Creeping fronts in degenerate reaction diffusion systems

S. Heinze[1], B. Schweizer[2]

[1] *Max-Planck-Institute for Mathematics in the Sciences, Inselstr. 22-26, D-04103 Leipzig, Germany*
[2] *Institut für Angewandte Mathematik, Im Neuenheimer Feld 294, D-69120 Heidelberg, Germany*

March 16, 2005

Abstract: We study systems of reaction diffusion type for two species in one space dimension and investigate the dynamics in the case that the second species does not diffuse. We consider competing species with two stable equilibria and front solutions that connect the two stable states. A free energy function determines a preferred state. If the diffusive species is preferred, traveling waves may appear. Instead, if the non-diffusive species is preferred, stationary fronts are the only monotone traveling waves. We show that these fronts are unstable and that the non-diffusive species can propagate at a logarithmic rate.

1 Introduction

Reaction diffusion equations are a common model in the description of nonlinear systems such as chemical processes, ecological systems, or nerve-pulse propagation. Recently, Luckhaus and Triolo [7] studied a discrete stochastic model for tumor growth and derived a limiting continuous model for the densities u and v of malignant and healthy tissue. In one space dimension the system can be written as

$$\begin{aligned} \partial_t u &= \partial_x^2 u + f(u,v), \\ \partial_t v &= g(u,v). \end{aligned} \tag{1.1}$$

In the discrete system the malignant cells are mobile which results in a positive diffusivity in the first equation, but the healthy cells are immobile and there is no diffusion in the second equation. A feature of the nonlinearities in (1.1) is competition, which results from the fact that in the discrete model the two species are competing for space. In the continuous model competition is expressed by

$$\partial_v f(u,v) \le 0, \quad \partial_u g(u,v) \le 0. \tag{1.2}$$

Due to competition, (1.1) defines a monotone dynamical system. The monotonicity has many analytical consequences such as a comparison principle for solutions. In the case of a positive diffusion in the second equation the existence and stability of traveling fronts is well-understood [6, 9]. We will analyze traveling waves for (1.1) and see, how the missing diffusivity changes the qualitative picture.

Our setting will be such that at position $x = -\infty$ only healthy cells are present, while at $x = +\infty$ only tumor cells are present. We are therefore interested in front solutions that connect the states $S_- = (0,1)$ (healthy tissue) with $S_+ = (1,0)$ (tumor tissue),

$$\begin{aligned} (u,v)(x) &\to S_- = (0,1) \qquad \text{for } x \to -\infty, \\ (u,v)(x) &\to S_+ = (1,0) \qquad \text{for } x \to +\infty. \end{aligned} \tag{1.3}$$

It is possible to associate to every state $(u,v) \in [0,1]^2$ a free energy $H(u,v)$. We will see that the energy difference between the states S_- and S_+ determines the dynamics of the system. In the case $H(S_-) > H(S_+)$, the system prefers the tumor (state $S_+ = (1,0)$) over the healthy state $S_- = (0,1)$. In fact, in this case there exist traveling wave solutions with non-vanishing speed, i.e. the tumor cells can invade at a finite rate. Furthermore, these waves are unique and stable (compare [5]).

In this work we study the opposite inequality,

$$H(S_-) < H(S_+). \tag{1.4}$$

In this case one may expect waves traveling right, corresponding to an invasion of healthy cells. Instead, no traveling waves with finite speed exist and there are only stationary front solutions. The phenomenon has some similarity to the blocking of propagation in an inhomogeneous medium with a highly varying diffusivity, where the existence of steady states prevents propagation [4]. We analyze the stability of the blocked waves. It turns out that the stationary fronts are unstable and that a sublinear penetration may occur. A global analysis shows that, with an arbitrarily small perturbation of the front, an invasion of the v-component at a logarithmic rate is possible.

For a similar competitive system we refer to Aronson, Tesei and Weinberger [1] concerning a model for pattern formation. They considered (1.1) on a bounded domain and with a diffusion coefficient in the first equation that depends on the second species. They show the existence of many discontinuous steady states and verify their stability. In their model, the assumptions on the reaction term differ from ours, in particular, the bistable case is not considered.

Assumption 1.1. *The nonlinearities $f, g \in C^2([0,1]^2, \mathbb{R})$ satisfy*

1. *Preserving positivity: $f(0,v) = 0 = g(u,0)$ for all $u, v \in [0,1]$.*
2. *Bistability: There are exactly two stable equilibria $S_- = (0,1)$ and $S_+ = (1,0)$, and two linearly unstable equilibria $(0,0)$ and (u_{saddle}, v_{saddle}).*
3. *Strict Competition: $\partial_v f(u,v) < 0$ and $\partial_u g(u,v) < 0$ for $u, v \in (0,1)$.*
4. *All nontrivial solutions (u,v) of $g(u,v) = 0$ are given by $u = \Gamma(v)$ for a monotonically decreasing function Γ. Setting $\Gamma(1) = 0$ and $\Gamma(0) = u^\circ$, Γ has an inverse $\gamma \in C^1([0,u^\circ],[0,1])$, which we extend trivially to a function $\gamma \in C^0([0,1],[0,1])$.*
5. *Non-degeneracy: For some $c, C > 0$, and all $u \in [0, u^\circ)$,*

$$c \leq -\partial_u \gamma \leq C, \quad Dg(u,\gamma(u)) \neq 0, \quad \partial_u \partial_v g(u,0) < 0. \tag{1.5}$$

In item 2., linearly stable means that the two eigenvalues of the Jacobian $D_{(u,v)}(f,g)$ have negative real part, linearly unstable that at least one eigenvalue has positive real part.

Our main results are collected in the following theorem.

Theorem 1.2. *Let f, g satisfy Assumption 1.1 and let S_- be the preferred state, i.e. $H(S_-) < H(S_+)$ with H defined in (2.1). Then there exists a unique monotone traveling wave solution (u,v) of (1.1) with asymptotics (1.3). This wave is blocked, the wave speed is $c = 0$.*

The stationary solution is unstable. There are initial data, arbitrarily close to the stationary front in $L^2(\mathbb{R})$, such that the corresponding time-dependent solutions satisfy, for all $x \in \mathbb{R}$,

$$(u,v)(x,t) \to S_- \quad \text{for} \quad t \to \infty.$$

Proof. The first part of the theorem is a special case of Theorem 2.1, the second part is shown in Theorem 3.3, where, additionally, the rate of convergence is determined. □

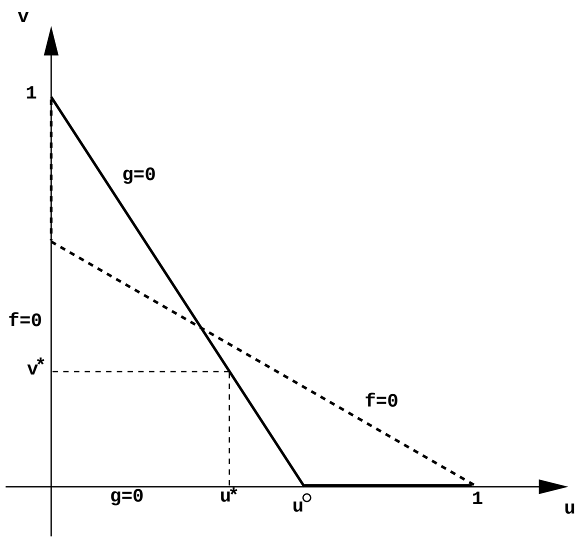

Figure 1: Possible zero-sets in the phase diagram

2 A Lyapunov function

The system (1.1) possesses a free energy function H, as was, to our knowledge, first noted by S. Luckhaus. The free energy allows to give a sufficient and necessary condition for the existence of standing waves. We set

$$H(u,v) = -\int_0^u f(\sigma,v)\,d\sigma - \int_v^1 \int_0^{\Gamma(\tau)} \partial_v f(\sigma,\tau)\,d\sigma\,d\tau, \tag{2.1}$$

and recall $g(\Gamma(\tau),\tau) = 0$. We have normalized H such that $H(S_-) = H((0,1)) = 0$. We want to verify that H has indeed the meaning of an energy. For functions $u, v \in L^\infty(\mathbb{R},[0,1])$ with $\partial_x u \in L^2(\mathbb{R})$ and $H(u,v) - H((1,0))\chi_+ \in L^1(\mathbb{R})$ we set

$$E((u,v)) := \int_{\mathbb{R}} \left\{ \frac{1}{2}|\partial_x u|^2 + H(u,v) - H((1,0))\chi_+ \right\} dx. \tag{2.2}$$

The term $H((1,0))\chi_+$ is inserted to ensure finiteness of the integral for solutions sharing the asymptotics of the traveling wave. Classical solutions satisfy

$$\partial_t E((u,v)) = \int_{\mathbb{R}} (-\partial_x^2 u)\partial_t u + \partial_u H \partial_t u + \partial_v H \partial_t v$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} (-\partial_x^2 u - f(u,v))\partial_t u \\
&\quad + \int_{\mathbb{R}} \left(- \int_0^u \partial_v f(\sigma, v)\ d\sigma + \int_0^{\Gamma(v)} \partial_v f(\sigma, v)\ d\sigma \right) \partial_t v \\
&= \int_{\mathbb{R}} \left\{ -|\partial_t u|^2 + \int_u^{\Gamma(v)} \partial_v f(\sigma, v)\ d\sigma \ \cdot\ g(u,v) \right\}.
\end{aligned}
$$

We exploit that $\partial_v f$ is non-positive and that

$$
g(u,v) \geq 0 \text{ for } u < \Gamma(v), \quad g(u,v) \leq 0 \text{ for } u > \Gamma(v),
$$

to conclude that E is decreasing.

2.1 Monotone traveling waves

Our next aim is to find stationary solutions of (1.1). In the stationary case, the second equation reduces to the algebraic relation $g(u,v) = 0$. For fixed $u \in [0,1]$, the equation $g(u,v) = 0$ admits only the solutions $v = 0$ and $v = \gamma(u)$ (not necessarily different), hence all stationary solutions of (1.1) can be found by solving with $\psi : \mathbb{R} \to \{0,1\}$ the scalar equation

$$
\begin{aligned}
\partial_x^2 u + f(u,v) &= 0, \\
v(x) &= \psi(x)\gamma(u(x)).
\end{aligned} \tag{2.3}
$$

If we furthermore demand that v is monotone and connects the states 1 and 0, we must chose for ψ the characteristic function of an interval $(-\infty, x^*)$. After a translation of the solution we may assume $x^* = 0$. By monotonicity of γ it suffices to study monotonically increasing functions u that connect 0 with 1 and solve (2.3) with $\psi = \chi_-$, the characteristic function of $(-\infty, 0)$.

Theorem 2.1 (Existence and uniqueness). *Assume 1.1, 1.-4., and*

$$
H(S_-) \leq H(S_+). \tag{2.4}
$$

There exists a monotone function $u \in L^\infty(\mathbb{R})$, $\partial_x u \in L^2(\mathbb{R})$, connecting 0 with 1 such that, with $v(x) = \gamma(u(x))\chi_-(x)$, the pair (u,v) is a weak stationary solution of (1.1). There exists no monotone traveling wave solution with speed $c \neq 0$ and, up to translations, only the above standing wave. The condition (2.4) is necessary for the existence of a stationary solution u.

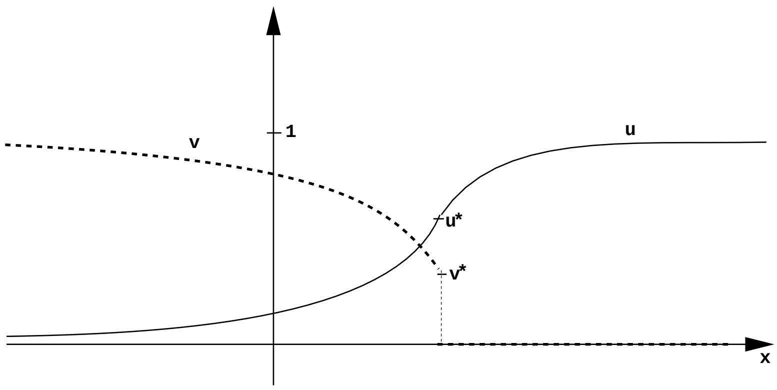

Figure 2: A standing wave solution

The qualitative behavior of the standing wave is depicted in Figure 2. Note that the profile v is non-smooth; there is a unique point where v or $\partial_x v$ has a discontinuity.

Proof. Step 1. Necessity. Let firstly $(u,v):\mathbb{R}\to\mathbb{R}^2$ be a monotone stationary solution of (1.1). After a shift we may assume that (u,v) solves (2.3) with $\psi = \chi_-$. We will calculate an integral formula for the solution. For $U\in\mathbb{R}$ we set

$$f_0(U) = f(U,0), \qquad f_1(U) = f(U,\gamma(U)), \tag{2.5}$$

and introduce

$$F_0(U) = -\int_U^1 f_0(\eta)\ d\eta, \qquad F_1(U) = \int_0^U f_1(\eta)\ d\eta. \tag{2.6}$$

By standard elliptic theory, the solution is smooth away from $x=0$, which allows the multiplication of $\partial_x^2 u + f(u,v) = 0$ with $\partial_x u$ and to conclude

$$\partial_x\left[\frac{1}{2}|\partial_x u(x)|^2 + F_0(u(x))\right] = 0 \quad \text{for } x>0, \tag{2.7}$$

$$\partial_x\left[\frac{1}{2}|\partial_x u(x)|^2 + F_1(u(x))\right] = 0 \quad \text{for } x<0. \tag{2.8}$$

We have defined F_0 and F_1 such that $F_0(1)=0$ and $F_1(0)=0$. Thus, the squared brackets of (2.7) and (2.7) both vanish on $\mathbb{R}$, and, in particular, they coincide in $x = 0$. Thus, a necessary condition for the existence of the solution u is the existence of $u^* = u(0) \in \mathbb{R}$ such that

$$A(u^*) := \int_0^{u^*} f_1 + \int_{u^*}^1 f_0 = F_1(u^*) - F_0(u^*) = 0. \tag{2.9}$$

It remains to study the function A. There holds $A(0) > 0$ by definition, A is monotonically decreasing on $U \in (0, u^\circ)$ by $f_1 \le f_0$, and $A(U) = A(u^\circ)$ for all $U \ge u^\circ$. Therefore the solvability of (2.3) implies $A(u^\circ) \le 0$.

In order to relate this condition to the energy, we calculate $H(S_+) = H((1,0))$ as

$$\begin{aligned} H(S_+) &= -\int_0^1 f(u,0)\, du - \int_0^1 \int_0^{\Gamma(v)} \partial_v f(u,v)\, du\, dv \\ &= -\int_0^1 f(u,0)\, du - \int_0^{u^\circ} f(u,\gamma(u))\, du + \int_0^{u^\circ} f(u,0)\, du \\ &= -\int_{u^\circ}^1 f_0(u)\, du - \int_0^{u^\circ} f_1(u)\, du = -A(u^\circ). \end{aligned}$$

Because of $H(S_-) = 0$, the necessary condition $A(u^\circ) \le 0$ is equivalent to (2.4).

Step 2. Existence and uniqueness of standing waves. We can define F_0, F_1, and A as in Step 1. Again, $A(0)$ is positive by definition. Assumption (2.4) yields that $A(u^\circ)$ is non-positive. We therefore find a point $u^* \in (0, u^\circ]$ with $A(u^*) = 0$. Because of

$$\partial_u A(u) = f(u,\gamma(u)) - f(u,0) < 0$$

for $u \in (0, u^\circ)$, this zero u^* is unique. We will construct a solution with $u(0) = u^*$. Equation (2.7) suggests to set

$$p := \left(2\int_{u^*}^1 f(s,0)\, ds\right)^{1/2} > 0,$$

and to define u as the solution of (2.3) on both $\mathbb{R}_+$ and $\mathbb{R}_-$ for the initial values $u(0) = u^*$ and $\partial_x u(0) = p$.

Let us first consider the set $x > 0$. The same calculation as before yields that u solves

$$\partial_x u(x) = \sqrt{2F_0(u(x))},$$

where F_0 is positive on $[u^*, 1)$ and $F_0(1) = 0$. Furthermore $f(1,0) = 0$ implies that $\sqrt{2F_0(U)}$ is a Lipschitz function in $U \in [u^*, 1]$. We conclude that the solution $u(x)$ is monotone and satisfies $u(x) \to 1$ for $x \to +\infty$.

The solution $u(x)$ on the set $x < 0$ is analyzed in the same way. We consider u as the solution of

$$\partial_x u(x) = \sqrt{2F_1(u(x))}.$$

We claim that $F_1(U) = 0$ has no solution on $(0, u^*]$. Once this is shown, the proof is finished as for $x > 0$.

Since $F_1(u^*) > 0$ and $F_1(0) = 0$, it suffices to assume that F_1 has a non-positive minimum $\tilde{u}$ on $(0, u^*)$, and to find a contradiction. In the minimum we have

$$F_1(\tilde{u}) \leq 0,\ 0 = F_1'(\tilde{u}) = -f(\tilde{u}, \gamma(\tilde{u})),$$

whence $\tilde{u}$ is the unique value for u such that in $S = (\tilde{u}, \gamma(\tilde{u}))$ there holds $f(S) = 0$. In this point we have $(\partial_u + \gamma'\partial_v)f(S) > 0$ by Assumption 1.1, 2. This yields a contradiction, since in a minimum of F_1 necessarily

$$0 \leq F_1''(\tilde{u}) = -(\partial_u + \gamma'\partial_v)f(S).$$

The uniqueness of the solution u follows from the uniqueness of the values $u(0) = u^* \in [0, u^\circ]$ and $\partial_x u(0) = p$, and from the Lipschitz properties of the functions $\sqrt{2F_i}$. It remains to note that, in the case $H(S_+) = H(S_-)$, the choice of any other $u^* \in [u^\circ, 1]$ yields only a shifted version of the above solution, since F_0 and F_1 coincide on $[u^\circ, u^*]$.

Step 3. Other traveling wave solutions. A traveling wave solution is a pair $u, v : \mathbb{R} \to \mathbb{R}$ together with a wave-speed $c \in \mathbb{R}$, such that $(x,t) \mapsto (u,v)(x+ct)$ is a weak solution of the time-dependent problem. For the proof of Theorem 2.1 it remains to exclude the existence of monotone traveling wave solutions with $c \neq 0$. To this end, let $(u,v) : \mathbb{R} \ni \xi \mapsto (u(\xi), v(\xi))$ be monotone with the asymptotics (1.3), and $c \in \mathbb{R}$ such that

$$\begin{aligned} c\partial_\xi u &= \partial_\xi^2 u + f(u,v), \\ c\partial_\xi v &= g(u,v). \end{aligned} \tag{2.10}$$

We note that, in the case $c \neq 0$, an iteration argument yields that every weak solution of system (2.10) is necessarily C^∞.

a) Exclusion of positive c. To derive a sign condition on $c \neq 0$ we calculate

$$\begin{aligned} c\partial_\xi\left(-\frac{1}{2}|\partial_\xi u|^2 + H(u,v)\right) &= c((-\partial_\xi^2 u + \partial_u H)\partial_\xi u + \partial_v H \partial_\xi v) \\ &= c(-\partial_\xi^2 u - f(u,v))\partial_\xi u + \int_u^{\Gamma(v)} \partial_v f(s,v)\, ds \ \cdot\ g(u,v) \\ &= -c^2|\partial_\xi u|^2 - \int_{\Gamma(v)}^u \partial_v f(s,v)\, ds \ \cdot\ \int_{\Gamma(v)}^u \partial_u g(s,v)\, ds. \end{aligned}$$

This expression is non-positive on $\mathbb{R}$, since, for $u(\xi) > \Gamma(v(\xi))$, both integrals of the last line are positive, in the opposite case both integrals are negative. Since the traveling wave connects the equilibrium $S_- = (0,1)$ with $S_+ = (1,0)$, an integration over $\mathbb{R}$ yields

$$c \cdot [H(S_+) - H(S_-)] \leq 0. \tag{2.11}$$

In the case of interest, $H(S_-) \leq H(S_+)$, we conclude $c \leq 0$. This is the information that we can expect from the energy difference: Since the state S_- is preferred, waves can, if they have a non-trivial speed, only travel to the right.

b) Exclusion of negative c. Assume now $c < 0$. We will find a contradiction by exploiting the missing diffusivity in the v-equation of (2.10). The monotonicity of v implies $g(u(\xi), v(\xi)) = c\partial_\xi v(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, and hence $v \leq \gamma(u)$ on $\mathbb{R}$. Then $u(\xi) \to 1$ for $\xi \to +\infty$ implies $u \geq u^\circ$ on $[\xi_0, \infty)$ for some $\xi_0 \in \mathbb{R}$ and thus $v \equiv 0$ on $[\xi_0, \infty)$. But as a solution of the v-equation of (2.10) with initial values $v(\xi_0) = 0$, v must vanish identically and can not satisfy the asymptotic behavior at $\xi = -\infty$. A contradiction. □

2.2 Non-monotone waves: Lack of asymptotic stability

Due to the absence of diffusion in the v-equation we can characterize stationary solutions of (1.1) by the scalar equation (2.3). So far, we have restricted our analysis to monotone solutions and thus to $\psi = \chi_-$. We here want to show that, leaving the class of monotone solutions, there are many stationary solutions of (1.1). Theorem 2.1 provided a solution u corresponding to $\psi = \chi_-$, that is, $v(x) = \chi_-(x)\gamma(u(x))$. With the help of the implicit function theorem, we will show that there are many other solutions nearby. This proves that the monotone front is not asymptotically stable in any L^p-norm for $p < \infty$.

Proposition 2.2 (Many non-monotone stationary solutions). *Assume $H(S_-) < H(S_+)$. There exists $\bar{\delta} > 0$ such that for all $0 < \delta_1 < \delta_2 < \bar{\delta}$ there is a stationary solution of (1.1) with $v(x) = \chi(x)\gamma(u)$, where χ is the characteristic function of the set $(-\infty, 0) \cup (\delta_1, \delta_2)$, that is, a function u solving*

$$\partial_\xi^2 u + f(u, \chi\gamma(u)) = 0. \tag{2.12}$$

Proof. With $P = (0, 0, u^*, u^*, u^*)$ we construct a function

$$G : \mathbb{R}^5 \supset B_\rho(P) \to \mathbb{R}^3, \quad G(\delta_1, \delta_2, \bar{u}_0, \bar{u}_1, \bar{u}_2) \in \mathbb{R}^3,$$

with $G(P) = 0$ and such that zeros of G correspond to solutions of (2.12). We then use the implicit function theorem to find points $(\bar{u}_0, \bar{u}_1, \bar{u}_2)$ solving $G(\delta_1, \delta_2, \bar{u}_0, \bar{u}_1, \bar{u}_2) = 0$ for small δ_1, δ_2.

We consider the two $(i = 0, 1)$ autonomous equations

$$\begin{aligned} &\partial_\xi^2 u + f_i(u) = 0, \qquad \text{for} \\ &f_0(u) := f(u, 0),\ f_1(u) := f(u, \gamma(u)). \end{aligned}$$

The two equations define two flow maps,

$$\Phi_x^i : \mathbb{R}^2 \ni (u(0), \partial_\xi u(0)) \mapsto ((\Phi_x^i)_1, (\Phi_x^i)_2) := (u(x), \partial_\xi u(x)) \in \mathbb{R}^2.$$

Given $(\delta, \bar{u}) = (\delta_1, \delta_2, \bar{u}_0, \bar{u}_1, \bar{u}_2)$ we set

$$p := \left(-2 \int_0^{\bar{u}_0} f_1(s)\, ds \right)^{1/2},$$

which we expect to be the derivative of u in $x = 0$. We furthermore set

$$\begin{aligned} G_1(\delta, \bar{u}) &:= (\Phi_{\delta_1}^0)_1(\bar{u}_0, p) - \bar{u}_1, \\ G_2(\delta, \bar{u}) &:= (\Phi_{\delta_2 - \delta_1}^1)_1 \circ \Phi_{\delta_1}^0(\bar{u}_0, p) - \bar{u}_2, \\ G_3(\delta, \bar{u}) &:= \frac{1}{2} \left[(\Phi_{\delta_2 - \delta_1}^1)_2 \circ \Phi_{\delta_1}^0(\bar{u}_0, p) \right]^2 - \int_{(\Phi_{\delta_2-\delta_1}^1)_1 \circ \Phi_{\delta_1}^0(\bar{u}_0, p)}^1 f_0(s)\, ds. \end{aligned}$$

By definition, $G(P) = 0$. If we find a point $(\delta, \bar{u})$ with $G(\delta, \bar{u}) = 0$, we can construct a solution to (2.12) by gluing together the above flows. It remains to calculate the derivatives of G in the point P.

$$\begin{aligned} \partial_{\bar{u}} G_1(P) \cdot \langle \bar{w}_0, \bar{w}_1, \bar{w}_2 \rangle &= \bar{w}_0 - \bar{w}_1, \\ \partial_{\bar{u}} G_2(P) \cdot \langle \bar{w}_0, \bar{w}_1, \bar{w}_2 \rangle &= \bar{w}_0 - \bar{w}_2, \\ \partial_{\bar{u}} G_3(P) \cdot \langle \bar{w}_0, \bar{w}_1, \bar{w}_2 \rangle &= p \cdot \partial_{\bar{u}_0} p \cdot \bar{w}_0 + f_0(u^*) \cdot \bar{w}_0 \\ &= \partial_{\bar{u}_0} \frac{1}{2} |p|^2 + f_0(u^*) \cdot \bar{w}_0 \\ &= (-f_1(u^*) + f_0(u^*)) \cdot \bar{w}_0. \end{aligned}$$

The strict inequality $H(S_-) < H(S_+)$ implies $u^* < u^\circ$ and thus $f_1(u^*) < f_0(u^*)$. It follows that the 3×3-matrix $\partial_{\bar{u}} G(P)$ is invertible. The implicit function theorem can be applied and provides the solutions. $\square$

Note that, with the same proof, one can construct solutions with any finite (odd) number of interfaces.

3 Orbital stability of a creeping front

In this section we study again the strict inequality $H(S_-) < H(S_+)$. We have seen that there exists a monotone stationary solution (u_s, v_s) of (1.1) of the special form $v_s(x) = \gamma(u_s(x))\chi_-(x)$, i.e. u_s solves

$$\partial_x^2 u_s + f(u_s, \gamma(u_s)\chi_-) = 0. \tag{3.1}$$

and $u_s(x) \to 0$ for $x \to -\infty$, $u_s(x) \to 1$ for $x \to +\infty$. By constructing other stationary solutions that are nearby we have seen in Subsection 2.2 that (u_s, v_s) is not asymptotically stable for equation (1.1). We will see that the solution is indeed unstable. This is shown by constructing small perturbations of (u_s, v_s) such that the corresponding solutions $(u, v)(t)$ of the initial value problem move arbitrarily far from (u_s, v_s).

More precisely, we will construct (u, v) which keep the *shape* of (u_s, v_s) for all times, but the functions are shifted by a penetration depth $X(t) \in \mathbb{R}$, i.e. the deviations $u(X(t)+., t) - u_s(.)$ and $v(X(t)+., t) - v_s(.)$ remain small. By showing that the position $X(t)$ satisfies $X(t) \to \infty$ for $t \to \infty$, we prove Theorem 1.2.

3.1 The linearized operator

In this subsection we derive some properties of the linearized system. They will be the key for our construction of orbitally stable solutions. We linearize (2.12) in u_s. The corresponding linear operator is

$$L : H^1(\mathbb{R}) \ni w \mapsto Lw \in H^{-1}(\mathbb{R}), \quad Lw(x) = -\partial_x^2 w(x) - \mu(x) w(x) \tag{3.2}$$

with

$$\mu(x) = \begin{cases} (\partial_u + \gamma' \partial_v) f(u_s(x), v_s(x)) & \text{for } x < 0, \\ \partial_u f(u_s(x), 0) & \text{for } x > 0. \end{cases}$$

Note that $\mu(x) \to \mu_- < 0$ for $x \to -\infty$ and $\mu(x) \to \mu_+ < 0$ for $x \to +\infty$.

The comparison principles. In order to have comparison principles, one needs the existence of a non-negative function w with $Lw \geq 0$. Such a function is given to us by construction, we can choose $w = \partial_x u_s$. We have the following two comparison principles.

1. Elliptic equation. Assume that there exists $w \in H^1(\mathbb{R})$, $w > 0$, with $Lw \geq 0$. Then every function $z \in H^1(\mathbb{R})$ solving

$$Lz \geq 0,$$

$$\limsup_{x\to\pm\infty} \frac{z(x)}{w(x)} \geq 0,$$

satisfies $z \geq 0$.

2. Parabolic equation. Every function $z : [0,\infty) \to H^1(\mathbb{R})$ solving $\partial_t z + Lz \geq 0$ and $z(0) \geq 0$ satisfies $z(t) \geq 0$ for all $t \geq 0$.

We refer to [8] for proofs. We will actually not use the elliptic comparison principle in the above form, but a variant which is developed in the next two paragraphs.

Positivity of the bilinear form. We study the bilinear form corresponding to L,

$$\Phi(u) := \int_{\mathbb{R}} |\partial_x u(x)|^2 - \mu(x)|u(x)|^2 \, dx. \tag{3.3}$$

Lemma 3.1. *Assume that there exists a non-negative function $w \in H^1(\mathbb{R})$ with $Lw \geq 0$, $Lw \neq 0$. Then the bilinear form Φ is positive,*

$$\Phi(u) > 0 \qquad \forall u \in H^1(\mathbb{R}), u \neq 0, \tag{3.4}$$

and coercive, i.e. for a constant $c > 0$ we have

$$\Phi(u) \geq c\|u\|_{H^1}^2 \qquad \forall u \in H^1(\mathbb{R}). \tag{3.5}$$

Proof. The definition of Φ implies that for some constants $c, C > 0$

$$\Phi(u) \geq -C\|u\|_{L^2}^2 + c\|u\|_{H^1}^2 \qquad \forall u \in H^1(\mathbb{R}). \tag{3.6}$$

Step 1. In order to show (3.4), let us assume that for some $\bar{u} \neq 0$ we have $\Phi(\bar{u}) \leq 0$. We minimize Φ on the 1-sphere of L^2 and find a sequence of functions u_n with $\|u_n\|_{L^2} = 1$. Without loss of generality we can assume $\Phi(u_n) \leq 0$. Replacing u_n by either its positive part $(u_n)_+$ or by $(-u_n)_+$, we can additionally assume that all u_n are non-negative.

Estimate (3.6) yields $c\|u_n\|_{H^1}^2 \leq C\|u_n\|_{L^2}^2 + \Phi(u_n) \leq C$. We choose a subsequence $u_n \rightharpoonup u$ in H^1 and $u_n \to u$ in L^2_{loc}. We can assume in the following that $\Phi(u_n) \leq -\delta < 0$, since in the case $\Phi \geq 0$ we only need to normalize a solution u of $\Phi(u) = 0$ and can proceed with Step 2.

There holds $\Phi(u) \leq \liminf \Phi(u_n)$ by the weak lower semicontinuity of norms and the strong local convergence. We claim that $\|u\|_{L^2} = 1$. Indeed, $\|u\|_{L^2} \leq 1$ is clear by the weak convergence $u_n \rightharpoonup u$ in L^2. Furthermore $\|u\| > 0$ is clear by $\Phi(u) \leq -\delta$. If $c_0 := \|u\|_{L^2} < 1$ we consider $\tilde{u} = u/c_0$ with

$\Phi(\tilde{u}) = \Phi(u)/c_0^2 < \liminf \Phi(u_n)$, again impossible, since u_n was a minimizing sequence.

Step 2. We found a minimizer u of Φ on the 1-sphere of L^2. As such, u is an H^1-eigenfunction of L with non-positive eigenvalue,

$$Lu = \lambda_0 u, \quad u \geq 0,\ \lambda_0 \leq 0.$$

The sign of λ_0 follows easily by testing the eigenvalue equation with u. We next multiply the eigenvalue equation with w and find, using the non-negativity of u and w,

$$0 \geq \lambda_0 \langle u, w \rangle = \langle Lu, w \rangle = \langle u, Lw \rangle \geq 0,$$

which yields $\lambda_0 = 0$. As a solution of $Lu = 0$ the function u is C^1. It can not vanish anywhere, since otherwise also the derivative vanishes by non-negativity, and as a solution of a linear ordinary differential equation u had to vanish identically. We can repeat the above calculation and find

$$0 = \lambda_0 \langle u, w \rangle = \langle u, Lw \rangle > 0,$$

by $Lw \neq 0$. We found a contradiction.

Step 3. We now show (3.5). Let us assume the contrary, the existence of a sequence u_n with $\|u_n\|_{H^1} = 1$ and with $\Phi(u_n) \to 0$. For a subsequence we find a limit u such that $u_n \rightharpoonup u$ in H^1 and $u_n \to u$ in L^2_{loc}. Estimate (3.6) yields $c = c\|u_n\|_{H^1}^2 \leq C\|u_n\|_{L^2}^2 + \Phi(u_n)$ and therefore a lower bound for $\|u_n\|_{L^2}^2$.

We claim that u is not trivial. For arbitrary $\varepsilon > 0$ the convergence $\Phi(u_n) \to 0$ yields for large n

$$\varepsilon + \int_{\mathbb{R}} (\mu(.))_+ |u_n|^2 \geq \int_{\mathbb{R}} |\partial_x u_n|^2 + \int_{\mathbb{R}} (-\mu(.))_+ |u_n|^2.$$

The function $(\mu(.))_+$ is supported on a compact subset $I \subset \mathbb{R}$ and we have L^2 convergence $u_n \to u$ on I. We find n_0 such that

$$2\varepsilon + \int_{\mathbb{R}} (\mu(.))_+ |u|^2 \geq \int_{\mathbb{R}} |\partial_x u_n|^2 + \int_{\mathbb{R}} (-\mu(.))_+ |u_n|^2 \quad \forall n \geq n_0.$$

Let us assume $u = 0$. Then, for appropriate choice of $\delta > 0$ and $M > 0$,

$$2\varepsilon \geq \delta \int_{\mathbb{R}\setminus(-M,M)} |u_n|^2 = \delta \|u_n\|_{L^2}^2 - \delta \int_{-M}^{M} |u_n|^2. \quad \forall n \geq n_0.$$

The second term on the right tends to zero by $u = 0$. For $\varepsilon > 0$ small we find a contradiction to the lower bound for $\|u_n\|_{L^2}$.

The lower semicontinuity of Φ for the above convergence yields for the limit u that $\Phi(u) \leq 0$, a contradiction to (3.4). $\square$

Positivity of solutions. We have seen that L defines a symmetric non-negative functional. We next study positivity of solutions, not only for L, but also for positive perturbations of L. We assume that we are given a bounded function $\psi \in C(\mathbb{R}, [0,\infty))$ and define the operator $L_\psi = L + \psi$.

The bilinear form corresponding to L_ψ is again positive, coercive, and symmetric. This implies that $L_\psi : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ is invertible. We now study positivity of solutions. We claim that for $\varphi \in H^{-1}(\mathbb{R})$

$$L_\psi z = \varphi \geq 0 \quad \Rightarrow \quad z \geq 0. \tag{3.7}$$

The function z is the minimizer of the coercive functional

$$\Phi_\varphi(z) := \frac{1}{2} \langle z, L_\psi z \rangle - \langle \varphi, z \rangle .$$

Let us assume z were negative somewhere. We then write $z = z_+ + z_-$ with $z_+(x) := \max(0, z(x))$ and $z_- \neq 0$ and calculate

$$\Phi_\varphi(z) = \frac{1}{2} \langle z_+, L_\psi z_+ \rangle - \langle \varphi, z_+ \rangle + \frac{1}{2} \langle z_-, L_\psi z_- \rangle - \langle \varphi, z_- \rangle > \Phi_\varphi(z_+),$$

using that L_ψ is a positive form. We find that the comparison function z_+ yields a lower value of the functional, a contradiction.

We claim that the solution z of (3.7) with $0 \neq \varphi \geq 0$ is everywhere positive, $z(x) > 0$ for all $x \in \mathbb{R}$. For a short proof we assume that $\varphi \geq \bar\varphi = \varepsilon \delta_{x_0}$, which will actually be the case in our application. Since L_ψ^{-1} preserves non-negativity, the solution $\bar z$ of $L_\psi \bar z = \bar\varphi$ satisfies $0 \leq \bar z \leq z$. Assume that $z(x) = 0$ for some $x \in \mathbb{R}$. Then also $\bar z(x) = 0$. In the case $x \neq x_0$ we find $\partial_x \bar z(x) = 0$ and as a solution of an ordinary differential equation, $\bar z(x_0) = 0$. The jump condition $[\partial_x \bar z](x_0) = -\varepsilon$ yields a contradiction to non-negativity. In the case $x = x_0$, the jump condition yields immediately the contradiction.

We finally study the asymptotic behavior of non-negative solutions u of

$$Lu \geq \varphi \quad \text{for} \quad \varphi(x) = e^{-\eta x} \chi_{\{x > M_0\}}.$$

We claim that for some $\delta = \delta(M_0) > 0$ the inequality $u(x) \geq \delta \varphi(x)$ holds for all $x > M_0$. We study for $\bar\mu = \inf(\mu)$ the operator $\bar L := -\partial_x^2 - \bar\mu$. The Green's function of $\bar L$ is given by $G(x, y) = e^{-\lambda|x-y|}/2\lambda$ with $\lambda = \sqrt{-\bar\mu}$. We compare u with the positive solution $\bar u$ of $\bar L \bar u = \varphi$, given by

$$\bar u(x) = \int_{\mathbb{R}} G(x, \xi)\, \varphi(\xi)\, d\xi.$$

There holds

$$L(u - \bar{u}) = Lu - \bar{L}\bar{u} + (\bar{L} - L)\bar{u} \geq (-\bar{\mu} + \mu)\bar{u} \geq 0.$$

Therefore $u \geq \bar{u}$, since L^{-1} is positivity preserving. The asymptotic behavior of $\bar{u}$ follows from the explicit form of the Green's function.

The comparison function. Let again a bounded function $\psi \in C(\mathbb{R}, [0, \infty))$ be given. We write χ_- and χ_+ for the characteristic function of $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively. We recall that we set $w = \partial_x u_s$.

Lemma 3.2. *For appropriate $\varepsilon > 0$ and $\eta > 0$, depending on ψ, the function*

$$U_0(x) := w(x) + \varepsilon e^{-\eta|x|}$$

and the H^1-solution $\bar{U}$ of

$$L_\psi \bar{U} = \psi U_0 \chi_- \tag{3.8}$$

satisfy

$$\Theta := \sup \left\{ \frac{\bar{U}(x)}{U_0(x)} \middle| x \in \mathbb{R} \right\} < 1. \tag{3.9}$$

Proof. We first note that U_0 is positive, that $\bar{U}$ exists by the invertibility of L_ψ, and that $\bar{U} \neq 0$ is non-negative, since U_0 is.

We calculate

$$Lw = (-\partial_x^2 - \mu)\partial_x u_s = [f(u^*, 0) - f(u^*, v^*)]\delta_0 =: f_0\delta_0,$$

where δ_0 is the Dirac distribution in $x = 0$ and $f_0 \in \mathbb{R}$, $f_0 > 0$. Furthermore

$$Le^{-\eta|.|}(x) = (-\eta^2 - \mu(x))e^{-\eta|x|} + 2\eta\delta_0.$$

We introduce $z_\varepsilon = U_0 - \bar{U}$ and our aim is to show

$$\inf_{x \in \mathbb{R}} \left\{ \frac{z_\varepsilon(x)}{U_0(x)} \right\} > 0. \tag{3.10}$$

In terms of z_ε equation (3.8) yields

$$L_\psi z_\varepsilon = LU_0 + \psi U_0 - \psi U_0 \chi_- \ ,$$

and therefore

$$L_\psi z_\varepsilon = (f_0 + 2\eta\varepsilon)\delta_0 + \varepsilon(-\eta^2 - \mu(x))e^{-\eta|x|} + \psi U_0 \chi_+. \tag{3.11}$$

We find that in the case $\varepsilon = 0$ the right hand side is non-negative. In the last paragraph on positivity of solutions we have seen that $z_0 > 0$ for $\varepsilon = 0$.

Let us consider now a small $\varepsilon > 0$. This means that we enlarge z_0 by the solution $\hat{z} = z_\varepsilon - z_0$ of

$$L_\psi \hat{z} = \varepsilon \hat{f},$$

with $\hat{f} = 2\eta\ \delta_0 + (-\eta^2 - \mu(x))e^{-\eta|x|}$. For η^2 small compared to the limit values of $-\mu$, we have seen that the solution $\hat{z}$ behaves like $+\delta e^{-\eta|x|}$ outside a compact interval. Therefore, outside this interval, $\hat{z}$ is positive. Inside the interval $\hat{z}$ might be negative, but since $z_0 > 0$ in the interval, we can choose $\varepsilon > 0$ small to have z_ε positive.

In particular, z_ε satisfies for some $\delta > 0$ and $M > 0$ the asymptotic estimate $z_\varepsilon(x) \geq \varepsilon\delta e^{-\eta|x|}$ for all $|x| > M$. We can therefore compare z_ε to U_0. Result (3.10) and the lemma follow. □

For later convenience we note that we also have an upper bound for solutions u. With some constant $C = C(\psi, U_0)$ there holds

$$L_\psi u = U_0 \quad \Rightarrow \quad u \leq CU_0. \tag{3.12}$$

Outside a large interval this estimate follows from the fact that both U_0 and u have the decay of $e^{-\eta|x|}$. Inside the interval the estimate follows immediately from the positivity of U_0 and the boundedness of u.

3.2 Orbital stability of a creeping front

The aim in this section is to study solutions (u, v) of the time dependent problem with initial values that are close to the stationary profiles (u_s, v_s). We recall some steps of the construction of the stationary solution. Recall that nontrivial solutions $(u, v) \in [0, 1]^2$ of $g(u, v) = 0$ are parametrized by $v = \gamma(u) = \Gamma^{-1}(u)$. The solution u_s was found by setting $v_s(x) = \gamma(u_s(x))\chi_-(x)$ and solving (2.12). u_s is monotonically increasing with $u_s(0) = u^*$, and $v_s(x)$ is monotonically decreasing with $v_s(0^-) = \gamma(u^*) =: v^*$ and $v_s \equiv 0$ on $\mathbb{R}_+$.

We will see that, for appropriate initial values, the instationary solution (u, v) remains in shape near to (u_s, v_s), but shifts its position. In order to associate a position to (u, v) we choose a small number $\rho \in (0, 1)$ (the smallness will be specified later), and define $X(t) \in \mathbb{R}$ to be the position with $v(X(t), t) = \rho v^*$. We will see that the position $X(t)$ is uniquely determined.

In order to verify that $(u, v)(X(t) + ., t)$ is close to $(u_s, v_s)(.)$, we study the differences,

$$U(\xi, t) = u(X(t) + \xi, t) - u_s(\xi), \quad V(\xi, t) = v(X(t) + \xi, t) - v_s(\xi). \tag{3.13}$$

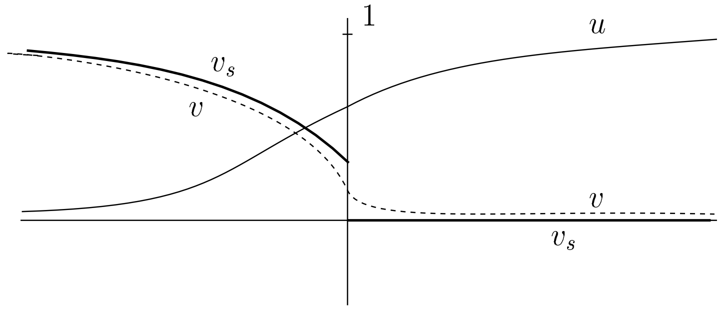

Figure 3: A solution $(u, v)(., t)$ and the stationary solution (u_s, v_s)

Our results on shape-stability and on instability of the stationary solution are summarized in the following theorem.

Theorem 3.3 (Creeping front). *We assume 1.1 and $H(S_-) < H(S_+)$ and study the unique stationary front solution (u_s, v_s). There exist initial values (u_0, v_0), arbitrarily close in the L^2-norm to (u_s, v_s), such that for the corresponding solution (u, v) of the initial value problem the functions $(U, V)(t)$ remain small in $L^\infty \times L^2$ for all t. The front travels to infinity, $X(t) \to +\infty$ for $t \to \infty$, with a logarithmic rate.*

The theorem is proved on the basis of the time-dependent system for U, V, and X,

$$\partial_t U = \partial_\xi^2 U + \dot{X} \cdot \partial_\xi u + f(u, v) - f(u_s, v_s) \qquad \text{on } \mathbb{R}, \tag{3.14}$$

$$\partial_t V = \dot{X} \cdot \partial_\xi v + g(u, v) \qquad \text{on } \mathbb{R} \setminus \{0\}, \tag{3.15}$$

$$V(0^-) = (\rho - 1)v^*, \tag{3.16}$$

$$V(0^+) = \rho v^*. \tag{3.17}$$

In the equations appear u and v; in this context $u(\xi)$ and $v(\xi)$ are short notations for the expressions $u_s(\xi) + U(\xi, t)$ and $v_s(\xi) + V(\xi, t)$.

We can interpret this system in the spirit of free boundary problems: There appear two conditions for $V(t, 0)$. This overdetermination of V determines the position $X(t)$.

We make the following choice of initial values.

$$U(\xi, 0) = 0 \qquad \forall \xi \in \mathbb{R}, \tag{3.18}$$

$$V(\xi, 0) = \rho v^* e^{-\lambda \xi} \qquad \forall \xi \in (0, \infty), \tag{3.19}$$

$$V(\xi, 0) = -(1 - \rho)v^* e^{\lambda \xi} \qquad \forall \xi \in (-\infty, 0). \tag{3.20}$$

We will choose $\rho > 0$ small, and, more importantly, $\lambda > 0$ large. More general initial values can be treated. Essential are the properties $v(.,0)$ smooth and monotonically decreasing, smallness of $V(.,0)$ in L^2 and of $U(.,0)$ in L^∞, monotonicity of $u(.,0)$.

The proof of Theorem 3.3 is based on the fact that a weighted $L^\infty(\mathbb{R}) \times L^2(\mathbb{R})$-smallness condition for (U,V) remains satisfied for all times. In particular, this implies that the front has always a positive speed (compare Lemma 3.6 for estimates). Indeed, in the point $\xi = 0$ we have $(u,v)(\xi,t) = (u_s(0) + U(0,t), v_s(0^+) + V(0^+,t)) = (u^* + U(0,t), \rho v^*)$. Smallness of $|U(0,t)|$ implies that $(u,v)(\xi = 0,t)$ is close to $(u^*, \rho v^*)$, hence $g((u,v)(\xi = 0,t))$ is always positive. We evaluate (3.15) in $\xi = 0^+$, where the left hand side vanishes. By monotonicity of the system, the solution (u,v) remains monotone in x. In particular, we have $\partial_\xi v(0,t) \le 0$ for all times. This shows $\dot{X}(t) > 0$.

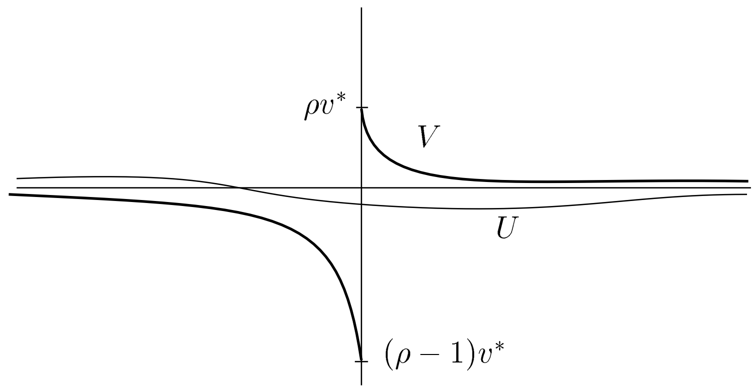

Figure 4: The solutions $(U,V)(.,t)$.

The proof of the theorem is based on the subsequent lemmas 3.4 to 3.6. The lemmas make the following heuristical observations precise.

1) For L^2-small V and small $\dot{X}$, (3.14) provides a small uniform bound for U.

2) For λ large, $\dot{X}$ bounded from below, and $|U|$ small, (3.15) implies that $V(.,t)$ is small in L^2 for all times.

3) The bounds for U imply a smallness of $\dot{X}(t) \in \mathbb{R}$.

4) As a tool in the above results: The speed $\dot{X}$ is bounded from below.

The key for the proof of the theorem is the following: the smallness of U found in 1) improves the smallness assumption on U in 2). Vice versa, the smallness of V found in 2) improves the smallness assumption on V in 1). These improvements are based on the fact that the factor Θ from (3.9) satisfies $\Theta < 1$. We can then conclude as follows: As long as U and V satisfy their smallness conditions, they satisfy even sharper conditions. By continuity, the estimates remain valid for all times.

Some of the constants below depend only on the functions f and g and on the position of the jump, u^*. We will call such constants *geometric* in the following.

Ad 1) Results on the stabilty of $U = 0$ exploit the properties of the operator $L = -\partial_\xi^2 - \mu(.)$ and its perturbations. We consider the positive perturbation of L given by $\psi(\xi) = \gamma'(u_s)\partial_v f(u_s, v_s)\ \chi_-$. We use U_0 from Lemma 3.2 and $\Theta < 1$ from property (3.9). We set $\Theta' = (1 + \Theta)/2 < 1$ to have some space in the calculations.

Lemma 3.4. *Given $C_0 > 0$ and $c_2 > 0$, there exist constants $C_{u,0}, C_{v,0} > 0$ (small), and $\lambda_0 > 0$ (large) such that the following holds. Every solution (U, V, X) of (3.14)-(3.20) that satisfies for some $T > 0$, $\lambda > \lambda_0$ and $C_v < C_{v,0}$ the estimates*

$$\begin{aligned}
\dot{X}(t) &\le C_0/\lambda, \qquad |U(\xi,t)| \le C_{u,0}U_0(\xi) && \forall\ \xi \in \mathbb{R}, t \in [0,T],\\
|V(\xi,t)| &\le e^{-c_2\lambda\xi} && \forall\ \xi > 0, t \in [0,T],\\
|V(\xi,t)| &\le \max\left\{e^{c_2\lambda\xi}, (-\gamma'(u_s(\xi)))C_vU_0(\xi)\right\} && \forall\ \xi < 0, t \in [0,T],
\end{aligned}$$

satisfies for all times $t \in [0,T]$ and all $\xi \in \mathbb{R}$ additionally

$$|U(\xi,t)| \le (\Theta' C_v + C(\lambda))U_0(\xi) \tag{3.21}$$

with some constant $C(\lambda) = C(\lambda; C_0, C_{v,0}, C_{u,0}, \lambda_0, c_2)$. This coefficient satisfies $C(\lambda) \to 0$ for $\lambda \to \infty$.

Proof. We write (3.14) in the form

$$\partial_t U = \partial_\xi^2 U + \dot{X}\partial_\xi U + f_0(\xi,t) + f_1(\xi,t).$$

Here on $\xi > 0$ we set

$$\begin{aligned}
f_0(\xi,t) &= f(u,v) - f(u,0) = \partial_v f(u(\xi,t), \zeta(\xi,t)) \cdot V(\xi,t),\\
f_1(\xi,t) &= f(u,0) - f(u_s,0) = \partial_u f(\eta(\xi,t),0) \cdot U(\xi,t) =: \tilde{\mu}(\xi,t) \cdot U(\xi,t),
\end{aligned}$$

for values $\zeta = \zeta(\xi,t) \in [0,V]$ and $\eta = \eta(\xi,t)$ with $|\eta(\xi,t) - u_s(\xi)| \le |U(\xi,t)|$. On the half line $\xi < 0$ we write similarly

$$\begin{aligned} f_0(\xi,t) &= f(u,v) - f(u,\gamma(u)) \\ &= \partial_v f(u,\zeta)(v - v_s + \gamma(u_s) - \gamma(u)) \\ &= \partial_v f(u(\xi,t),\zeta(\xi,t)) \cdot [V(\xi,t) - \gamma'(\eta_1(\xi,t))U(\xi,t)], \\ f_1(\xi,t) &= f(u,\gamma(u)) - f(u_s,v_s) \\ &= (\partial_u f + \gamma'(\eta_2(\xi,t))\partial_v f)(\eta_2(\xi,t),\gamma(\eta_2(\xi,t))) \cdot U(\xi,t) \\ &=: \tilde{\mu}(\xi,t) \cdot U(\xi,t). \end{aligned}$$

Again, η_1 and η_2 satisfy $|\eta_i(\xi,t) - u_s(\xi)| \le |U(\xi,t)|$. We have now written the equation for U in the form

$$\partial_t U + LU = f_0(\xi,t) + (\tilde{\mu} - \mu)U + \dot{X}\partial_\xi U,$$

with $|\mu - \tilde{\mu}| \le CC_{u,0}U_0$. We next have to analyze f_0. We set $\psi_0(\xi) = \partial_v f(u_s(\xi), v_s(\xi))$ and recall $\psi(\xi) = \gamma'(u_s(\xi))\psi_0(\xi)\ \chi_-(\xi)$. We can write

$$f_0(\xi,t) = \psi_0(\xi)V(\xi,t) - \psi(\xi)U(\xi,t) + R_U(\xi,t)\cdot U(\xi,t) + R_V(\xi,t)\cdot V(\xi,t),$$

with $|R_U(.,t)| + |R_V(.,t)| \le C(|V| + |U|)$ small in L^2 for $C_{u,0}$ and C_v small, and λ large. Our equation becomes

$$\begin{aligned} \partial_t U + LU + \psi(\xi)U &= \psi_0(\xi)V(\xi,t) \\ &\quad + R_U(\xi,t)\cdot U(\xi,t) + R_V(\xi,t)\cdot V(\xi,t) + (\tilde{\mu} - \mu)U + \dot{X}\partial_\xi U. \end{aligned}$$

To make the arguments clear, we first consider the equation without the error terms. With the help of the parabolic comparison principle we can compare U with the stationary solutions u of $L_\psi u = \sup_t(\psi_0 V)$. The assertion $|V| \le -\gamma' C_v U_0 \chi_-$ implies that $|U|$ remains below the comparison solution $\bar{U}_1$ of

$$L\bar{U}_1 + \psi(.)\bar{U}_1 = |\psi_0\gamma' C_v U_0 \chi_-| = \psi C_v U_0.$$

By definition of Θ we have the pointwise bound $\bar{U}_1 \le \Theta C_v U_0$.

We now consider the error terms. They allow only for the estimate $U \le \bar{U} = \bar{U}_1 + \bar{U}_2$ for every function $\bar{U}_2$ satisfying

$$L_\psi \bar{U}_2 \ge C(|U| + |V|)^2.$$

We use $L_\psi^{-1}U_0 \le CU_0$ from (3.12) and see that we can choose $\bar{U}_2$ with the estimate

$$\begin{aligned} |\bar{U}_2| &\le C\ \tilde{L}_\psi^{-1}\left[C_{u,0}U_0|\bar{U}_1 + \bar{U}_2| + C_v U_0 C_{u,0} + e^{-c_2\lambda|.|} + C_{v,0}C_v U_0\right] \\ &\le CC_{u,0}C_v U_0 + CC_{u,0}\|\bar{U}_2\|_\infty U_0 \\ &\qquad + C(C_{u,0} + C_{v,0})C_v U_0 + O(\lambda^{-1/2})e^{-c_2\lambda|.|}. \end{aligned}$$

For small $C_{u,0}$ we can absorb the second term in the left hand side. This shows the claim. ☐

Ad 2) In order to motivate and to describe the V-estimate, let us assume for the moment $\dot{X} > 0$ were constant in time and $u \equiv u^*$ were constant in time and space. We study the ordinary differential equation for $v(x,.)$ of (1.1). The equation has the two stationary points $v = 0$ and $v^* = \gamma(u^*)$. If we assume $v(x,0) \in (0,v^*)$ for all $x > 0$, then $v(x,.)$ is (possibly) a long time in the vicinity of 0, then a finite time in an interval $(\varepsilon, v^* - \varepsilon)$, then for all further times close to v^*. Translating this into the moving co-ordinate system with origin in $X(t)$ and using $v(X(t),t) = \rho v^*$, we find values $v \in (\varepsilon, v^* - \varepsilon)$ only in a finite neighborhood $I = (-\delta, \delta)$ of $\xi = 0$, whereas to the left (the right) of I the solution v is close to v^* (to 0). For small $\dot{X} > 0$, the number δ becomes arbitrarily small.

The subsequent lemma verifies this picture in the perturbed situation of u not being constant (but, instead, with $|U|$ small), and with $\dot{X}$ not constant (but, instead, with a lower and a small upper bound). By showing (3.23) and (3.24) for small β, we find an exponential estimate in the vicinity of $\xi = 0$, while far from $\xi = 0$ the estimate for V is as good as U permits.

Lemma 3.5. *Let $c_0, C_0, \beta > 0$ be given. There is a geometric constant $\Delta x = x^\circ/2 > 0$ and constants $C_u = C_u(c_0,\beta), \rho = \rho(c_0), c_2 = c_2(c_0, C_0, \beta) > 0$ that can be chosen arbitrarily small, and $\lambda_0 = \lambda_0(c_0, C_0, \beta) > 0$ such that the following holds.*

Given $\lambda \geq \lambda_0$ and $U(.,.)$, $X(.)$ satisfying for all $t \in [0,T]$

$$|U(\xi,t)| \leq C_u U_0(\xi),$$
$$0 < \dot{X}(t) \leq \frac{C_0}{\lambda},$$

and for $t + \Delta t \leq T$

$$X(t + \Delta t) = X(t) + \Delta x \quad \text{implies} \quad \Delta t \leq \frac{\lambda + t}{c_0}, \tag{3.22}$$

then the solution V of (3.15) satisfies for all $t \in [0,T]$ on $\{\xi > 0\}$

$$0 \leq V(\xi,t) \leq \rho v^* e^{-c_2 \lambda \xi}, \tag{3.23}$$

and on $\{\xi < 0\}$

$$|V(\xi,t)| \leq \max\left\{-\gamma'(u_s(\xi))(1+\beta)C_u U_0(\xi), (1-\rho)v^* e^{c_2\lambda\xi}\right\}. \tag{3.24}$$

Proof. We denote by $x^\circ > 0$ the position with $u_s(x^\circ) = u^\circ$. At (approximately) the position $\xi = x^\circ$ the solution $v = 0$ looses its stability in the v-equation, since u has (approximately) the critical value u°. We set $\Delta x = x^\circ/2$.

We study an arbitrary point $x_1 \in [0, \infty)$. In the case $X(T) \geq x_1$ we choose t_1 such that $x_1 := X(t_1)$, in the opposite case we set $t_1 = T$. We derive results for $v(x_1, t)$ for all $t \leq t_1$. They are then interpreted as results for

$$\begin{aligned} v(x_1, t) &= v(X(t) + x_1 - X(t), t) = v_s(x_1 - X(t)) + V(x_1 - X(t), t) \\ &= V(x_1 - X(t), t). \end{aligned}$$

Step 1. Choice of t_- and t_+. In order to analyze $v(x_1, t)$ we must understand what happens in the time t_0 when $v = 0$ looses stability in x_1, that is, $u_s(x_1 - X(t_0)) = u^\circ$, or $x_1 = X(t_0) + x^\circ$. Our first aim is to choose $x_+ > x^\circ$, $x_- < x^\circ$, and $C_{u,1}$ small with

$$\begin{aligned} C_g &:= \frac{1}{2} \inf \{ -\partial_v g(u, 0) \,|\, u \geq u_s(x_+) - C_{u,1} \} > 0, \\ C'_g &:= 2 \sup \{ \partial_v g(u, 0) \,|\, u \geq u_s(x_-) - C_{u,1} \} > 0, \\ & u_s(x_-) + C_{u,1} < u^\circ < u_s(x_+) - C_{u,1}, \end{aligned}$$

such that $|x_+ - x_-| < x^\circ/2$ and with

$$C'_g \leq c_0/2 \min\{C_g, x_1\}.$$

This is possible by setting $x_+ = 5x^\circ/4$, and choosing $C_{u,1}$ small compared to x° and $|x_- - x^\circ| = O(C_{u,1})$, since then $C'_g = O(C_{u,1})$. The constant $C_{u,1}$ depends on c_0. We impose on C_u that $2C_u \|U_0\|_\infty \leq C_{u,1}$.

We want to study time instances t_- and t_+ such that $v = 0$ is stable before t_- and $v = 0$ is unstable after t_+. If possible, we choose them to satisfy

$$X(t_-) = x_1 - x_+, \quad X(t_+) = x_1 - x_-.$$

Note that t_- can not always be defined as above. In the case $x_1 < x_+ = 5x^\circ/4$ we set $t_- = 0$ and in the case $X(T) < x_1 - x_+$ we set $t_- = T$. In the same way we set $t_+ = 0$ for $x_1 < x_-$ and $t_+ = T$ for $X(T) < x_1 - x_-$.

Step 2. The time span $(0, t_-)$. We exploit that for small initial values the function $v(x_1, .)$ is exponentially decreasing in this time span.

We have initially $v(x_1, 0) = \rho v^* e^{-\lambda x_1}$ and $\partial_t v(x_1, t) = g(u, v) \leq -C_g v(x_1, t)$ for all $t \leq t_-$ (assuming $\lambda \geq \lambda_0$ for a large geometric constant λ_0 in order to have small values of v). Therefore

$$v(x_1, t) \leq \rho v^* e^{-\lambda x_1 - C_g t} \quad \forall t \leq t_-,$$

and we trivially find at the position $\xi = x_1 - X(t)$ for $t \le t_-$

$$V(\xi, t) = v(x_1, t) \le \rho v^* e^{-\lambda x_1 - C_g t} \le \rho v^* e^{-\lambda \xi}.$$

Step 3. The time span (t_-, t_+). In the time span (t_-, t_+) the function $v(x_1, .)$ may be exponentially increasing. To derive bounds, we exploit assumption (3.22), which yields a bound on the duration $\Delta t = t_+ - t_-$. Using $x_+ - x_- < \Delta x$, we find $\Delta t \le (\lambda + t_-)/c_0$.

We start from $v(x_1, t_-) \le \rho v^* e^{-\lambda x_1 - C_g t_-}$. For large λ_0, there is a slow growth rate, and we find

$$\begin{aligned} v(x_1, t) &\le \rho v^* e^{-\lambda x_1 - C_g t_-} e^{C_g' \Delta t} \\ &\le \rho v^* \exp\left(-\lambda \left(x_1 - C_g'/c_0\right) - t_- \left(C_g - C_g'/c_0\right)\right) \\ &\le \rho v^* \exp(-\lambda x_1/2). \end{aligned}$$

We conclude the bound for V by writing again $\xi = x_1 - X(t)$,

$$V(\xi, t) = v(x_1, t) \le \rho v^* e^{-\lambda x_1/2} \le \rho v^* e^{-\lambda \xi/2} \quad \forall t \in (t_-, t_+).$$

Step 4. The time span (t_+, t_1). The argument for the v-estimate is now of a different nature. We exploit that v grows exponentially and the fact that $v(x_1, t_1) \le \rho v^*$, with equality if $X(t_1) = x_1$.

Since U is small, $u(x_1, t) \le u_s(x_-) + C_{u,1} < u^\circ$ for all $t \in (t_+, t_1)$. For small ρ (of the order of $C_{u,1}$, and thus depending on c_0), there is $c_g > 0$ (of the order of $C_{u,1}$) such that $g(u, v) \ge c_g \cdot v$ for all $v \le \rho v^*$ and $u \le u_s(x_-) + C_u$. This implies for $t \ge t_+$

$$v(x_1, t_1) \ge v(x_1, t) e^{c_g (t_1 - t)}.$$

Recalling that $v(x_1, t_1) \le \rho v^*$ we find

$$V(x_1 - X(t), t) = v(x_1, t) \le \rho v^* e^{-c_g (t_1 - t)}.$$

To relate $\xi = x_1 - X(t)$ and $\Delta t = t_1 - t$ we use the upper bound for the velocity, $\xi/\Delta t \le C_0/\lambda$. This yields

$$V(\xi, t) \le \rho v^* e^{-c_g \Delta t} \le \rho v^* e^{-c_g \lambda \xi / C_0} \quad \forall t \in [t_+, t_1].$$

With $c_2 := \min(c_g/C_0, 1/2)$ we find the result (3.23).

Step 5. Inequality (3.24). We now consider arbitrary $x_1 \le X(T)$ and $t_1 = X^{-1}(x_1)$ if possible and set $t_1 = 0$ for $x_1 < 0$. We have to analyze $v(x_1, t)$ for $t > t_1$. We exploit that after time t_1 the solution $v(x_1, .)$ runs towards the solution $v = \gamma(u)$ of $g(u, v) = 0$. Once v is in the vicinity of the graph of γ, denoted in this proof by Γ, the stability of Γ implies that v can not escape again.

We start from the equation for V,

$$\partial_t V(\xi, t) = \dot{X}(t)\partial_\xi v(X(t) + \xi, t) + g(u(X(t) + \xi, t), v(X(t) + \xi, t))$$

which we write in the form

$$\begin{aligned}(\partial_t - \dot{X}(t)\partial_\xi)V(\xi, t) = \dot{X}(t)\partial_\xi v_s(\xi) \\ + g(u_s(\xi) + U(\xi, t), v_s(\xi) + V(\xi, t)).\end{aligned} \tag{3.25}$$

(i) We first analyze the case of large $|V|$. For a concise definition we use the constant $c_\gamma := -2\gamma'(u^*) = 2|\gamma'(u_s(0))|$. We consider the case

$$V(\xi, t) \leq -c_\gamma C_{u,1}. \tag{3.26}$$

We achieve for small $C_{u,1} > 0$

$$\begin{aligned}C_{g,1} := \inf \{g(u, v)|u^*/2 \leq u \leq u^* + C_{u,1}, \\ \rho v^* \leq v \leq \gamma(u) - c_\gamma C_{u,1}\} > 0,\end{aligned}$$

and calculate in the case of (3.26) for $-\xi$ not too large,

$$(\partial_t - \dot{X}(t)\partial_\xi)V(\xi, t) \geq \frac{C_0}{\lambda}\partial_\xi v_s(\xi) + C_{g,1}.$$

For large λ we find that the material derivative above is positive. Together with $V(0, .) \equiv (\rho-1)v^*$ and the initial values $V(., 0) \geq (\rho-1)v^* e^{\lambda\xi}$, this shows the uniform decay of $|V(\xi, t)|$. By the upper bound for the velocity $\dot{X}$, the slope near $\xi = 0$ is of order $C_{g,1}\lambda/C_0$ and we find (3.24) with $c_2 = c_2(c_0, C_0)$.

(ii) We now consider the next case,

$$-c_\gamma C_{u,1} \leq V(\xi, t) \leq \gamma'(u_s(\xi))(1 + \beta)C_u U_0(\xi), \tag{3.27}$$

where the right hand side is negative. Again, we must exploit (3.25). We use the $\mathbb{R}^2$ distance function d and the non-degeneracy assumption (1.5) to find

$$C_{g,2} := \inf \left\{ \frac{g(u, v)}{d((u, v), \Gamma)} \middle| 0 \leq u \leq u^* + C_{u,1}, \rho v^* \leq v < \gamma(u) \right\} > 0,$$

depending on ρ. With this constant we can now calculate

$$\begin{aligned}g(u_s(\xi) + U(\xi, t), v_s(\xi) + V(\xi, t)) &\geq C_{g,2} d((u_s + U, v_s + V), \Gamma) \\ &\geq C C_{g,2} \, |\langle (U, V), (-\gamma', 1)\rangle| \\ &= C C_{g,2} \, |V(\xi, t) - \gamma' U(\xi, t)|,\end{aligned}$$

where γ' is evaluated at a point η such that $(\eta, \gamma(\eta))$ is the orthogonal projection of (u, v) to Γ. If we replace in the above expression η by u_s, we introduce an error that we denote by R. We can calculate for V

$$(\partial_t - \dot{X}(t)\partial_\xi)V(\xi,t) \geq \dot{X}(t)\partial_\xi v_s(\xi) - CC_{g,2}(V(\xi,t) - \gamma'(u_s)U(\xi,t)) + R.$$

We distinguish two cases. If $V < 2\gamma'(u_s)|U|$ we estimate the error by $|R| \leq CC_{u,1}|V|$ and write

$$(\partial_t - \dot{X}(t)\partial_\xi)V(\xi,t) \geq \dot{X}(t)\partial_\xi v_s(\xi) - \frac{1}{2}CC_{g,2}V(\xi,t) - CC_{u,1}|V|.$$

Imposing that $C_{u,1}$ is small compared to $C_{g,2}$ we can absorb the error term. If, instead, $V \geq 2\gamma'(u_s)|U|$ we estimate the error by $|R| \leq CC_u^2$ and find

$$(\partial_t - \dot{X}(t)\partial_\xi)V(\xi,t) \geq \dot{X}(t)\partial_\xi v_s(\xi) + CC_{g,2}|\gamma'(u_s)|\beta C_u U_0(\xi) - CC_u^2 U_0(\xi).$$

We impose that C_u is small compared to $\beta C_{g,2}$ such that the third term is again small compared to the second. In both cases we find a linear decay of $-V$ with a rate $c_2\lambda$ with $c_2 = c_2(\beta, C_u, \rho)$.

(iii) It remains to consider the upper limit case of (ii), $V(\xi,t) = -|\gamma'(u_s(\xi))|(1+\beta)C_u U_0(\xi)$. We evaluate once more (3.25),

$$\begin{aligned}
&(\partial_t - \dot{X}(t)\partial_\xi)\left[V(\xi,t) + |\gamma'(u_s)|(1+\beta)C_u U_0(\xi)\right] \\
&\quad \geq -\frac{C_0}{\lambda}C[|\partial_\xi v_s| + |\partial_\xi u_s|] + CC_{g,2}|\gamma'(u_s)|\beta C_u U_0(\xi) - C\frac{C_0}{\lambda}\partial_\xi U_0(\xi).
\end{aligned}$$

For large λ, this is positive for all (ξ, t) since U_0 and $\partial_\xi U_0$ decay not faster than u_s and $\partial_\xi u_s$ for $\xi \to -\infty$. We find that $V(\xi,t) \geq \gamma'(u_s(\xi))(1+\beta)C_u U_0(\xi)$ remains valid for all times. The upper bound follows in the same way. □

Ad 3) Upper estimates for $\dot{X}(t)$ are the consequence of lower estimates for $-\partial_x v$. The following lemma exploits the non-degeneracy assumption $\partial_v \partial_u g(u, 0) < 0$.

Lemma 3.6. *For C_u small, $T > 0$ fixed, and λ_0 large compared to geometric constants, there is $C_0 > 0$ such that for all $\lambda \geq \lambda_0$ the following holds. If the solution $U(.,.)$ of (3.14) satisfies*

$$|U(\xi,t)| \leq C_u U_0(\xi)$$

for $t \in [0,T]$ and every $\xi \in \mathbb{R}$, then the solution (V, X) of (3.15)-(3.17) satisfies

$$\dot{X}(t) < \frac{C_0}{\lambda} \tag{3.28}$$

for all $t \in [0,T]$.

Proof. Given $t_1 \in [0,T]$ we consider $x_1 := X(t_1)$. We will find C_0 such that, if (3.28) holds with the constant $2C_0$ on $[0,t_1]$, then (3.28) holds also with the constant C_0 on $[0,t_1]$. By continuity of $\dot{X}$ and the small initial value, the C_0-estimate then remains valid for all times.

The interval $[0,t_-]$**.** We assume that $x_1 > 2x^\circ$, else we proceed with step 2. We define $t_- < t_1$ to be the time-instance at which the front has reached the distance $2x^\circ$ from x_1, that is, $x_1 = X(t_-) + 2x^\circ$. Our first aim is to compare $\partial_x v(x_1,t_-)$ with $\lambda v(x_1,t_-)$.

The equation for $t \mapsto y(t) := -\partial_x v(x_1,t)$ is

$$\begin{aligned} y(0) &= -\partial_x v(x_1,0) = \lambda v(x_1,0),\\ \partial_t y(t) &= -\frac{d}{dx} g(u(x_1,t), v(x_1,t))\\ &= -\partial_u g(u,v)\partial_x u(x_1,t) + \partial_v g(u,v)\cdot y(t)\\ &\geq \partial_v g(u,v)\cdot y(t). \end{aligned}$$

We set $\mu(t) := \partial_v g(u(x_1,t), v(x_1,t))$ and estimate with the variation of constants formula

$$y(t) \geq y(0)\exp\left(\int_0^t \mu(\tau)\, d\tau\right).$$

In a similar way we calculate the evolution equation for $z(t) = \lambda v(x_1,t)$ as

$$\begin{aligned} z(0) &= \lambda v(x_1,0) = y(0),\\ \partial_t z(t) &= \lambda(g(u,v) - g(u,0)) = \lambda \partial_v g(u,\zeta)\cdot v(x_1,t)\\ &\leq \partial_v g(u,v)\cdot z(t) + Cv(x_1,t)z(t) \end{aligned}$$

for a geometric constant C, and use again the variations of constants formula. We find for the ratio $y(t)/z(t)$ for $t < t_-$.

$$\begin{aligned} \frac{y(t)}{z(t)} &\geq \exp\left(\int_0^t \mu(\tau)\, d\tau\right) \Big/ \left[\exp\left(\int_0^t (\mu(\tau) + Cv(x_1,\tau))\, d\tau\right)\right]\\ &= \exp\left(-\int_0^t Cv(x_1,\tau)\, d\tau\right). \end{aligned}$$

Our aim is to show that $y(t)/z(t)$ is bounded from below, independent of x_1. We can exploit that the distance to the front is more than $2x^\circ$, whence v is uniformly decaying. We have $\partial_v g(u,v) \leq -C_g < 0$ for a geometric constant C_g and find

$$\int_0^t v(x_1,\tau)\, d\tau \leq e^{-\lambda x_1}\int_0^t e^{-C_g\tau}\, d\tau \leq e^{-\lambda x_1}\frac{1}{C_g}.$$

For large λ we find $y(t_-) \geq z(t_-)/2$.

The interval $[t_-, t_+]$**.** We choose $\eta > 0$ small compared to a geometric constant and denote by t_+ the time instance with $z(t_+) = \lambda v(t_+) = \eta$. We claim that for large λ we have $x_1 - X(t_+) < x^\circ/2$.

Indeed, for t with $X(t) \geq x_1 - x^\circ/2$, using the smallness of U, the function z grows exponentially. $z(t_1) = \lambda\rho v^*$ implies

$$z(t) \leq \lambda\rho v^* e^{-C(t_1-t)} \quad \text{for } t \in [t_+, t_1],$$

t with $X(t) \geq x_1 - x^\circ/2$. We have to transform this into an estimate in terms of spatial distances. We assumed the upper bound for the velocity $\dot{X} \leq 2C_0/\lambda$, by which we can compare spatial and temporal distances as $\Delta x \leq \Delta t\ 2C_0/\lambda$. We write for t as above

$$\eta = z(t_+) \leq z(t) \leq \lambda\rho v^* e^{-C'\lambda(x_1 - X(t))}.$$

For large λ, this yields a small upper bound for $x_1 - X(t_+) > 0$.

On (t_-, t_+) we use the non-degeneracy of g,

$$\begin{aligned} \partial_t y(t) &\geq \partial_v g(u,v) \cdot y(t) + c_u v(x_1, t) \\ \partial_t z(t) &\leq \partial_v g(u,v) \cdot z(t) + C v(x_1,t) z(t). \end{aligned}$$

Using $z(t) \leq \eta$ and imposing smallness of $\eta > 0$, we find $(y - z/2)(t_-) \geq 0$ and $\partial_t(y - z/2) \geq \mu(t)(y - z/2)$. This yields $y(t_+) \geq z(t_+)/2 = \eta/2$.

The interval $[t_+, t_1]$. By the estimate for $x_1 - X(t_+)$ and smallness of U we find in this time span $\partial_v g(u,v) \geq C_g > 0$ for a geometric constant C_g. The function z grows from η to $\lambda\rho v^*$. Since z grows at most exponentially, we find $|t_1 - t_+| = O(\log(\lambda))$.

In the same time, the function $y(t)$ grows at least exponentially with initial value $y(t_+) \geq \eta/2$. We find that $y(t_1) \geq c\lambda$ for a geometric constant c. This concludes the proof, since

$$\dot{X}(t_1) = -\frac{\partial_t v(x_1,t_1)}{\partial_x v(x_1,t_1)} \leq \frac{\sup(g)}{c\lambda}.$$

□

Ad 4) Lower bounds for $\dot{X}$. We perform two calculations which demonstrate

$$\dot{X}(t) \sim \frac{1}{\lambda + t}. \tag{3.29}$$

The precise statements are in (3.30) and (3.31). In particular, we will verify assumption (3.22) of lemma 3.5, which can be interpreted as a lower bound for $\dot{X}$.

We set $\Delta x = x^\circ/2$ and consider a pair (t_1, x_1) with $X(t_1) = x_1 > \Delta x$, and the time instance $T < t_1$ with $x_1 = X(T) + \Delta x$. We assume in this calculation $|U| \le C_u$ and smallness of ρ. We can introduce

$$C_1 := \inf\{\partial_v g(u,v) | u \ge u^*, v \le \rho v^*\} < 0,$$
$$C_2 := \inf\{\partial_v g(u,v) | u \le u_s(x^\circ/2) + C_u, v \le \rho v^*\} > 0.$$

We find for $t \in (0,T)$

$$\partial_t v(x_1,t) = g(u(x_1,t), v(x_1,t)) = \partial_v g(u(x_1,t),\zeta) v(x_1,t) \ge C_1 v(x_1,t).$$

The initial values are $v(x_1, 0) = \exp(-\lambda x_1)$ whence

$$v(x_1,T) \ge \exp(-\lambda x_1 + C_1 T).$$

The same calculation on (T, t_1) yields now

$$\rho v^* = v(x_1,t_1) \ge \exp(-\lambda x_1 + C_1 T + C_2 \Delta t)$$

with $\Delta t = t_1 - T$. This results in

$$\Delta t \le \frac{\lambda + T}{c_0} \qquad \text{for } \Delta x = x^\circ/2. \tag{3.30}$$

In particular, we see that the front travels to infinity, $X(t) \to +\infty$ for $t \to \infty$. We have furthermore verified assumption (3.22).

In a similar fashion we can also calculate an upper bound as also indicated by (3.29). This result improves the upper bound of lemma 3.6, but it gives only an integrated version of the estimate.

We consider the distance $\Delta x = 2x^\circ$. Our aim is to calculate how long it takes the front to travel the distance Δx. We set $X(t_1) = x_1$ and $X(T) = x_1 - \Delta x$ and want to calculate $\Delta t = t_1 - T$. Using

$$C_1 := \sup\{\partial_v g(u,v) | u \ge u^* + 2x^\circ - C_u, v \le \rho v^*\} < 0$$

we find on $(0,T)$ the inequality $\partial_t v(x_1,t) \le C_1 v(x_1,t)$, whence

$$v(x_1,T) \le \exp(-\lambda x_1 + C_1 T).$$

On the time interval (T, t_1) we have $\partial_t v(x_1, t) \leq C_2 v(x_1, t)$ with $C_2 := \sup\{\partial_v g(u, v)\} > 0$. We conclude

$$\rho v^* = v(x_1, t_1) \leq \exp(-\lambda x_1 + C_1 T + C_2 \Delta t).$$

For λ large we find

$$\Delta t > \frac{\lambda + T}{c_0} \qquad \text{for } \Delta x = 2x^\circ. \tag{3.31}$$

This shows (3.29) in an integrated sense and shows that the front can propagate only at a logarithmic rate.

Proof of Theorem 3.3. By the continuity of U, V, X, and $\dot{X}$, the assumptions of all lemmas are satisfied on a short time interval $(0, T_0)$. Let us consider the largest time instance $T > 0$ such that all assumptions are satisfied up to time T. Assuming $T < \infty$, by continuity of u and v, one inequality assumption of the lemmas is indeed an equality at time T. We will lead this to a contradiction, thus showing $T = \infty$.

For small C_u and ρ, and large λ, inequality (3.30) and Lemma 3.6 provide us with constants c_0 and C_0 regarding the velocity of the front. We use $\Theta < 1$ from (3.9) and $\Theta' = (1 + \Theta)/2$.

Our aim is to combine the U-estimate (3.21),

$$|U(\xi, t)| \leq (\Theta' C_v + C(\lambda)) U_0(\xi)$$

with the V-estimate (3.24),

$$|V(\xi, t)| \leq \max\left\{-\gamma'(u_s(\xi))(1 + \beta) C_u U_0(\xi), e^{c_2 \lambda |\xi|}\right\}.$$

We use the estimates with $C_u = (\Theta' C_v + C(\lambda))$ and $C_v = (1 + 2\beta) C_u$. If we satisfy

$$\Theta' C_v + C(\lambda) = \Theta'(1 + 2\beta) C_u + C(\lambda) < C_u, \tag{3.32}$$

then the U and the V-inequalities hold strictly. We choose $\beta > 0$ such that $\Theta'(1 + 2\beta) < 1$. With this β and c_0, C_0 as above we use Lemma 3.5 which yields C_u, ρ, c_2 (arbitrarily small) and λ_0 (large). If necessary, we decrease C_u and C_v, such that also Lemma 3.4 is applicable. If necessary, we further increase λ, such that (3.32) is satisfied. □

References

[1] D.G. Aronson, A. Tesei, and H. Weinberger. A density-dependent diffusion system with stable discontinuous stationary solutions. *Ann. Mat. Pura Appl.* (4) 152:259–280, 1988.

[2] D.G. Aronson and H.F. Weinberger. Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In J. Goldstein, editor, *Partial differential Equations and related Topics*, Lect. Notes Math. 446, pages 5–49. Springer, 1975.

[3] P.C. Fife and J.B. McLeod. The approach of solutions of nonlinear diffusion equations to traveling front solutions. *Arch. Rat. Mech. Anal.*, 65:335-361, 1977.

[4] S. Heinze. Diffusion-advection in cellular flows with large Péclet numbers. *Arch. Ration. Mech. Anal.*, 168:329-342, 2003.

[5] S. Heinze, B. Schweizer, and H. Schwetlick. Existence of front solutions in degenerate reaction diffusion systems. *Preprint 2004-03, SFB 359, Uni Heidelberg,* 2004.

[6] Y. Hosono and M. Mimura. Singular perturbation approach to traveling waves in competing and diffusing species models. *J. Math. Kyoto Univ.*, 22(3):435–461, 1982.

[7] S. Luckhaus and L. Triolo, The continuum reaction-diffusion limit of a stochastic cellular growth model. *Rend. Acc. Lincei. (to appear).*

[8] M. Protter and H. Weinberger. Maximum principles in differential equations. *Prentice-Hall, Inc., Englewood Cliffs, N.J.*, 1967.

[9] A.I. Volpert, V.A. Volpert, and V.A. Volpert. Traveling wave solutions of parabolic systems, *Translations of Mathematical Monographs*, 140. American Mathematical Society, Providence, RI, 1994.