

Dividing by Δt and defining the data $(f_j)_{j<m}$ by the averages of f that appear on the right hand side, the scheme coincides with (2.1) for the weights $\theta = \int_0^1 s(1-s)\,ds = 1/6$ and $(1-2\theta) = 2\int_0^1 s^2\,ds = 2/3$.

Remark 2.2 (The weights for a stable discretization)**.** *The time discretization (2.1) is a standard scheme to solve the wave equation and is analyzed, e.g., in [14]. Using the notation of that reference, the scheme is a Newmark scheme with $\gamma = 1/2$. Theorem 4.1*

of [14] *yields that a discrete energy is conserved and that the scheme is unconditionally stable for* $\theta \geq 1/4$.

We note that the natural discretization of Remark 2.1 is with $\theta = 1/6$ *and does therefore not satisfy the stability condition.*

In the remainder of this article we will study the scheme (2.1). When w can be chosen arbitrarily in $H_0^1(\tilde{\Omega})$, then the equation is the weak form of

$$\frac{u^{j+1} - 2u^j + u^{j-1}}{\Delta t^2} = \nabla \cdot a\nabla(\theta u^{j+1} + (1-2\theta)u^j + \theta u^{j-1}) + f_j \,. \tag{2.3}$$

On the other hand, if we demand that u^j is contained in some subspace of $H_0^1(\tilde{\Omega})$ and the equation must hold also for all w in the same subspace, then relation (2.1) can have a different meaning, it can describe a finite element scheme and it can also incorporate boundary conditions. It will therefore be important to analyze (2.1) for an arbitrary subspace $X \subset \mathcal{X} := H_0^1(\tilde{\Omega})$.

Stability of the scheme. We will apply the time discretization scheme (2.1) for various spaces X and various right hand sides f. The solvability of the scheme and its stability is guaranteed by the subsequent lemma. The lemma does not provide new results, but it puts the properties of the scheme in a form that is useful for our further analysis.

Lemma 2.3 (Existence and stability)**.** *Let* $X \subset \mathcal{X} = H_0^1(\tilde{\Omega})$ *be a closed subspace, we use the Hilbert space structure of* $\mathcal{X}$ *on* X*. For fixed* $m \in \mathbb{N}$*, let* $(f_1, ..., f_{m-1})$ *be a right hand side with* $f_j \in \mathcal{X}'$ *for every* $j < m$*. Let initial data be given by* $u^0 = u^1 = 0$*. We consider the scheme* (2.1) *in* X *for* $\theta \geq 1/4$*. More precisely: For every* $j \geq 1$ *we look for* $u^{j+1} \in X$ *such that* (2.1) *holds for every* $w \in X$*. Then there exists a unique solution* $(u^1, ..., u^m)$ *and the solution satisfies the stability estimate*

$$\sup_{k\leq m} \left\| \frac{u^k - u^{k-1}}{\Delta t} \right\|_{L^2(\tilde{\Omega})} + \sup_{k\leq m} \|u^k + u^{k-1}\|_{\mathcal{X}} \leq C \sum_{j=1}^{m-1} \Delta t \; \|f_j\|_{L^2(\tilde{\Omega})} \,. \tag{2.4}$$

The constant C *depends on the cross section* Γ *of the domain, but it does not depend on* m *or* T *as long as* $\Delta t \leq 1$ *is satisfied.*

For $\theta > 1/4$ *there also holds, with* $C = C(\theta)$*,*

$$\sup_{k\leq m} \left\| \frac{u^k - u^{k-1}}{\Delta t} \right\|_{L^2(\tilde{\Omega})} + \sup_{k\leq m} \|u^k\|_{\mathcal{X}} \leq C \sum_{j=1}^{m-1} \|f_j\|_{\mathcal{X}'} \,. \tag{2.5}$$

Proof. The bilinear form $b : X \times X \to \mathbb{R}$

$$b(u, w) := \frac{1}{\Delta t^2} \int_{\tilde{\Omega}} u \cdot w + \theta \int_{\tilde{\Omega}} \nabla u \cdot a\nabla w \tag{2.6}$$

is continuous and coercive. The Lax-Milgram theorem provides the existence of a unique solution $u^{j+1} \in X$ of the (2.1).

In order to obtain the stability, we use $w = u^{j+1} - u^{j-1}$ as a test function. For the first term we obtain

$$\int_{\tilde{\Omega}} \frac{u^{j+1} - 2u^j + u^{j-1}}{\Delta t^2}(u^{j+1} - u^{j-1}) = \int_{\tilde{\Omega}} \left| \frac{u^{j+1} - u^j}{\Delta t} \right|^2 - \left| \frac{u^j - u^{j-1}}{\Delta t} \right|^2 \,.$$

Below, we will fix an index $k \leq m$ and perform a summation over $j \in \{1, ..., k-1\}$; this makes all contributions of the above expression vanish except for the boundary terms $j = k-1$ and $j = 1$.

Let us now turn to the second term. We calculate with the short hand notation $\langle u, w\rangle_a = \int \nabla u \cdot a \nabla w$ and $\|u\|_a^2 = \langle u, u\rangle_a$. We have to evaluate the expression

$$\begin{aligned} &\int_{\tilde{\Omega}} \nabla(\theta u^{j+1} + (1-2\theta)u^j + \theta u^{j-1}) \cdot a\nabla(u^{j+1} - u^{j-1}) \\ &\quad = \theta\|u^{j+1}\|_a^2 + (1-2\theta)\langle u^j, u^{j+1} - u^{j-1}\rangle_a - \theta\|u^{j-1}\|_a^2 \,, \end{aligned} \tag{2.7}$$

and its summation over $j \in \{1, ..., k-1\}$.

In order to simplify the summation of (2.7), we expand the energy term

$$\begin{aligned} E_j &:= (\theta - \tfrac{1}{4})\|u^{j+1} - u^j\|_a^2 + \frac{1}{4}\|u^{j+1} + u^j\|_a^2 \\ &= \theta\left(\|u^{j+1}\|_a^2 - 2\langle u^j, u^{j+1}\rangle_a + \|u^j\|_a^2\right) + \langle u^j, u^{j+1}\rangle_a \,. \end{aligned}$$

This implies

$$E_j - E_{j-1} = \theta\left(\|u^{j+1}\|_a^2 - \|u^{j-1}\|_a^2\right) + (1-2\theta)\langle u^j, u^{j+1} - u^{j-1}\rangle_a \,,$$

which is the expression of (2.7). We conclude that the summation of (2.7) over $j \in \{1, ..., k-1\}$ yields

$$E_{k-1} - E_0 = \left(\theta - \tfrac{1}{4}\right)\|u^k - u^{k-1}\|_a^2 + \frac{1}{4}\|u^k + u^{k-1}\|_a^2 \,. \tag{2.8}$$

We can now combine our findings: Relation (2.1) with $w = u^{j+1} - u^{j-1}$ and a summation over $j \in \{1, ..., k-1\}$ yields,

$$\begin{aligned} &\int_{\tilde{\Omega}} \left|\frac{u^k - u^{k-1}}{\Delta t}\right|^2 + \left(\theta - \tfrac{1}{4}\right)\|u^k - u^{k-1}\|_a^2 + \frac{1}{4}\|u^k + u^{k-1}\|_a^2 \\ &\quad = \sum_{j=1}^{k-1} \int_{\tilde{\Omega}} f_j \,(u^{j+1} - u^{j-1}) \,. \end{aligned} \tag{2.9}$$

A Poincaré inequality holds in $\mathcal{X} = H_0^1(\tilde{\Omega})$ and hence also in every subspace. This shows that the norm $\|.\|_a$ is equivalent to the full $H^1(\tilde{\Omega})$-norm. Taking the supremum over $k \in \{1, ..., m\}$ we obtain

$$\begin{aligned} \sup_k \left\|\frac{u^k - u^{k-1}}{\Delta t}\right\|_{L^2(\tilde{\Omega})}^2 + \sup_k \|u^k + u^{k-1}\|_{\mathcal{X}}^2 &\leq C \sum_{j=1}^{m-1} \|f_j\|_{L^2(\tilde{\Omega})} \|u^{j+1} - u^{j-1}\|_{L^2(\tilde{\Omega})} \\ &\leq C \left(\sup_j \|u^{j+1} - u^{j-1}\|_{L^2(\tilde{\Omega})}\right) \sum_{j=1}^{m-1} \|f_j\|_{L^2(\tilde{\Omega})} \,. \end{aligned}$$

With Young's inequality we obtain (2.4).

In the case $\theta > 1/4$, we can additionally use the second term in (2.9) to estimate the functions u^k in $\mathcal{X}$ and obtain, with $C = C(\theta)$,

$$\sup_k \left\| \frac{u^k - u^{k-1}}{\Delta t} \right\|^2_{L^2(\tilde{\Omega})} + \sup_k \|u^k\|^2_{\mathcal{X}} \leq C \sum_{j=1}^{m-1} \|f_j\|_{\mathcal{X}'} \, \|u^{j+1} - u^{j-1}\|_{\mathcal{X}}$$
$$\leq C \left(\sup_j \|u^j\|_{\mathcal{X}} \right) \sum_{j=1}^{m-1} \|f_j\|_{\mathcal{X}'} \, .$$

This shows (2.5). □

3. The domain truncation scheme

We can now describe the domain truncation method with the help of the general time discretization scheme. We only have to apply it successively for different spaces X and different right hand sides f. In this section, no structural assumptions on a (such as periodicity) must be assumed.

As a base space we have chosen $\mathcal{X} := H^1_0(\tilde{\Omega})$. We assume that $X_{\tilde{\Omega}} \subset \mathcal{X}$ is some linear subspace. The choice $X_{\tilde{\Omega}} = \mathcal{X}$ implies that we solve the continuous elliptic problem in each time step. When $X_{\tilde{\Omega}} \subset \mathcal{X}$ is the span of a finite number of ansatz functions, then (2.1) is a finite element discretization of the wave equation on the unbounded domain $\tilde{\Omega}$.

After the domain truncation, we want to seek for a function that lives only on Ω. In order to incorporate a Dirichlet-to-Neumann operator, we have to solve equations on the unbounded exterior domain R. To this end, we define

$$X_\Omega := \{u \in H^1(\Omega) \,|\, u = \tilde{u}|_\Omega \text{ for some } \tilde{u} \in X_{\tilde{\Omega}}\} \, , \tag{3.1}$$

$$X_R := \{\tilde{u} \in X_{\tilde{\Omega}} \,|\, \tilde{u}|_\Omega = 0\} \, . \tag{3.2}$$

We emphasize that the definitions are not symmetric: The functions in X_Ω can have arbitrary values on Γ_0, the functions in X_R must vanish on Γ_0.

We additionally assume that we are given a bounded linear extension operator

$$E : X_\Omega \to X_{\tilde{\Omega}} \, . \tag{3.3}$$

We say that E is an extension operator when $(Eu)|_\Omega = u$ holds for every $u \in X_\Omega$. If we choose to work with the full space, $X_{\tilde{\Omega}} = \mathcal{X} = H^1_0(\tilde{\Omega})$, then we can define an operator E with the help of a symmetric extension across Γ_0, combined with a smooth cut-off function. In a finite element setting, $X_{\tilde{\Omega}}$ is a space that is spanned by ansatz functions and X_Ω is the restriction of these functions to Ω. In this case, the extension operator of $u \in X_\Omega$ is given in a natural way: For Eu, one uses the same linear combination of ansatz functions as for u.

For fixed m and a Hilbert space X we use the notation

$$Z := \ell^\infty(\{0, 1, ..., m\}; X) = \big\{(u^0, ..., u^m) \big| u^j \in X \ \forall j \leq m\big\} \ , \ \|u\|_Z := \sup_{j \leq m} \|u^j\|_X \, . \tag{3.4}$$

When X is one of the Hilbert spaces $X_{\tilde{\Omega}}$, X_Ω, or X_R, then the corresponding space Z is denoted as $Z_{\tilde{\Omega}}$, Z_Ω, and Z_R, respectively.

Lemma 3.1 (Dirichlet-to-Neumann construction)**.** *We assume that the coefficient a, the time horizon T, the time discretization $t_k = k\Delta t$ with $T = m\Delta t$ and $\Delta t \leq 1$,*

$\theta > 1/4$, and the right hand side $f = (f_1, ..., f_{m-1})$ are fixed. We consider spaces $Z_{\tilde{\Omega}}$, Z_Ω, and Z_R as above. There holds:

(i) Original problem. *There exists a uniquely defined function $\tilde{u} \in Z_{\tilde{\Omega}}$ satisfying* (2.1) *in the space $X_{\tilde{\Omega}}$ with right hand side f.*

(ii) Outer domain problem. *Given $U \in Z_{\tilde{\Omega}}$, which we interpret as boundary data on Γ_0, there exists a uniquely defined function $v \in Z_{\tilde{\Omega}}$ such that the following conditions hold: (a) v satisfies* (2.1) *with vanishing right hand side and for test functions w in the space X_R. (b) The boundary condition is satisfied in the sense that $v - U \in Z_R$.*

(iii) Incorporation of a Neumann condition. *Given $v \in Z_{\tilde{\Omega}}$, we define a right hand side $\hat{f} = (\hat{f}_j)_j \in Z'_\Omega$ as follows: For $j = 0$ we set $\hat{f}_0 = 0$ and for $j \in \{1, ..., m-1\}$ and $w \in Z_\Omega$ we set*

(3.5)
$$\langle \hat{f}_j, w\rangle := \int_R \frac{v^{j+1} - 2v^j + v^{j-1}}{\Delta t^2}\,(Ew) + \int_R \nabla(\theta v^{j+1} + (1-2\theta)v^j + \theta v^{j-1}) \cdot a\nabla(Ew)\,.$$

A continuous linear operator

$$F : Z_\Omega \ni u \mapsto \hat{f} \in Z'_\Omega \tag{3.6}$$

is defined as the concatenation of the following maps:

(a) $Z_\Omega \ni u \mapsto U := Eu \in Z_{\tilde{\Omega}}$ *(extension of a solution in Ω)*
(b) $Z_{\tilde{\Omega}} \ni U \mapsto v \in Z_{\tilde{\Omega}}$ *(solving the outer domain problem for U as in (ii))*
(c) $Z_{\tilde{\Omega}} \ni v \mapsto \hat{f} \in Z'_\Omega$ *(evaluation of the Neumann data of v as in (iii)).*

Proof. The claim of (i) follows directly from Lemma 2.3.

To verify (ii), we consider $\tilde{u} = v - U \in Z_R$ as unknown and consider (2.1) with test functions $w \in X_R$. The right hand side $\tilde{f} = (\tilde{f}_j)_j \in Z'_R$ is defined as follows: For $w \in X_R$ we set

$$\langle \tilde{f}_j, w\rangle := -\int_{\tilde{\Omega}} \frac{U^{j+1} - 2U^j + U^{j-1}}{\Delta t^2}\, w - \int_{\tilde{\Omega}} \nabla(\theta U^{j+1} + (1-2\theta)U^j + \theta U^{j-1}) \cdot a\nabla w\,. \tag{3.7}$$

Let $\tilde{u} \in Z_R$ be the solution for the right hand side $\tilde{f}$, which exists by Lemma 2.3. The function $v := U + \tilde{u}$ then satisfies all desired relations.

Regarding (iii), we only have to observe that the objects are well defined. □

Theorem 3.2 (Transparent boundary condition)**.** *With $F : u \mapsto \hat{f}$ defined in* (3.6)*, we consider the following evolution equation on Ω: We seek for $u \in Z_\Omega$ such that*

(3.8)
$$\int_\Omega \frac{u^{j+1} - 2u^j + u^{j-1}}{\Delta t^2}\, w + \int_\Omega \nabla(\theta u^{j+1} + (1-2\theta)u^j + \theta u^{j-1}) \cdot a\nabla w = \int_\Omega f_j\, w - \langle F(u)_j, w\rangle$$

holds for every $j < m$ and for every $w \in X_\Omega$. The problem (3.8) *is a problem on Ω with an exact transparent boundary condition.*

More precisely, the following holds: Assume that we can solve system (3.8) *with $u \in Z_\Omega$. Let furthermore $\tilde{u} \in Z_{\tilde{\Omega}}$ be the original solution as defined in (i) of Lemma 3.1. Then there holds*

$$\tilde{u}|_\Omega = u\,. \tag{3.9}$$

Proof. We use the quantities that are defined in (ii) and (iii) of Lemma 3.1: u is the solution of problem (3.8) on Ω with right hand side $f - \hat{f}$. The extension is $U = Eu$. The solution on the outer domain with boundary data U is v. By construction, v is a solution of the equations on R. Furthermore, there holds $v = U = u$ on Ω. Our aim is to verify that v is a solution on the entire domain $\tilde{\Omega}$. Once this is shown, by uniqueness, there holds $\tilde{u} = v$ and the claim is shown.

Let w be an arbitrary test function in $X_{\tilde{\Omega}}$. We decompose w into two parts according to the definition of X_Ω and X_R. With $w_\Omega := E(w|_\Omega)$ and $w_R = w - w_\Omega$ there holds $w = w_\Omega + w_R$ with $w_\Omega \in X_\Omega$ and $w_R \in X_R$.

For a fixed index j, we write symbolically $\langle \bar{\partial}_t^2 u, w\rangle + \langle u, w\rangle_A$ for the left hand side of (2.1). When the integrations in this expression are performed only over Ω, then we write $\langle \bar{\partial}_t^2 u, w\rangle_\Omega + \langle u, w\rangle_{A,\Omega}$. In order to check the equations for v on $\tilde{\Omega}$, we fix $j \le m-1$ and $w \in X_{\tilde{\Omega}}$ and calculate

$$\begin{aligned}
\langle \bar{\partial}_t^2 v, w\rangle + \langle v, w\rangle_A &= \langle \bar{\partial}_t^2 v, w_\Omega\rangle + \langle v, w_\Omega\rangle_A + \langle \bar{\partial}_t^2 v, w_R\rangle + \langle v, w_R\rangle_A \\
&= \langle \bar{\partial}_t^2 v, E(w|_\Omega)\rangle + \langle v, E(w|_\Omega)\rangle_A \\
&= \langle \bar{\partial}_t^2 v, E(w|_\Omega)\rangle_\Omega + \langle v, E(w|_\Omega)\rangle_{A,\Omega} + \langle \hat{f}, w|_\Omega\rangle \\
&= \langle \bar{\partial}_t^2 u, w|_\Omega\rangle_\Omega + \langle u, w|_\Omega\rangle_{A,\Omega} + \langle \hat{f}, w|_\Omega\rangle \\
&= \langle f, w|_\Omega\rangle = \langle f, w\rangle\,.
\end{aligned}$$

The first equality is obtained from $w = w_\Omega + w_R$. The second equality uses that v is a solution of the homogeneous problem on R, and the definition of w_Ω. In the third equality, the domain of integration is split into an integral over Ω and an integral over R; the integral over R is recognized as the definition of $\hat{f}$, see (3.5). In the fourth equality we only omit the extension operator, which is possible since only Ω-integrals are evaluated. In the last line, the solution property (3.8) of u is exploited. The last equality uses the fact that f is supported in Ω.

The result of the above calculation is that v is a solution of the original problem in $\tilde{\Omega}$. As already noted, this shows $\tilde{u} = v$ and hence $\tilde{u}|_\Omega = u$. □

Lemma 3.3 (Simplification by time-shift invariance)**.** *For every $k \in \{1, ..., m-1\}$ we define an operator N_k as follows: Given $u^2 \in X_\Omega$ we construct u as the trivial extension in the sense that $u = (u^0, u^1, u^2, u^3, ..., u^m)$ with $u^i = 0\ \forall i \neq 2$. With F of* (3.6) *we define*

$$N_k : X_\Omega \to X_\Omega'\,, \qquad u^2 \mapsto F(u)_k\,. \tag{3.10}$$

The operator F can be reconstructed from the operators N_k:

$$F : Z_\Omega \to Z_\Omega'\,, \qquad F(u)_j = \sum_{i=0}^{j-1} N_{j-i}(u^{i+2})\,. \tag{3.11}$$

Proof. We note that $\hat{f} := F(u)$ has only the non-trivial entries $\hat{f}_1, ..., \hat{f}_{m-1}$ in X_Ω'. These entries define $N_1(u^2) := \hat{f}_1, ..., N_{m-1}(u^2) := \hat{f}_{m-1}$.

In order to check the reconstruction formula we have to consider again general solution vectors $u = (0, 0, u^2, u^3, ...)$. We first observe for $j = 1$ that, by causality and definition of N_1, there holds $F(u)_1 = F((0,0,u^2,u^3,...))_1 = F((0,0,u^2,0,...))_1 = N_1(u^2)$. By linearity, causality, and time shift invariance of solutions we can calculate

for $j = 2$

$$\begin{aligned} F(u)_2 &= F((0,0,u^2,u^3,...))_2 = F((0,0,u^2,0,...))_2 + F((0,0,0,u^3,0,...))_2 \\ &= N_2(u^2) + N_1(u^3)\,, \end{aligned}$$

which provides (3.11). The analogous calculation can be performed for $j > 2$. □

Remark 3.4 (Boundary operator property and causality)**.** *The operators N_k depend only on boundary data in the sense that: For every $u^2 \in X_\Omega$ with $(Eu^2)|_R = 0$ there holds $N_k(u^2) = 0$. Furthermore, for every $u^2 \in X_\Omega$ and w with $Ew = 0$, the application is also vanishing: $\langle N_k(u^2), w\rangle = 0$. The operator F is causal in the sense that $F(u)_j$ depends only on the entries $u^0, ..., u^{j+1}$ of u.*

All claims follow immediately from the definition of F in (3.5).

Proposition 3.5 (Decomposition into explicit and implicit part)**.** *In order to find a solution u to* (3.8)*, we have to solve, in each time step j, the following problem for $u^{j+1} \in X_\Omega$:*

$$\int_\Omega \frac{u^{j+1}}{\Delta t^2}\, w + \theta \int_\Omega \nabla u^{j+1} \cdot a \nabla w + \langle N_1(u^{j+1}), w\rangle = \langle b_j, w\rangle \qquad \forall w \in X_\Omega\,. \tag{3.12}$$

The right hand side b_j depends on the previously calculated solution entries $u^0, u^1, ..., u^j$, and is calculated with the operators N_k, $k \geq 2$, by evaluating (3.8) *and* (3.11)*.*

In problem (3.12)*, the left hand side defines a continuous and coercive bilinear form on X_Ω. In particular, problem* (3.12) *possesses a unique solution.*

Proof. Since the bilinear form on the left hand side is the same for every $j \geq 1$, we consider here $j = 1$. The leading two integrals read

$$\int_\Omega \frac{u^2}{\Delta t^2}\, w + \theta \int_\Omega \nabla u^2 \cdot a \nabla w\,,$$

they define a coercive bilinear form in u^2 and w on X_Ω. It is therefore sufficient to show non-negativity of N_1. We use the test function $w = u^2 \in X_\Omega$ and calculate, using the definition of $\hat{f}_1$ in (3.5) and $v^0 = v^1 = 0$,

$$\begin{aligned} \langle N_1(u^2), u^2\rangle = \langle \hat{f}_1, u^2\rangle &= \frac{1}{\Delta t^2}\int_R v^2\, Eu^2 + \theta \int_R \nabla v^2 \cdot a \nabla (Eu^2) \\ = \frac{1}{\Delta t^2}\int_R v^2\,(Eu^2 - v^2) + \theta \int_R \nabla v^2 \cdot a \nabla (Eu^2 - v^2) \\ + \frac{1}{\Delta t^2}\int_R v^2\, v^2 + \theta \int_R \nabla v^2 \cdot a \nabla v^2 \geq 0\,, \end{aligned}$$

where we exploited in the last step that v^2 is a solution of the homogeneous wave equation in R and $Eu^2 - v^2$ is an admissible test function in X_R. Finally, we used the non-negativity of the quadratic terms. □

4. Calculation of the operator family N

The result of the last section is that an exact domain truncation scheme is found when the Dirichlet-to-Neumann operator family $N = (N_1, ..., N_{m-1})$, $N_k : X_\Omega \to X'_\Omega$, $k \in \{1, ..., m-1\}$ is calculated. By definition, the calculation of N involves, for each argument $u^2 \in X_\Omega$, the solution of a wave equation on the unbounded domain R.

In practice, the calculation of N is performed with a boundary box B. Let us recall some notation from the introduction. The boundary box is $B := (0,1) \times \Gamma \subset \tilde{\Omega}$, its left boundary is $\Gamma_0 := \{0\} \times \Gamma$, its right boundary is $\Gamma_1 := \{1\} \times \Gamma$. Dirichlet-to-Neumann boundary operators on Γ_0 are denoted by N_{left}. We can also treat the interface Γ_1 as a right boundary of a truncated domain and perform the same construction; this defines Dirichlet-to-Neumann boundary operators on Γ_1, denoted by N_{right}.

We make the assumption that the function spaces are defined with a finite element scheme. Let us furthermore assume that the discretization of Γ_0 and its neighboring elements is identical to the discretization of Γ_1 and its neighbors (up to the obvious shift by e_1). In this case, we can apply an operator N_{right} also along Γ_0 and we can apply an operator N_{left} along Γ_1.

From now on, we denote by N_{exact} the exact Dirichlet-to-Neumann boundary operator on Γ_0 as defined in Theorem 3.2. Upon identification of boundaries, we may regard N_{exact} also as the exact boundary operator on Γ_1. The aim in the following is to develop, in finite element spaces, a practical scheme for the calculation of N_{exact} or of an approximation N.

4.1. **Calculation of N with an iteration.** We recall the quite natural idea of calculating N with an iteration. Assume that an approximate operator family $N^{(i)}$ is given. We can interpret $N^{(i)}$ as boundary operators on Γ_1, as N_{right}. The corresponding boundary condition allows to solve wave equations on B. When $N^{(i)}$ is a good approximation of N, then the solutions on B will approximate the (restrictions of) solutions on the unbounded domain R.

If we insert all possible boundary data $u^2|_{\Gamma_0}$ and solve these equations on B with the boundary condition on Γ_1 given by $N^{(i)}$, we find new boundary operators $N^{(i+1)}$ on Γ_0. Upon an identification of boundaries, the scheme defines an iteration

$$N^{(i)} \mapsto N^{(i+1)} .$$

A fixed point N of this iteration provides the exact Dirichlet-to-Neumann operator N_{exact}. Such an observation is the starting point of related constructions, see [9].

4.2. **One-swipe calculation of N.** Our aim is to calculate an approximate domain truncation operator family $N = (N_1, ..., N_{m-1})$ without performing an iteration over boundary operators. All calculations are performed only on the radiation box $B = (0,1) \times \Gamma$. All quantities are kept fixed: Coefficients a, time interval length T and its discretization with m and $\Delta t = T/m$. We furthermore assume that the discretization of the truncated domain is fixed by the choice of a finite element space $X_\Omega \subset H^1(\Omega)$. We also fix a discretization of B by choosing a finite element space $X_B \subset H^1(\Omega)$.

The important assumption for the subsequent construction is that the right boundary of Ω and the right boundary of B can be identified. More precisely, we assume the following: For every function $u \in X_\Omega$ that is supported only on finite elements that are neighboring a boundary point $x \in \Gamma_0$, the shifted function $u(. - e_1)$ is an element of X_B. Vice versa, for every function $v \in X_B$ that is localized around a point $y \in \Gamma_1$, we assume that $v(. + e_1)$ is an element of X_Ω.

Algorithm. We now describe the algorithm for the calculation of operators N_k in the one-swipe scheme. In each time level, numbered with j, we consider all possible boundary data g. A boundary datum is given by a function $g \in X_B$, but we only have to consider the finite element basis functions $g \in X_B$ with $g|_{\Gamma_0} \neq 0$. For every g we

calculate the solution v of a wave equation with the boundary data g; more precisely, we demand $v^i = 0\ \forall i \neq 2$ and $v^2 = g$ on Γ_0. No restriction is imposed on the right boundary Γ_1 and the right hand side f is always vanishing on B. For each level j and each g, two objects are evaluated: The function $v^{j+1} =: P_{j+1}(g) \in X_B$, which is a solution of a discrete wave equation on B at time t_{j+1}, and the desired operator $N_j(g)$. The time steps $j = 1$ and $j \geq 2$ are slightly different, we have to describe them separately.

Initialization: $j = 1$. We set $v^1 := v^0 := 0$. We determine $v^2 \in X_B$ with the scheme (2.1). The scheme reads, because of trivial initial data and trivial right hand side,

$$\int_B \frac{v^2}{\Delta t^2}\, w + \int_B \theta \nabla v^2 \cdot a \nabla w = 0 \tag{4.1}$$

for all $w \in X_B$ with $w|_{\Gamma_0} = 0$. The solution is non-trivial since the boundary condition $v^2 = g$ on Γ_0 is incorporated. The Dirichlet condition is implemented just as in Step (ii) in Lemma 3.1. We note that (4.1) encodes, implicitly, a homogeneous Neumann condition on Γ_1.

Equation (4.1) defines v^2 for every g, hence it defines an operator $P_2 : g \mapsto v^2$. We next define the operator N_1 from v^2. Essentially, we want to define $N_1(g)$ as the element $\hat{f}_1$ from (3.5). The test functions should be identical to the extensions of their boundary data, we demand $w = E(w|_\Omega)$. For such a test function we define

$$\langle N_1(g), w\rangle := \langle \hat{f}_1, w\rangle = \frac{1}{\Delta t^2}\int_B v^2\, w + \theta \int_B \nabla v^2 \cdot a \nabla w\,. \tag{4.2}$$

Time step: $j \geq 2$. We assume that N_k is already calculated for $k \leq j-1$. The data v^j and v^{j-1} are given. We recall that, in the calculation of v^{j+1}, we consider a homogeneous Dirichlet condition on Γ_0, no restrictions on Γ_1, a vanishing right hand side f. Essentially, we obtain $P_{j+1}(g) = v^{j+1}$ as the solution of the transparent boundary condition scheme (3.8). The right hand side of (3.8) reads, using $f_j = 0$ and the formula (3.11) for F_j:

$$\int_\Omega f_j\, w - \langle F(v)_j, w\rangle = -\left\langle \sum_{i=0}^{j-1} N_{j-i}(v^{i+2}), w \right\rangle .$$

The term corresponding to $i = j-1$ reads $N_1(v^{j+1})$; this is an implicit term and we write it on the left hand side. The term for $i = 0$ is $N_j(v^2)$; since the operator N_j is not yet calculated, we simply omit this term. Since we calculate a discretized wave equation in B, the operators N_k must be considered as operators on the right boundary. In order to highlight this interpretation, we write $N_k(v^i|_{\Gamma_1})$. Altogether, we solve the following system for $v^{j+1} \in X_B$:

$$\begin{aligned} \int_B \frac{v^{j+1} - 2v^j + v^{j-1}}{\Delta t^2}\, w + \int_B \nabla(\theta v^{j+1} + (1-2\theta)v^j + \theta v^{j-1}) \cdot a\nabla w \\ + \langle N_1(v^{j+1}|_{\Gamma_1}), w\rangle = -\sum_{i=1}^{j-2} \langle N_{j-i}(v^{i+2}|_{\Gamma_1}), w\rangle \end{aligned} \tag{4.3}$$

for all $w \in X_B$ with $w|_{\Gamma_0} = 0$. This defines $P_{j+1}(g) = v^{j+1}$.

As before, the operator N_j is determined by evaluating $\hat{f}_j$ in (3.5). When w describes boundary data on Γ_0 in the sense that $w = E(w|_\Omega)$, we set

$$\langle N_j(g), w\rangle := \int_B \frac{v^{j+1} - 2v^j + v^{j-1}}{\Delta t^2}\, w + \int_B \nabla(\theta v^{j+1} + (1-2\theta)v^j + \theta v^{j-1}) \cdot a\nabla w\,. \tag{4.4}$$

The algorithm is defined with the above four equations: For every g, (4.1) is used to calculate v^2 and (4.2) defines N_1. Then, for every $j \geq 2$ and every g, (4.3) is used to calculate v^{j+1} and (4.4) defines N_j.

Complexity. Let us briefly discuss the computational complexity of the scheme. In the two-dimensional case, when $\Delta x > 0$ stands for the typical diameter of elements, the number of nodes in B is $O((\Delta x)^{-2})$ and the number of nodes on Γ_0 is $O((\Delta x)^{-1})$. Solving one time step of the wave equation in B requires solving a linear system with $O(h^{-2})$ unknowns. We have to do that for every time step and for every boundary element function, i.e., $m \cdot O((\Delta x)^{-1})$ times. When the matrix solver is linear in the number of unknowns, the total complexity to calculate the family N is of the order $m \cdot O((\Delta x)^{-3})$.

In three dimensions, the same calculation yields the complexity $m \cdot O((\Delta x)^{-5})$.

4.3. **An error estimate for the one-swipe calculation of N.** In this section, we denote the exact domain truncation operator family as $N_{\text{exact}} = \tilde{N} = (\tilde{N}_1, ..., \tilde{N}_{m-1})$. The operators N_k are defined with the unbounded domain R and with a finite element discretization X_R; we assume that X_R extends the discretization of B in the sense that $u \in X_R$ implies $u|_B \in X_B$. The operator family $\tilde{N}$ is exact in the sense of Theorem 3.2: The calculation with $\tilde{N}$ on Ω provides the same solution sequence (on Ω) as the calculation of the corresponding discretization on $\tilde{\Omega} = \Omega \cup \Gamma_0 \cup R$.

We emphasize that all discretizations are fixed. Based on the discretizations, we have the exact transparent boundary operators $\tilde{N}$ and the approximate boundary operators N as obtained from the one-swipe calculation scheme (4.1)–(4.4). The operators are maps

$$\tilde{N}_k, N_k : X_\Omega \to X_\Omega'\,,$$

and we will measure differences in the corresponding operator norm.

Theorem 4.1 (Error in the one-swipe calculation of N)**.** *Let the spatial discretization be fixed with the spaces X_B and X_R, allowing for an identification of the boundaries Γ_0 and Γ_1 as described above. Let T, m, $\Delta t = T/m$, and $\theta > 1/4$ be fixed. Then there exists a constant C, depending on the discretization and on Δt, such that the following error estimate holds:*

$$\|N_k - \tilde{N}_k\| \leq C \sup_{g, \|g\| \leq 1} \|v^2|_{\Gamma_1}\|\,. \tag{4.5}$$

We recall that v^2 is the solution in the second time instance, determined by (4.1)*.*

Estimate (4.5) has on the right hand side the term $\|v^2|_{\Gamma_1}\|$. We emphasize the following fact: If the numerical scheme had a finite speed of propagation property, then we could assume $v^2|_{\Gamma_1} = 0$ for every g. This shows the strength of (4.5).

The weakness of (4.5) consists in the fact that C depends on both, the discretization in time and the discretization in space. That C is independent of both cannot be expected, see the discussion of function spaces in the introduction. On the other

hand, we would expect that an independence of either time or space discretization is possible; we did not succeed in proving this.

Proof. We fix $g \in X_B$ to impose a Dirichlet boundary condition on Γ_0 at time t_2. The system for v^{j+1} is given in (4.3). The system for $\tilde{v}^{j+1}$ is identical, just that all N_k are replaced by $\tilde{N}_k$ and that the term $\tilde{N}_j(v^2)$ is included on the right hand side, compare Theorem 3.2 and (3.11). We denote the difference of the two solutions as $u^j = v^j - \tilde{v}^j$. The equation for u^{j+1} then reads

$$\begin{aligned}
&\int_B \frac{u^{j+1} - 2u^j + u^{j-1}}{\Delta t^2}\, w + \int_B \nabla(\theta u^{j+1} + (1-2\theta)u^j + \theta u^{j-1}) \cdot a \nabla w \\
&\quad + \sum_{i=0}^{j-1} \langle \tilde{N}_{j-i}(u^{i+2}|_{\Gamma_1}), w\rangle = \sum_{i=0}^{j-1} \langle \tilde{N}_{j-i}(v^{i+2}|_{\Gamma_1}), w\rangle - \sum_{i=1}^{j-1} \langle N_{j-i}(v^{i+2}|_{\Gamma_1}), w\rangle \\
&\quad = \langle \tilde{N}_j(v^2|_{\Gamma_1}), w\rangle - \sum_{i=1}^{j-1} \langle (\tilde{N}_{j-i} - N_{j-i})(v^{i+2}|_{\Gamma_1}), w\rangle\,.
\end{aligned} \tag{4.6}$$

The a priori estimate for u is derived from (4.6) by inserting $w = u^{j+1} - u^{j-1}$ and summing over j.

Step 1: Positivity of the left hand side of (4.6). We claim that, for $w = u^{j+1} - u^{j-1}$, the left hand side of (4.6) produces positive terms. In order to verify this, we must investigate the last term of the left hand side. A comparison with (3.11) and (3.5) shows that the term reads

$$\begin{aligned}
\sum_{i=0}^{j-1} \langle \tilde{N}_{j-i}(u^{i+2}|_{\Gamma_1}), w\rangle = \langle \tilde{F}(u)_j, w\rangle &= \int_{R\setminus B} \frac{\hat{u}^{j+1} - 2\hat{u}^j + \hat{u}^{j-1}}{\Delta t^2}\, (Ew) \\
&+ \int_{R\setminus B} \nabla(\theta \hat{u}^{j+1} + (1-2\theta)\hat{u}^j + \theta \hat{u}^{j-1}) \cdot a \nabla(Ew)\,,
\end{aligned} \tag{4.7}$$

where $\hat{u}^j$ is a solution of the discretized wave equation on $R \setminus B$ with Dirichlet data on Γ_1 given by u^j, E is the extension of a function on B to a function on R. Inserting the test function $w = u^{j+1} - u^{j-1}$ into (4.7) and exploiting that $\hat{u} - Eu$ can be used in the wave equation on $R \setminus \bar{B}$ for $\hat{u}$, we obtain

$$\begin{aligned}
\sum_{i=0}^{j-1} \langle \tilde{N}_{j-i}(u^{i+2}|_{\Gamma_1}), u^{j+1} - u^{j-1}\rangle &= \int_{R\setminus B} \frac{\hat{u}^{j+1} - 2\hat{u}^j + \hat{u}^{j-1}}{\Delta t^2}\, (\hat{u}^{j+1} - \hat{u}^{j-1}) \\
&+ \int_{R\setminus B} \nabla(\theta \hat{u}^{j+1} + (1-2\theta)\hat{u}^j + \theta \hat{u}^{j-1}) \cdot a \nabla(\hat{u}^{j+1} - \hat{u}^{j-1})\,.
\end{aligned} \tag{4.8}$$

We find that the test function $w = u^{j+1} - u^{j-1}$ in (4.6) provides, on the left hand side, integrals over R, namely

$$\begin{aligned}
&\int_R \frac{\hat{u}^{j+1} - 2\hat{u}^j + \hat{u}^{j-1}}{\Delta t^2}\, (\hat{u}^{j+1} - \hat{u}^{j-1}) \\
&\qquad + \int_R \nabla(\theta \hat{u}^{j+1} + (1-2\theta)\hat{u}^j + \theta \hat{u}^{j-1}) \cdot a \nabla(\hat{u}^{j+1} - \hat{u}^{j-1}) \\
&\quad = \int_R \left| \frac{\hat{u}^{j+1} - \hat{u}^j}{\Delta t} \right|^2 - \left| \frac{\hat{u}^j - \hat{u}^{j-1}}{\Delta t} \right|^2 + \hat{E}_j - \hat{E}_{j-1}\,,
\end{aligned} \tag{4.9}$$

where we used

$$\hat{E}_j := (\theta - \tfrac{1}{4})\|\hat{u}^{j+1} - \hat{u}^j\|_a^2 + \frac{1}{4}\|\hat{u}^{j+1} + \hat{u}^j\|_a^2$$

with the property

$$\hat{E}_j - \hat{E}_{j-1} = \theta\|\hat{u}^{j+1}\|_a^2 + (1-2\theta)\langle \hat{u}^j, \hat{u}^{j+1} - \hat{u}^{j-1}\rangle_a - \theta\|\hat{u}^{j-1}\|_a^2$$

as in the proof of Lemma 2.3. We sum over $j \in \{1, ..., k\}$ for arbitrary $k \leq m$ and obtain, because of the trivial initial conditions, on the left hand side of (4.6),

$$\int_R \left|\frac{\hat{u}^{k+1} - \hat{u}^k}{\Delta t}\right|^2 + (\theta - \tfrac{1}{4})\|\hat{u}^{k+1} - \hat{u}^k\|_a^2 + \frac{1}{4}\|\hat{u}^{k+1} + \hat{u}^k\|_a^2 .$$

Step 2: The right hand side of (4.6) *and conclusion.* We recall that we use $w = u^{j+1} - u^{j-1}$ in (4.6), and that we sum over $j \in \{1, ..., k\}$. Finally, we take the maximum over all $k \leq k_0$. With a trivial insertion of Δt and with C depending on $\theta > 1/4$ we obtain, using also the equivalence of the norms $\|.\|_a$ and $\|.\|_{X_B}$,

$$\begin{aligned}
&\max_{k\leq k_0}\left\|\frac{u^{k+1}-u^k}{\Delta t}\right\|_{L^2(B)}^2 + \max_{k\leq k_0}\|u^k\|_{X_B}^2 \\
&\quad\leq C\sum_{j=1}^{k}\Delta t\left|\left\langle \tilde{N}_j(v^2|_{\Gamma_1}), \frac{u^{j+1}-u^{j-1}}{\Delta t}\right\rangle\right| \\
&\qquad + C\sum_{j=1}^{k}\sum_{i=1}^{j-1}\Delta t\left|\left\langle (\tilde{N}_{j-i} - N_{j-i})(v^{i+2}|_{\Gamma_1}), \frac{u^{j+1}-u^{j-1}}{\Delta t}\right\rangle\right| \\
&\quad\leq C\max_{j\leq k_0}\left\|\tilde{N}_j(v^2|_{\Gamma_1})\right\|_{X_B'}\max_{k\leq k_0}\left\|\frac{u^{k+1}-u^k}{\Delta t}\right\|_{X_B} \\
&\qquad + C\max_{j\leq k_0}\left\|\sum_{i=1}^{j-1}(\tilde{N}_{j-i} - N_{j-i})(v^{i+2}|_{\Gamma_1})\right\|_{X_B'}\max_{k\leq k_0}\left\|\frac{u^{k+1}-u^k}{\Delta t}\right\|_{X_B} .
\end{aligned}$$

In order to absorb the last factor in the left hand side, we use the equivalence of the two norms $\|.\|_{L^2(B)}$ and $\|.\|_{X_B}$ on X_B with a constant $C_1(X_B)$. The above calculation then provides

$$\begin{aligned}
&\max_{k\leq k_0}\left\|\frac{u^{k+1}-u^k}{\Delta t}\right\|_{X_B}^2 + \max_{k\leq k_0}\|u^k\|_{X_B}^2 \\
&\leq C_1^2(X_B)\left(\max_{j\leq k_0}\left\|\tilde{N}_j(v^2|_{\Gamma_1})\right\|_{X_B'}^2 + m\max_{j\leq k_0-1}\max_{i\leq k_0-1}\left\|(\tilde{N}_j - N_j)(v^{i+2}|_{\Gamma_1})\right\|_{X_B'}^2\right) .
\end{aligned} \tag{4.10}$$

We define the expression $Y(k_0) := \max_{k\leq k_0}\|N_k - \tilde{N}_k\|^2$, recalling that operators N_k are measured as maps $X_B \to X_B'$. The left hand side of (4.10), by definition of u^k, gives an estimate for $N_k - \tilde{N}_k$, when the supremum over all g of X_B-norm at most 1 is taken. We emphasize that the operator $N_k - \tilde{N}_k$ contains also second time derivatives of u^j. This implies that, in the estimate

$$Y(k_0) \leq CC_2(\Delta t)\sup_g\left(\max_{k\leq k_0}\left\|\frac{u^{k+1}-u^k}{\Delta t}\right\|_{X_B}^2 + \max_{k\leq k_0}\|u^k\|_{X_B}^2\right) , \tag{4.11}$$

the constant $C_2(\Delta t) \sim m^2$ depends on the discretization in time. We introduce the quantity $\rho := \sup_{g,\|g\|\leq 1} \max_j \|\tilde{N}_j(v^2|_{\Gamma_1})\|^2$. In terms of Y, estimates (4.10) and (4.11) provide

$$\begin{aligned} Y(k_0) &\leq CC_1^2(X_B)C_2(\Delta t)\times \\ &\qquad \sup_g \left(\max_{j\leq k_0} \left\| \tilde{N}_j(v^2|_{\Gamma_1}) \right\|_{X_B'}^2 + m \max_{j\leq k_0-1} \max_{i\leq k_0-1} \left\| (\tilde{N}_j - N_j)(v^{i+2}|_{\Gamma_1}) \right\|_{X_B'}^2 \right) \\ &\leq CC_1^2(X_B)C_2(\Delta t)\left(\rho + mY(k_0-1)\right), \end{aligned}$$

where we exploited that the solution entries v^{i+2} are bounded. A finite number, $m-1$ to be precise, of applications of this estimate shows $Y(k_0) \leq C\rho$ with C depending on the discretization. It remains to note that, since all the operators $\tilde{N}_j$ are bounded, there holds $\rho = \sup_{g,\|g\|\leq 1} \max_j \|\tilde{N}_j(v^2|_{\Gamma_1})\| \leq C \sup_{g,\|g\|\leq 1} \|v^2|_{\Gamma_1}\|$. This provides (4.5). □

We note that the above proof provides for the constant C of (4.5) the estimate $C^2 \leq (C_1^2(X_B)C_2(\Delta t)m)^m \leq C_0(m^3(\Delta x)^{-2})^m$. Luckily, our numerical experiments show much better approximation results.

5. Numerical results

We implemented the numerical scheme presented in this article and solved wave equations in periodic media with the box-operator N. The goal of this section is to illustrate the method, to check it in terms of practical utility and, in particular, to measure computing times. All calculations are performed on a desktop computer and the Matlab computing environment is used. We performed tests in dimensions $n = 1$ and $n = 2$. In each case, we calculate a numerical solution that we call the "reference solution": We compute on a very large domain with Neumann conditions on the artificial boundaries; the truncation is such that the waves are not reaching the artificial boundaries. For clarity, we should mention that, strictly speaking, the numerical scheme has an infinite speed of propagation; this means that there are errors in the reference solution beyond discretization errors.

In our calculations we fix the parameters for the radiation box B, calculate in a pre-processing step the operator N for this radiation box, and solve then the wave equation in Ω with the transparent boundary condition introduced by N. The numerical solution obtained in this way is called the "radiation box solution" or only the "box solution". In all test cases, we compare the radiation box solution with the reference solution and compare the corresponding calculation times.

5.1. **Dimension $n = 1$.** In the one-dimensional case we investigate the wave equation (1.1) with the ε-periodic positive coefficient $a(x) = \sqrt{2} + \sin(2\pi\frac{x}{\varepsilon})$ for all $x \in \mathbb{R}$. We use $\varepsilon = 0.2$, a vanishing source term, a homogeneous initial condition $u(x,0) = 0$ in $(-3,\infty)$, the initial condition

$$\partial_t u(x,0) = \cos(-\pi x) \qquad \text{for } x \in (-2.5,-0.5),$$

and $\partial_t u(x,0) = 0$ for $x \in (-3,\infty) \setminus (-2.5,-0.5)$. We use the homogeneous Neumann condition $\partial_x u(-3,t) = 0$ for all $t > 0$ at $x = -3$.

The time discrete equation is given by (2.3). We always use $\theta = 1/4$ to ensure that the scheme is unconditionally stable, see Remark 2.2. The space discretization is given by the finite element method, we use equidistant points and the discretization

parameters $\Delta t = 0.001$ and $\Delta x = 0.002$. The solution is calculated up to time $T = 6$. We note that, in this 1D case, the finite element discretization is equivalent to a finite difference method.

The first step of the scheme is to calculate the operator N. We recall that the calculation of N is equivalent to solving a wave equation in the radiation box B for every point on the left boundary (only one point in the one-dimensional case). We perform the one-swipe calculation of N with the interval $B = (0, \varepsilon)$, a single periodicity cell, and use the same discretization parameters in B as in the rest of the domain. The calculation of N took 0.27 seconds. Using N as a transparent boundary condition on Γ_0 (of convolution type), we obtain the box solution with a calculation only on $(-3, 0)$, for results see Figure 1. Errors are shown in Figure 2. The L^2-norms for $T = 6$ are

$$\|u_{\text{ref}}(., T)\|_{L^2(\Omega)} = 1.37 \times 10^{-3},$$
$$\|u_{\text{ref}}(., T) - u_{\text{box}}(., T)\|_{L^2(\Omega)} = 1.72 \times 10^{-7}.$$

We recall that the boundary operator N is introduced in order to calculate the solution in the domain of interest Ω by discretizing only Ω and by calculating discrete solutions only on Ω. Nevertheless, the operator N also allows to extend the solution to all of R. We used this extension for illustration purposes in Figure 1, but we can, of course, also evaluate errors outside Ω. Results are shown in Figure 3. We observe that a reasonable approximation is obtained also outside of Ω, but errors are several magnitudes larger outside Ω than inside Ω.

Regarding calculation times we observed the following numbers: The computation of the reference solution took 100 seconds, the computation of the box solution took 12.38 seconds. This advantage of the domain truncation was to be expected since the calculation domain is smaller. The numbers show that, even when the calculation time for N of 0.29 seconds is taken into account, the box solution is calculated faster by a factor of more than 7. We note that, when multiple calculations in Ω must be performed, the preprocessing step of calculating N still has to be done only once.

5.2. **Dimension $n = 2$.** We start with the description of the coefficient a in the two-dimensional test case. In the periodicity cell $Y := [0, 1)^2 \subset \mathbb{R}^2$ we consider the disk $Q := B_{0.3}((0.5, 0.5)) = \{(y_1 - 0.5)^2 + (y_2 - 0.5)^2 \leq 0.09\} \subset Y$ and set

$$\tilde{a}(y_1, y_2) := \begin{cases} 1.5 & \text{for } (y_1, y_2) \in Q\,, \\ 1 & \text{for } (y_1, y_2) \in [0, 1]^2 \setminus Q\,. \end{cases}$$

For positive ε, an ε-periodic coefficient field a is defined by $a(x_1, x_2) := \tilde{a}(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})$. In our numerical experiments we use $\varepsilon = 0.2$.

We consider the domain $\Omega := (-3, 0) \times (0, 0.2)$. We always use a vanishing source $f = 0$, homogeneous initial data $u(., 0) = 0$ in Ω, vanishing initial velocity $\partial_t u(., 0) = 0$ outside a disk $\Sigma := B_{0.03}((-2, 0.15)) = \{(x_1, x_2) | (x_1 + 2)^2 + (x_2 - 0.15)^2 < (0.03)^2\}$, and

$$\partial_t u(x, 0) = \cos\left(\frac{\pi}{0.06}\sqrt{(x_1 + 2)^2 + (x_2 - 0.15)^2}\right) \text{ if } (x_1, x_2) \in \Sigma\,.$$

As discretization parameters we choose $\Delta t = 0.005$ and we compare three different mesh sizes for the spatial triangulation, namely $\Delta x = 0.04$, $\Delta x = 0.02$ and $\Delta x = 0.01$. In the numerical tests, we impose homogeneous Neumann boundary conditions on the left, upper, and lower boundaries.

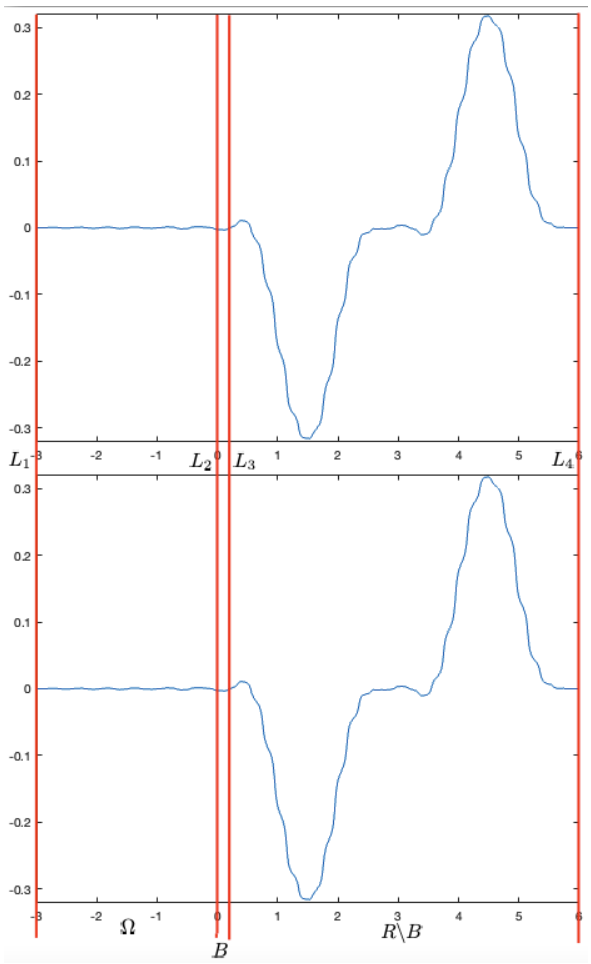

FIGURE 1. Two numerical solutions of a one-dimensional wave equation. The interval $(-3, 6)$ is decomposed into three parts: $\Omega = (-3, 0) = (L_1, L_2)$ is the domain of interest, $B = (0, \varepsilon) = (L_2, L_3)$ is the radiation box on which N is computed, $R \setminus B = (\varepsilon, 6) = (L_3, L_4)$ is the outer part that is used for the calculation of the reference solution. All solutions are shown at time $T = 6$. Top: The reference solution, calculated on $(-3, 6)$. Bottom: The box solution, obtained by solving a wave equation on $(-3, 0)$ with boundary condition given by N. For the above illustration, the box solution is extended to all of $(-3, 6)$ according to the operator N.

Figure 4 shows the reference solution and the box solution, calculated with $B = (0, \varepsilon) \times (0, \varepsilon) = (0, 0.2) \times (0, 0.2)$. The box solution is shown after its extension, which is calculated in a post-processing step from the boundary data and N. We see a good agreement of the two solutions. The absolute values of the difference is shown in Figure 6 for the time instances $T = 3$ and $T = 6$. We notice once more that the error propagates from the artificial boundary. The error for time instance $T = 6$ is given by Table 1.

Figure 5 and 6 shows the absolute values of the difference of the reference and the box solution both in Ω and in the extended domain. Again we see that the error is much higher on the right hand side where we perform the reconstruction of the

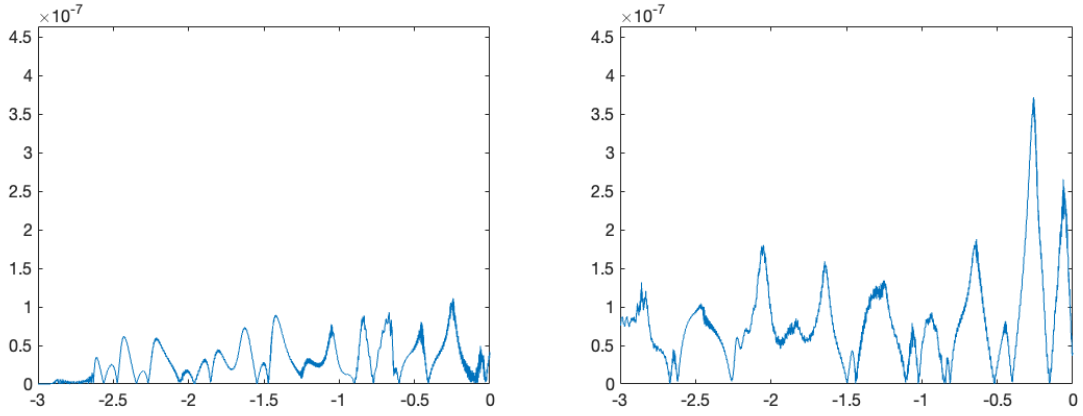

FIGURE 2. Errors in one space dimension: Absolute values of the difference between reference solution and box solution in $\Omega = (-3, 0)$ for the times $T = 3$ (left) and $T = 6$ (right). The maximal error is about 10^{-7} for $T = 3$ and about $4 \cdot 10^{-7}$ for $T = 6$.

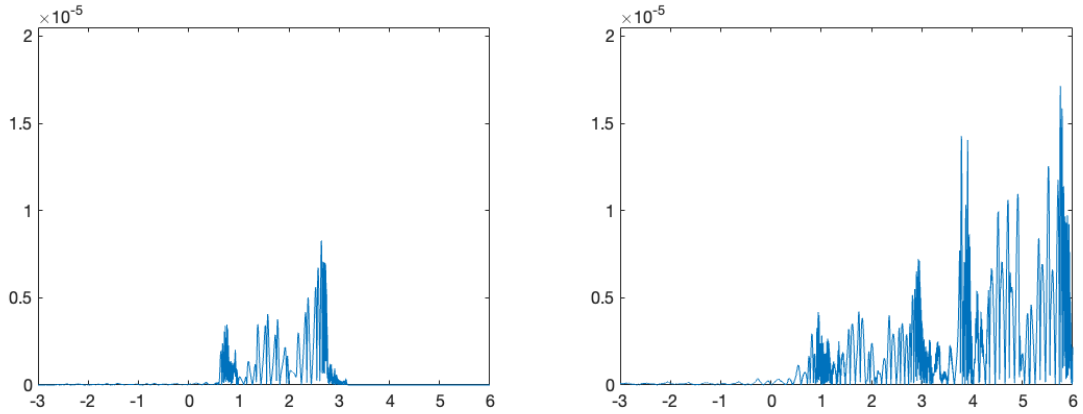

FIGURE 3. Errors outside the domain Ω in one space dimension: The figure shows the absolute values of the difference between the reference solution and the (extension of) the box solution. The whole interval $(-3, 6)$ is shown here. By comparing the errors at time $T = 3$ (left) and at time $T = 6$ (right), we see that the error is produced at the artificial boundary and propagates to the left and to the right. The error remains of the order 10^{-6} even outside Ω.

solution. Here we consider a reconstruction domain $R := (0, 7) \times (0, 0.2)$ and not $(0, 6) \times (0, 0.2)$ since we notice a reflection from the boundary which is again due to the infinite speed of propagation of the numerical scheme. This effect disappears when we consider a larger domain.

The time consumption for the different discretization parameters are given in Table 2. For $\Delta x = 0.02$, the box solution is calculated faster by a factor larger than 13. For $\Delta x = 0.01$, we gain by a factor larger than 20. Even when we consider the time to compute N, the time to obtain the box solution takes less time than the reference

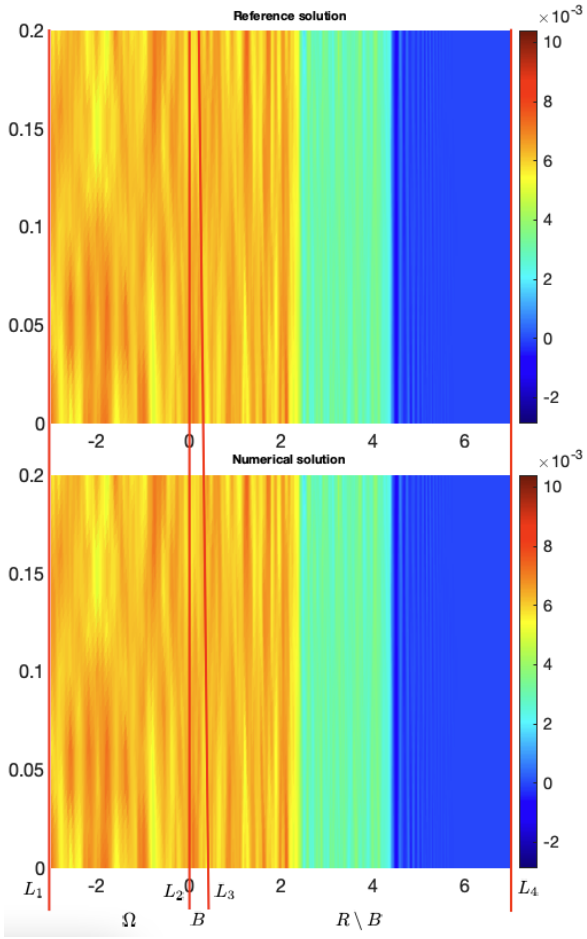

FIGURE 4. Two solutions at time $T = 6$, calculated with $\Delta x = 0.02$. Similar to the one-dimensional setting shown in Figure 1 the domain of interest is $\Omega = \{x_1 \in (L_1, L_2)\}$, the radiation box is $B = \{x_1 \in (L_2, L_3)\}$, and the outer domain is $\{x_1 \in (L_3, L_4)\}$ with $L_1 = -3$, $L_2 = 0$, $L_3 = \varepsilon$, $L_4 = 6$. Top: The reference solution. Bottom: The box solution, which is calculated in $\Omega = (-3, 0) \times (0, 0.2)$, and concatenated with its extension based on its boundary values.

solution. We recall that the preprocessing step has to be done only once when many computations with the same coefficient field have to be performed.

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	$\|u_{\text{ref}}(.,T)\|_{L^2(\Omega)}$	$\|u_{\text{ref}}(.,T) - u_{\text{box}}(.,T)\|_{L^2(\Omega)}$
$\Delta x = 0.04$	5.06×10^{-3}	1.67×10^{-5}
$\Delta x = 0.02$	4.81×10^{-3}	3.21×10^{-9}
$\Delta x = 0.01$	4.72×10^{-3}	1.07×10^{-13}

TABLE 1. Norm of the solution and norm of the difference in two dimensions for different discretization parameters, calculated with $\Delta t = 0.005$ and for the time instance $T = 6$. Remarkable is the very strong decrease of the error with the refinement of the grid. This can be understood as follows: Errors are produced by the infinite speed of propagation of the numerical scheme. When the space resolution is fine enough, almost no information from the left boundary of B reaches the right boundary of B after time Δt. Errors are produced by this information exchange, compare Theorem 4.1.

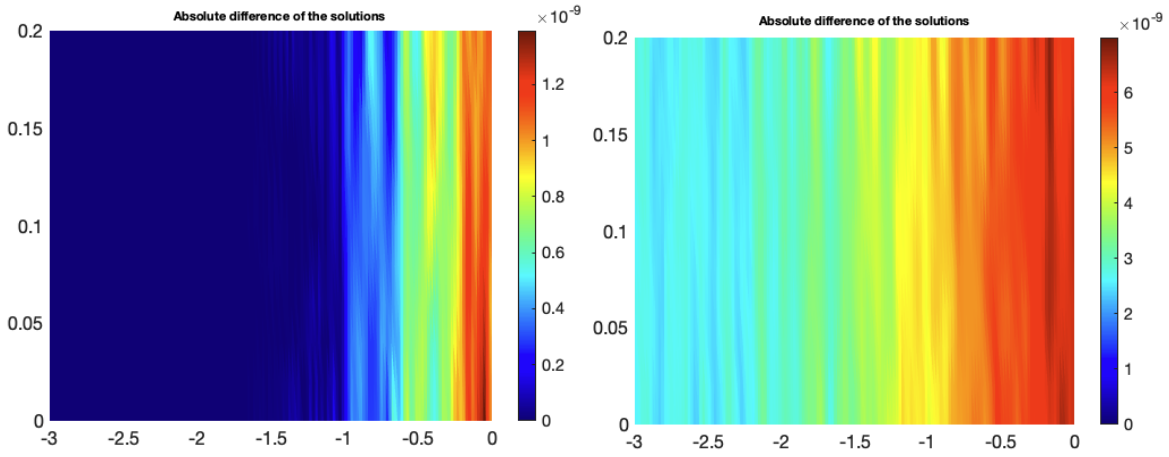

FIGURE 5. The absolute values of the difference between the reference and the box solution in two dimensions for $\Delta x = 0.02$. Left: $T = 3$. Right: $T = 6$.

	t_{ref}	t_N		t_{box}	
		t_{right}	t_{inv}	t_{right}	t_{inv}
$\Delta x = 0.04$	1.37	3.38	0.02	1	0.09
$\Delta x = 0.02$	29.85	4.81	0.24	1.38	0.81
$\Delta x = 0.01$	385	7.99	1.16	1.48	16.3

TABLE 2. Time consumption for different discretization parameters, calculated in two dimensions with $\Delta t = 0.005$ and $T = 6$. The numbers t_{ref}, t_N, and t_{box} refer to the time to compute the reference solution, the N operator, and the radiation box solution respectively, all in seconds. The number t_{right} gives the time to construct the right hand side and t_{inv} the time to solve the system of linear equations.

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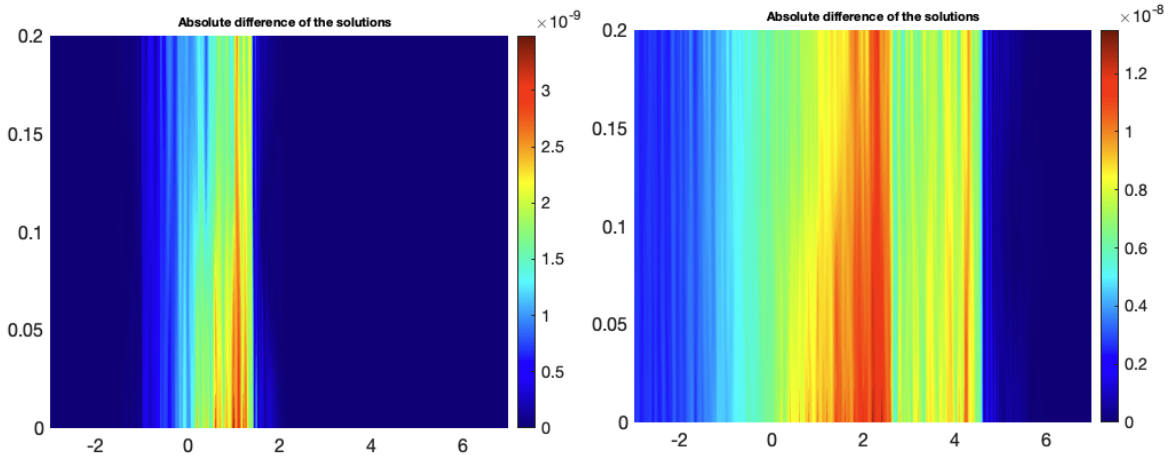

FIGURE 6. The error outside the domain Ω. We see again that the error propagates from the artificial boundary to the left and to the right. Left: $T = 3$. Right: $T = 6$.

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