Relaxation analysis in a data driven problem with a single outlier

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Abstract: We study a scalar elliptic problem in the data driven context. Our interest is to study the relaxation of a data set that consists of the union of a linear relation and single outlier. The data driven relaxation is given by the union of the linear relation and a truncated cone that connects the outlier with the linear subspace.

MSC: 35J20, 49J45

1 Introduction

The data driven perspective is new in the field of material science and partial differential equations, we mention [16] and [6] as the two fundamental contributions of this young field. In the data driven perspective certain laws of physics are accepted as invariable, e.g. balance of forces or compatibility. On the other hand, material laws (such as Hooke's law) can be questionable. In the classical approach, measurements are used to estimate constants of material laws. The new paradigm is to use a set of data points, obtained from measurements; the data points are not interpreted as realizations of some law, but calculations and analysis are based directly on the cloud of data points.

On a more formal level, one introduces a set $\mathcal{E}$ of functions that satisfy the invariable physical laws. A second set $\mathcal{D}$ denotes those functions that are consistent with the data. In this setting, the aim is to find functions in $\mathcal{E}$ that minimize the distance to the data set $\mathcal{D}$.

The emphasis in [16] was to derive computing algorithms for this new approach. The mathematical analysis in [6] establishes well-posedness properties and introduces, among other tools, data convergence and relaxation in the data driven context. It is shown that data driven relaxation differs markedly from traditional relaxation, see the discussion below.

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In the work at hand, we investigate a scalar setting, which can be used e.g. in the modelling of porous media. We seek for two functions, G (a gradient) and J (a negative flux). Given a domain $Q \subset \mathbb{R}^n$ and a source $f : Q \to \mathbb{R}$, the invariable physical laws are the compatibility $G = \nabla U$ for some U and the mass conservation $\nabla \cdot J = f$ (in other contexts, the second law is the balance of forces). We introduce

$$\mathcal{E} := \left\{ (G, J) \in L^2(Q; \mathbb{R}^n) \times L^2(Q; \mathbb{R}^n) \,|\, G = \nabla U \,,\, U \in H^1_0(Q, \mathbb{R}),\, \nabla \cdot J = f \right\} . \tag{1.1}$$

In the classical approach, one might be interested in the linear material law given by $J = AG$ for $A \in \mathbb{R}^{n \times n}$. We note that a pair $(G, J) \in \mathcal{E}$ with $J = AG$ can be found be solving the scalar elliptic equation $\nabla \cdot (A \nabla U) = f$.

In the data driven perspective, the material law is replaced by a data set $\mathcal{D}$. In a simple setting, we are given a local data set $\mathcal{D}_{\mathrm{loc}} := \{(g_i, j_i) \,|\, i \in I\} \subset \mathbb{R}^n \times \mathbb{R}^n$ for some index set I. This data set might be obtained by measurements, in this case the index set I is finite and $\mathcal{D}_{\mathrm{loc}}$ is a cloud of points in $\mathbb{R}^n \times \mathbb{R}^n$. The set of functions that respect the data is

$$\mathcal{D} := \left\{ (g, j) \in L^2(Q; \mathbb{R}^n) \times L^2(Q; \mathbb{R}^n) \,|\, (g(x), j(x)) \in \mathcal{D}_{\mathrm{loc}} \text{ for a.e. } x \in Q \right\} . \tag{1.2}$$

In the data driven perspective, the task is: *Find a pair $(G, J) \in \mathcal{E}$ that minimizes the distance to the set $\mathcal{D}$.*

We remark that we recover the classical problem if we introduce

$$\mathcal{D}^A_{\mathrm{loc}} := \{(g, j) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, j = Ag\} \tag{1.3}$$

and the corresponding set of functions $\mathcal{D}^A$ as in (1.2). For typical choices of Q, A, and f, the linear problem can be solved; in this case, there exists $(G, J) \in \mathcal{E} \cap \mathcal{D}^A$ and the minimization task has a solution that realizes the distance 0.

The advantage of the data driven perspective is the generality of the data set. In the minimization task above, an arbitrary data set $\mathcal{D}$ can be considered. Three different types of questions can be asked:

1. Minimality conditions: When $\mathcal{E} \cap \mathcal{D}$ is empty, what are conditions for minimizers of the distance?
2. Families of data sets: Given a family of data sets $\mathcal{D}_h$ and solutions (G_h, J_h) of the minimization problems, what can we say about limits?
3. Relaxation: Given $\mathcal{D}$ and sequences of pairs $(G^h, J^h) \in \mathcal{E}$ and $(g^h, j^h) \in \mathcal{D}$. Which limits are attainable in the sense of data convergence?

The present paper is devoted to the third question. We investigate a special data set: $\mathcal{D}_{\mathrm{loc}}$ is the union of $\mathcal{D}^A_{\mathrm{loc}}$ and $\mathcal{D}^B_{\mathrm{loc}}$, where $\mathcal{D}^A_{\mathrm{loc}}$ is as in (1.3) and $\mathcal{D}^B_{\mathrm{loc}}$ is a one-point set of a single outlier. In this setting, the minimization problem is solvable with distance 0 since $\mathcal{D}$ is larger than $\mathcal{D}^A$. Our interest is to study the relaxation problem.

The motivation to study the data set $\mathcal{D}_{\text{loc}} = \mathcal{D}^A_{\text{loc}} \cup \mathcal{D}^B_{\text{loc}}$ is to understand the effect of a single outlier in a cloud of measurement points. When an increasing number of data points approximates the plane of Hooke's law $\mathcal{D}^A_{\text{loc}}$, then the data driven solutions to these data sets approximate the classical solution with Hooke's law; this is one of the results in [6]. Our interest is an outlier: When the measurements contain a single point that is not in $\mathcal{D}^A_{\text{loc}}$, the data driven solutions can always use this data point in the further process. How far off can the data driven solutions be because of the single outlier? Our result characterizes the relaxed data set and shows that it is only changed locally in the vicinity of the outlier. In this sense, the outlier has only a limited effect on the data driven solutions.

In more mathematical terms, the analysis of this article is concerned with sequences of pairs $(G^h, J^h) \in \mathcal{E}$ and $(g^h, j^h) \in \mathcal{D}$ that converge in the sense of data convergence. We are interested in possible limit functions (g, j). Possible values of constant limit functions are denoted as $\mathcal{D}^{\text{relax}}_*$. The set $\mathcal{D}^{\text{relax}}_*$ contains $\mathcal{D}_{\text{loc}}$, it is the "data driven convexification" of $\mathcal{D}_{\text{loc}}$. Our main result is the characterization of $\mathcal{D}^{\text{relax}}_*$. We find that the set is strictly larger than $\mathcal{D}_{\text{loc}}$, but smaller than the convex hull of $\mathcal{D}_{\text{loc}}$. We will characterize $\mathcal{D}^{\text{relax}}_*$ as the union of $\mathcal{D}^A_{\text{loc}}$ with a truncated cone that connects the additional point $\mathcal{D}^B_{\text{loc}}$ with the hyperplane $\mathcal{D}^A_{\text{loc}}$. Denoting the truncated cone by C, our main result states $\mathcal{D}^{\text{relax}}_* = C \cup \mathcal{D}^A_{\text{loc}}$, see Theorem 1.2.

The proof consists of two parts. The inclusion $\mathcal{D}^{\text{relax}}_* \supset C \cup \mathcal{D}^A_{\text{loc}}$ requires a construction of a sequence of functions that use a fine mixture of materials. We will construct simple and iterated laminates. In order to realize a point on the lateral boundary of the cone C, it is sufficient to construct a simple laminate with phases A and B. For a point in the interior of C, an iterated laminate must be constructed. Such iterated laminates are quite standard, we mention [11] and [20].

The other part of the proof regards the inclusion $\mathcal{D}^{\text{relax}}_* \subset C \cup \mathcal{D}^A_{\text{loc}}$. We show this inclusion with an application of the div-curl lemma. In our context, the notion of data convergence of [6] provides exactly the prerequisites in order to use the div-curl lemma for data convergent sequences.

Literature. Relaxation is a classical problem in the calculus of variations. For a functional $I : X \to \bar{\mathbb{R}}$ on a Banach space X, one introduces the relaxed functional $I^{\text{relax}} : X \to \bar{\mathbb{R}}$ as $I^{\text{relax}}(u) := \inf \left\{ \liminf_k I(u^k) \,|\, u^k \rightharpoonup u \right\}$. A related notion is that of quasiconvexity; loosely speaking, quasiconvex functionals coincide with their relaxation. For fundamental results on these important concepts we refer to [2, 7, 10]. For a functional I which is not quasiconvex, one can construct laminates or more complex patterns in order to find the relaxed functional and/or the quasiconvex envelope of the integrand, see e.g. [3] and [5]. For an introduction we refer to [20].

The data driven perspective introduces a new concept of a relaxation. For a data set $\mathcal{D}$, the task is to study the relaxed data set, which consists of points that are attainable as limits in the sense of data convergence. A relaxed data set in this sense has been calculated in [6] for a problem in the vectorial case: For a data set that describes a non-monotone material law (corresponding to a non-convex energy), the authors determine the relaxed data set, compare (3.26) and Theorem 3.6 in [6]. The relaxed data set is larger than the original data set, but it is smaller than the convex hull of the original data set. A similar phenomenon appears in our main result.

We want to emphasize the close relation to homogenization. In the primal problem of homogenization, one *prescribes* different material laws in different points x of the macroscopic domain, and asks for the effective law for fine mixtures. Building upon such results, one then asks: With *any* material laws in different points x (material laws of some admissible set), which effective material laws can be obtained by homogenization? This leads to bounds for effective material laws as in [12, 13, 17] and to optimization of the distribution of the single material laws, see [1, 4]. For early results in this direction which also highlight the relation to relaxation see [18, 19].

Our main result may be interpreted in the perspective of homogenization. We use the two material laws $\mathcal{D}^A$ and $\mathcal{D}^B$ in different regions of the macroscopic domain, possibly in a fine mixture. We ask what effective laws can be obtained in the limit. The warning about this description is that $\mathcal{D}^B$ is not a linear relation and hence does not describe a material law in the classical setting of homogenization.

We will make use of the div-curl Lemma in the second part of the proof. This lemma is also used in the compensated compactness method of homogenization, see [14, 21]. Related concepts are those of Γ-convergence [8], Young-measures [10], and H-convergence [11].

For recent developments of the data driven approach we refer to [9] and [15], which are both concerned with numerical aspects.

1.1 The main result

Let $n \geq 2$ be the dimension, $Q \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $f \in H^{-1}(Q;\mathbb{R})$ a given source, and $A \in \mathbb{R}^{n\times n}$ a positive definite symmetric matrix. We consider the local material data sets

$$\mathcal{D}^A_{\text{loc}} := \{(g,j) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, j = Ag\}\,, \tag{1.4}$$

$$\mathcal{D}^B_{\text{loc}} := \{(g,j) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, g = 0, j = e_1\} = \{(0,e_1)\}\,, \tag{1.5}$$

and

$$\mathcal{D}_{\text{loc}} := \mathcal{D}^A_{\text{loc}} \cup \mathcal{D}^B_{\text{loc}}\,. \tag{1.6}$$

We therefore enrich the data set $\mathcal{D}^A_{\text{loc}}$ of the classical approach with the one point set $\mathcal{D}^B_{\text{loc}}$. We choose here $(0,e_1) \notin \mathcal{D}^A_{\text{loc}}$ as the position of the outlier; by elementary transformations, an arbitrary outlier can be analyzed. Functions with values in the data set are defined by

$$\mathcal{D} := \left\{(g,j) \in L^2(Q;\mathbb{R}^n) \times L^2(Q;\mathbb{R}^n) \,|\, (g(x),j(x)) \in \mathcal{D}_{\text{loc}} \text{ for a.e. } x\right\}\,. \tag{1.7}$$

We recall that the fundamental task in the data driven approach is to find $(G, J) \in \mathcal{E}$ from (1.1) that minimizes the distance to $\mathcal{D}$. In the above setting, a vanishing distance can be realized, since $\mathcal{D}$ is larger than $\mathcal{D}^A$.

Our interest is to study the relaxed data set. We focus here on constant states that can be approximated in the sense of data convergence with sequences in $\mathcal{E} \times \mathcal{D}$. We use the notion of data convergence of Definition 3.1 in [6].

Definition 1.1 (Relaxed data set)**.** *We use* $\mathcal{E}$ *from* (1.1) *and* $\mathcal{D}$ *from* (1.7)*. A pair* $(g, j) \in L^2(Q; \mathbb{R}^n) \times L^2(Q; \mathbb{R}^n)$ *is in the relaxed data set and we write* $(g, j) \in \mathcal{D}^{\mathrm{relax}}$ *if the following holds:*

There exist sequences (g^h, j^h) *and* (G^h, J^h) *and a limit* $(G, J) \in \mathcal{E}$ *such that, for every* h*,*

$$(G^h, J^h) \in \mathcal{E} \qquad \textit{and} \qquad (g^h, j^h) \in \mathcal{D}\,. \tag{1.8}$$

Furthermore, we demand that the pair $((g^h, j^h), (G^h, J^h))$ *converges in the sense of data convergence to* $((g, j), (G, J))$ *as* $h \to 0$*, which means that*

$$\begin{aligned} &g^h \rightharpoonup g\,, \quad j^h \rightharpoonup j\,, \quad G^h \rightharpoonup G\,, \quad J^h \rightharpoonup J \qquad \textit{in } L^2(Q; \mathbb{R}^n),\\ &g^h - G^h \to g - G\,, \qquad j^h - J^h \to j - J \qquad \textit{in } L^2(Q; \mathbb{R}^n)\,. \end{aligned}$$

We introduce the subset of attainable values,

$$\mathcal{D}^{\mathrm{relax}}_* := \left\{ (g, j) \in \mathbb{R}^n \times \mathbb{R}^n \middle| (g, j) \in \mathcal{D}^{\mathrm{relax}} \textit{ (as constant functions)} \right\}\,. \tag{1.9}$$

We remark that the relaxed data set $\mathcal{D}^{\mathrm{relax}}$ can also be characterized as a Kuratowski limit. The precise statement is provided in Lemma 1.4 below.

In our main result, we characterize the relaxed data set $\mathcal{D}^{\mathrm{relax}}_*$. We prove that it is the union of two sets: the hyperplane $\mathcal{D}^A_{\mathrm{loc}}$ and a truncated cone C with vertex in the outlier $\mathcal{D}^B_{\mathrm{loc}}$. The cone is truncated by the hyperplane $\mathcal{D}^A_{\mathrm{loc}}$.

We define the cone in the following steps. For $b \in [0, 1]$, we set

$$C_b := \Big\{ (g, j) \in \mathbb{R}^n \times \mathbb{R}^n \,\Big|\, g \cdot Ag \le (1 - b) g_1\,,\ j = be_1 + Ag \Big\}\,. \tag{1.10}$$

For fixed b, the set C_b is an n-dimensional closed ellipsoid in $\mathbb{R}^{n \times n}$. For $b = 1$, the ellipsoid degenerates to a point, $C_1 = \{(0, e_1)\} = \mathcal{D}^B_{\mathrm{loc}}$. On the other hand, for $b = 0$, every vector in C_0 satisfies $j = Ag$, hence $C_0 \subset \mathcal{D}^A_{\mathrm{loc}}$. We define the truncated cone C as

$$C := \bigcup_{b \in [0,1]} C_b\,. \tag{1.11}$$

Our main result is the characterization of the relaxed data set.

Theorem 1.2. *With the truncated cone* C *of* (1.11)*, the set* $\mathcal{D}^{\mathrm{relax}}_*$ *of Definition 1.1 is given by*

$$\mathcal{D}^{\mathrm{relax}}_* = C \cup \mathcal{D}^A_{\mathrm{loc}}\,. \tag{1.12}$$

Theorem 1.2 characterizes the relaxation of the data set in the context of data driven analysis. The convexification of a set consisting of an hyperplane and an outlier yields the union of the plane with a truncated cone that connects the outlier with the plane, compare Figure 2. In particular, the data driven relaxation does *not* yield the (classical) convexification of the original set, which is an infinite strip (the infinite strip can be regarded as the truncated cone with opening angle π; in this sense, the data driven relaxation yields a cone with smaller opening angle).

1.2 Comments on the main result

In this work, we concentrate on the study of constant functions g and j that can be approximated in the sense of data convergence. We therefore include the following open problem regarding the relaxed data set $\mathcal{D}^{\text{relax}}$.

Open Problem 1.3. *It is not clear whether or not $\mathcal{D}^{\text{relax}}$ is given by some local space $\mathcal{D}^{\text{relax}}_{\text{loc}}$ as in* (1.2)*. Furthermore, even if this is the case, it is not clear whether or not $\mathcal{D}^{\text{relax}}_{\text{loc}}$ coincides with $\mathcal{D}^{\text{relax}}_{*}$.*

Our definition of $\mathcal{D}^{\text{relax}}$ was given in terms of sequences. As noted above, the set $\mathcal{D}^{\text{relax}}$ can also be described in terms of a Kuratowski limit as in [6].

Lemma 1.4 (Kuratowski limit)**.** *Let data convergence be denoted as $\triangle - \lim$. We use Kuratowski convergence of sets, which coincides with Γ-convergence of the indicator functions. With these topological tools, the data relaxation can be written as a limit:*

$$\mathcal{D}^{\text{relax}} \times \mathcal{E} = K(\triangle)\text{-}\lim\ \mathcal{D} \times \mathcal{E}\,. \tag{1.13}$$

Proof. Similar to [6] the sequential characterization of the Kuratowski limit follows from an (equi-)transversality condition.

Step 1: Transversality. We claim that there exist constants $C_1, C_2 > 0$ such that every pair $z = (g, j) \in \mathcal{D}$ and $Z = (G, J) \in \mathcal{E}$ satisfies

$$\|z\|_{L^2(Q;\mathbb{R}^n)^2} + \|Z\|_{L^2(Q;\mathbb{R}^n)^2} \le C_1 \|z - Z\|_{L^2(Q;\mathbb{R}^n)^2} + C_2\,. \tag{1.14}$$

The inequality is concluded with the help of the positivity of $A \in \mathbb{R}^{n\times n}$, $\xi \cdot A\xi \ge c_0|\xi|^2$ for some $c_0 > 0$. From this estimate and the fact that $z \in \mathcal{D}$ implies $g = 0$ on $\{j \ne Ag\}$ we deduce

$$\begin{aligned} c_0 \int_Q |G|^2 \le \int_Q G \cdot AG = \int_Q G \cdot Ag + G \cdot A(G - g) \\ \le \int_{\{j=Ag\}} G \cdot j + |A| \|G\|_{L^2(Q;\mathbb{R}^n)} \|g - G\|_{L^2(Q;\mathbb{R}^n)}\,. \end{aligned} \tag{1.15}$$

Since $Z \in \mathcal{E}$ implies $G = \nabla U$ and $\nabla \cdot J = f$, we further obtain

$$\begin{aligned}\int_{\{j=Ag\}} G \cdot j \leq& \int_{\{j=Ag\}} G \cdot J + \|G\|_{L^2(Q;\mathbb{R}^n)} \|j - J\|_{L^2(Q;\mathbb{R}^n)} \\ =& \int_Q \nabla U \cdot J - \int_{\{j \neq Ag\}} G \cdot j + \|G\|_{L^2(Q;\mathbb{R}^n)} \|j - J\|_{L^2(Q;\mathbb{R}^n)} \\ =& -\int_Q Uf - \int_{\{j \neq Ag\}} (G - g) \cdot j + \|G\|_{L^2(Q;\mathbb{R}^n)} \|j - J\|_{L^2(Q;\mathbb{R}^n)} \\ \leq& C\|G\|_{L^2(Q;\mathbb{R}^n)} \|f\|_{H^{-1}(Q)} + \|j\|_{L^2(Q;\mathbb{R}^n)} \|g - G\|_{L^2(Q;\mathbb{R}^n)} \\ &+ \|G\|_{L^2(Q;\mathbb{R}^n)} \|j - J\|_{L^2(Q;\mathbb{R}^n)} , \end{aligned} \tag{1.16}$$

where we have used Poincaré's inequality and $G = \nabla U$ in the last step; here and below, C denotes a constant that depends only on A, Q, n and that may change from line to line. Together with (1.15) we deduce that

$$\begin{aligned}\|G\|^2_{L^2(Q;\mathbb{R}^n)} \leq& C\big(\|g - G\|^2_{L^2(Q;\mathbb{R}^n)} + \|j - J\|^2_{L^2(Q;\mathbb{R}^n)}\big) \\ &+ C\|f\|^2_{H^{-1}(Q)} + C\|j\|_{L^2(Q;\mathbb{R}^n)} \|g - G\|_{L^2(Q;\mathbb{R}^n)} . \end{aligned} \tag{1.17}$$

The triangle inequality yields an analogous inequality for g,

$$\begin{aligned}\|g\|^2_{L^2(Q;\mathbb{R}^n)} \leq& C\big(\|g - G\|^2_{L^2(Q;\mathbb{R}^n)} + \|j - J\|^2_{L^2(Q;\mathbb{R}^n)}\big) \\ &+ C(\|f\|^2_{H^{-1}(Q)} + C\|j\|_{L^2(Q;\mathbb{R}^n)} \|g - G\|_{L^2(Q;\mathbb{R}^n)} . \end{aligned} \tag{1.18}$$

Since $j = e_1$ holds in $\{j \neq Ag\}$ we next observe that

$$\int_Q |j|^2 \leq \int_{\{j=Ag\}} |Ag|^2 + |\{j \neq Ag\}| \leq C\|g\|^2_{L^2(Q;\mathbb{R}^n)} + |Q| .$$

Using (1.18) and Young's inequality, this provides

$$\|j\|_{L^2(Q;\mathbb{R}^n)} \leq C\big(\|g - G\|_{L^2(Q;\mathbb{R}^n)} + \|j - J\|_{L^2(Q;\mathbb{R}^n)}\big) + C(1 + \|f\|_{H^{-1}(Q)}) . \tag{1.19}$$

This estimate can be inserted in (1.17) and we obtain the corresponding estimate for G. By the triangle inequality, we control all functions g, G, j, J in $L^2(Q;\mathbb{R}^n)$ by the right-hand side of (1.19). This proves the transversality (1.14).

Step 2: Sequential characterization of Kuratowski convergence. The Kuratowski limit $K(\triangle)$-$\lim\ \mathcal{D} \times \mathcal{E}$ is given by the domain of the Γ-limit of the (constant sequence of the) indicator function of $\mathcal{D} \times \mathcal{E}$. To characterize this set consider any point $(z_0, Z_0) \in L^2(Q;\mathbb{R}^n)^2$. Since Γ-convergence is a local property, when computing the Γ-limit in this point we may restrict ourselves to any neighborhood of (z_0, Z_0) with respect to the △-topology. In particular, we may choose a neighborhood in which all pairs $(z, Z) \in L^2(Q;\mathbb{R}^n)^2$ satisfy $\|(z - z_0) - (Z - Z_0)\|_{L^2(Q;\mathbb{R}^n)} < 1$ (note that strong convergence of differences is part of the definition of △-convergence). Then the transversality property implies that we can restrict the computation of the Gamma limit to a bounded set in

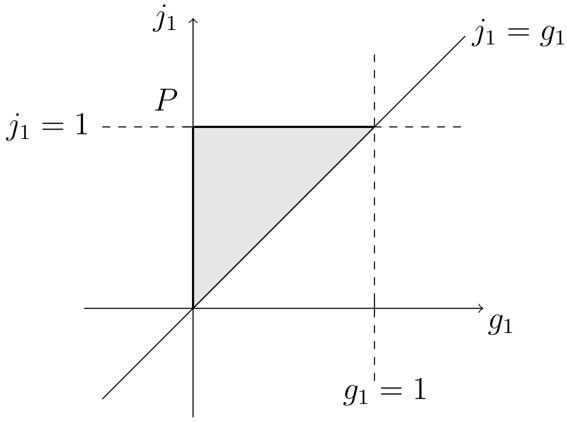

Figure 1: A sketch for $A = \mathrm{id}$, showing only the plane $(g_1, j_1) \in \mathbb{R}^2$. The diagonal line corresponds to the set $\mathcal{D}^A_{\mathrm{loc}}$ of points with $j = g$. The exceptional point P is $(g_1, j_1) = (0, 1)$, corresponding to the one-point set of additional data points, $\mathcal{D}^B_{\mathrm{loc}} = \{(0, e_1)\}$.

$L^2(Q;\mathbb{R}^n)^2$. On bounded sets the data convergence topology is metrizable. Hence the topological and the sequential characterization of Γ convergence coincide [8, Proposition 8.1].

The sequential characterization of the $\liminf$ and $\limsup$ inequalities that characterize Γ-convergence of the indicator function of $\mathcal{D} \times \mathcal{E}$ to $\mathcal{D}^{\mathrm{relax}} \times \mathcal{E}$ are described by the properties:

(i) For any sequence (z^h, Z^h) in $\mathcal{D} \times \mathcal{E}$ that $\triangle$-converges to a limit $(z, Z) \in L^2(Q;\mathbb{R}^n)^2$, there holds $(z, Z) \in \mathcal{D}^{\mathrm{relax}} \times \mathcal{E}$.

(ii) For any $(z, Z) \in \mathcal{D}^{\mathrm{relax}} \times \mathcal{E}$ there exists a sequence (z^h, Z^h) in $\mathcal{D} \times \mathcal{E}$ that $\triangle$-converges to (z, Z).

This is equivalent to the characterization of $\mathcal{D}^{\mathrm{relax}}$ given in Definition 1.1. □

1.3 Equivalent descriptions for the truncated cone C

A special case. In the case $n = 2$ and $A = \mathrm{id} \in \mathbb{R}^{2\times 2}$, the cone C is

$$C = \left\{ ((g_1, g_2), (g_1 + 1 - 2r, g_2)) \mid r \in [0, 1/2], (g_1 - r)^2 + g_2^2 \le r^2 \right\} . \qquad (1.20)$$

The last condition expresses that $g = (g_1, g_2)$ is contained in the disc $B_r((r, 0))$ with radius r and center $(r, 0)$. Because of $j_1 = g_1 + 1 - 2r$, the disc is mapped into an inclined plane.

In order to see the equivalence, it suffices to use the new variable $r = (1-b)/2$. The condition $g \cdot Ag \le (1 - b)g_1$ becomes $g_1^2 + g_2^2 \le 2rg_1$.

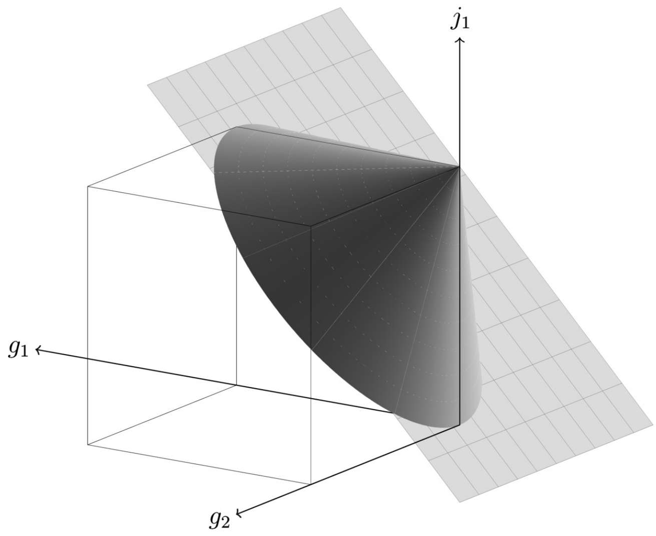

Figure 2: Three dimensional illustration of the cone C and part of the plane $\mathcal{D}^A_{\mathrm{loc}}$ in the g_1, g_2, j_1 space for $A = \mathrm{id}$.

The lateral boundary of C. For $b \in [0,1]$ fixed, the lateral boundary of C_b is

$$\partial_{\mathrm{lat}} C_b := \Big\{ (g,j) \in C \,\Big|\, g \cdot Ag = (1-b) g_1,\ j = be_1 + Ag \Big\} .$$

The lateral boundary of C can be expressed as $\partial_{\mathrm{lat}} C := \bigcup_{b\in[0,1]} \partial_{\mathrm{lat}} C_b$. With this notation, the boundary of C is given by

$$\partial C = \partial_{\mathrm{lat}} C \cup \left(\mathcal{D}^A_{\mathrm{loc}} \cap \left\{ g \cdot Ag \le g_1 \right\} \right) . \tag{1.21}$$

We can generalize (1.20) as follows. Let $y \in \mathbb{R}^n$ be a vector that satisfies $Ay = e_1$. We introduce the scalar product $\langle v_1, v_2 \rangle_A := v_1 \cdot Av_2$ and the associated norm $|\cdot|_A$. The corresponding sphere with center $\frac{1}{2} y$ that contains 0 is

$$S^A_y := \left\{ x \in \mathbb{R}^n \,\middle|\, \left| x - \tfrac{1}{2} y \right|_A = \tfrac{1}{2} \left| y \right|_A \right\} .$$

Then $(g,j) \in \partial_{\mathrm{lat}} C_b$ if and only if $g \in (1-b) S^A_y$ and $j = be_1 + Ag$. In fact, for $b = 1$, there holds $\partial_{\mathrm{lat}} C_b = \{(0, e_1)\}$ and the equivalence is valid. For $b \in [0,1)$, we find

$$\begin{aligned} g \cdot Ag = (1-b) g_1 &\iff \langle g, g - (1-b) y \rangle_A = 0 \\ &\iff \left| g - \tfrac{1-b}{2} y \right|_A^2 = \tfrac{(1-b)^2}{4} \left| y \right|_A^2 \\ &\iff \left| \frac{1}{1-b} g - \frac{1}{2} y \right|_A^2 = \frac{1}{4} \left| y \right|_A^2 . \end{aligned}$$

For later use we include the following alternative characterization of $\partial_{\mathrm{lat}} C_b$.

Lemma 1.5. *The lateral boundary can also be written as*

$$\partial_{\mathrm{lat}} C_b = \left\{ (g,j) \in \mathbb{R}^n \times \mathbb{R}^n \,\middle|\, \exists \nu \in \mathbb{R}^n \setminus \{0\} : \; g = (1-b) \frac{\nu_1}{\nu \cdot A\nu} \nu, \, j = b e_1 + A g \right\} . \tag{1.22}$$

Proof. We fix $b \in [0,1]$ and denote by K_b the right-hand side of (1.22). Consider any (g,j) with $j = be_1 + Ag$ and $g \neq 0$. Then

$$\begin{aligned} (g,j) \in \partial_{\mathrm{lat}} C_b &\iff g \cdot Ag = (1-b) g_1 \\ &\iff |g|_A^2 g = (1-b) g_1 g \\ &\iff g = (1-b) \frac{g_1}{|g|_A} \frac{g}{|g|_A} \\ &\implies (g,j) \in K_b \,, \end{aligned}$$

where the choice $\nu = g$ provides the last implication.

Vice versa, let (g,j) with $g \neq 0$ be in K_b. By definition, there exists $\nu \neq 0$ with $g = (1-b)\frac{\nu_1}{\nu \cdot A\nu}\nu$. We can calculate $g \cdot Ag = (1-b)^2 \nu_1^2/(\nu \cdot A\nu)$ and $(1-b)g_1 = (1-b)^2 \nu_1^2/(\nu \cdot A\nu)$, which shows $g \cdot Ag = (1-b)g_1$. □

2 Construction of approximating sequences

The goal of this section is to prove the inclusion

$$\mathcal{D}_*^{\mathrm{relax}} \supset C \cup \mathcal{D}_{\mathrm{loc}}^A \,, \tag{2.1}$$

which is one part of the claim of Theorem 1.2. Since trivial sequences can be chosen to verify $\mathcal{D}_{\mathrm{loc}}^A \subset \mathcal{D}_*^{\mathrm{relax}}$, we only have to show $C \subset \mathcal{D}_*^{\mathrm{relax}}$. For an arbitrary point on the lateral boundary of the cone C, we will use laminates to construct data convergent sequences (g^h, j^h) and (G^h, J^h).

In order to motivate the subsequent constructions, let us present what we can achieve in the case $A = \mathrm{id}$ with simple laminates of horizontal or vertical layers. With respect to Figure 1 we can say: The simple laminates show that all points in the vertical line of the cone and all points in the horizontal line of the cone can be constructed.

Remark 2.1 (Horizontal layers)**.** *We consider $A = \mathrm{id}$ and fix $b \in (0,1)$. We decompose Q into thin horizontal layers such that e_1 is a tangential vector of the interfaces. The layers have the width $(1-b)h$ and $b\,h$ in an alternating fashion. The layers with width $(1-b)h$ are called A-layers, the other layers are B-layers. In the A-layers, we set $J^h := j^h := G^h := g^h := 0$, in the B-layers we set $G^h := g^h := 0$ and $J^h := j^h := e_1$.*

By construction, $(g^h, j^h) \in \mathcal{D}$. Since layers are horizontal, J^h has a vanishing divergence. As a trivial function, G^h is a gradient. We find $(G^h, J^h) \in \mathcal{E}$. The functions converge weakly in $L^2(Q)$ and the differences $g^h - G^h$ and $j^h - J^h$ converge strongly. We therefore obtain that the vertical line $\{(g,j) \,|\, g = 0, j = (j_1, 0, ..., 0), j_1 \in [0,1]\}$ is contained in $\mathcal{D}_*^{\mathrm{relax}}$.

Remark 2.2 (Vertical layers). *We consider again $A = \mathrm{id}$. We proceed as in Remark 2.1, but we now decompose Q into thin layers with normal vector e_1. In the interior of Q, in the A-layers, we set $J^h := j^h := G^g := g^h := e_1$, in the B-layers we set $G^h := g^h := 0$ and $J^h := j^h := e_1$.*

Up to truncations near the boundary, one can verify $(G^h, J^h) \in \mathcal{E}$, $(g^h, j^h) \in \mathcal{D}$, and the convergence properties. We therefore obtain that the the horizontal line $\{(g, j) \,|\, g = (g_1, 0, ..., 0), g_1 \in [0, 1], j = 0\}$ is contained in $\mathcal{D}_*^{\mathrm{relax}}$.

After these motivating examples, we move on to the construction in the general case.

Lemma 2.3 (Simple laminates). *The set $\mathcal{D}_{\mathrm{loc}}^A$ and the boundary of the cone C of (1.11) are contained in $\mathcal{D}_*^{\mathrm{relax}}$,*

$$\mathcal{D}_{\mathrm{loc}}^A \cup \partial C \subset \mathcal{D}_*^{\mathrm{relax}}\,. \tag{2.2}$$

Proof. The inclusion $\mathcal{D}_{\mathrm{loc}}^A \subset \mathcal{D}_*^{\mathrm{relax}}$ holds trivially. Indeed, given $g \in \mathbb{R}^n$ and $j = Ag \in \mathbb{R}^n$, it suffices to use the constant functions $j^h = j$, $J^h = 0$, $g^h = g$, and $G^h = 0$. Analogously, the single point $\mathcal{D}_{\mathrm{loc}}^B$, which is the vertex of the cone, is contained in $\mathcal{D}_*^{\mathrm{relax}}$.

It therefore suffices to show that, for $b \in (0, 1)$, the set $\partial_{\mathrm{lat}} C_b$ belongs to $\mathcal{D}_*^{\mathrm{relax}}$. We consider a point $(g, j) \in \partial_{\mathrm{lat}} C_b$. By Lemma 1.5, we can express this point in the form

$$g = (1-b)\frac{\nu_1}{\nu \cdot A\nu}\nu\,, \quad j = be_1 + Ag$$

for some $\nu \in \mathbb{R}^n \setminus \{0\}$.

Step 1: Construction of approximating sequences. For $h > 0$, we consider the following layered subdivision of Q, using the direction ν,

$$\begin{aligned} B^h &:= \{x \in Q \,|\, x \cdot \nu \in [0, bh) + h\mathbb{Z}\}\,, \\ A^h &:= \{x \in Q \,|\, x \cdot \nu \in [bh, h) + h\mathbb{Z}\}\,. \end{aligned}$$

For the volume fractions we note that $|B^h| \to b|Q|$ and $|A^h| \to (1-b)|Q|$ as $h \searrow 0$. The field (g^h, j^h) is chosen as

$$g^h = 0 \quad \text{and} \quad j^h = e_1 \quad \text{in } B^h\,, \tag{2.3}$$

$$g^h = \frac{\nu_1}{\nu \cdot A\nu}\nu \quad \text{and} \quad j^h = Ag^h \quad \text{in } A^h\,. \tag{2.4}$$

By definition of the fields, $(g^h, j^h) \in \mathcal{D}$ is satisfied.

We note that the construction assures $j^h \cdot \nu = e_1 \cdot \nu$ in B^h and

$$j^h \cdot \nu = Ag^h \cdot \nu = A\left(\frac{\nu \cdot e_1}{\nu \cdot A\nu}\nu\right) \cdot \nu = (A\nu \cdot \nu)\,\frac{\nu \cdot e_1}{\nu \cdot A\nu} = e_1 \cdot \nu \quad \text{in } A^h\,.$$

This shows $\nabla \cdot j^h = 0$ in Q.

We want to find a function $u^h : Q \to \mathbb{R}$ that satisfies

$$g^h = \nabla u^h \quad \text{in } Q\,.$$

The function u^h can be constructed explicitly. We use the continuous (and piecewise affine) function $v^h : \mathbb{R} \to \mathbb{R}$ with $v^h(0) = 0$ and with the derivatives $\partial_\xi v^h(\xi) = 0$ for $\xi \in (0, bh) + h\mathbb{Z}$ and $\partial_\xi v^h(\xi) = \nu_1/(\nu \cdot A\nu)$ for $\xi \in (bh, h) + h\mathbb{Z}$. Using v^h, we set

$$u^h(x) := v^h(x \cdot \nu) \quad \text{ with } \ \nabla u^h(x) = \partial_\xi v^h(x \cdot \nu)\, \nu = g^h(x)\,.$$

We may introduce

$$u(x) := (1 - b)\frac{\nu_1}{\nu \cdot A\nu} x \cdot \nu \quad \text{for } x \in Q\,.$$

Then $u^h \rightharpoonup u$ and $\|u^h - u\|_{L^\infty} \le Ch$ hold for a constant C that does not depend on h.

In order to define a corresponding pair (G^h, J^h), we choose a cut-off function $\varphi_h \in C^1_c(Q)$ with values in $[0, 1]$, satisfying $\varphi_h = 1$ in $\{x \in Q \,|\, \mathrm{dist}(x, \partial Q) \ge 2h\}$ and $\varphi_h = 0$ in $\{x \in Q \,|\, \mathrm{dist}(x, \partial Q) \le h\}$ and $|\nabla \varphi_h| \le \frac{2}{h}$. Furthermore, we fix a function J_f with $\nabla \cdot J_f = f$. With these preparations we define

$$U^h := (u^h - u)\varphi_h\,, \quad G^h = \nabla U^h\,, \quad J^h := j^h + J_f\,. \tag{2.5}$$

Step 2: Verification of the properties. By definition, G^h is a gradient of a function in $H^1_0(Q)$. The field J^h has the divergence $\nabla \cdot J^h = \nabla \cdot j^h + \nabla \cdot J_f = f$. This shows $(G^h, J^h) \in \mathcal{E}$.

We now verify the data convergence property. We clearly have

$$g^h \rightharpoonup g := (1 - b)\frac{\nu_1}{\nu \cdot A\nu}\nu\,, \tag{2.6}$$

$$j^h \rightharpoonup j := be_1 + (1 - b)\frac{\nu_1}{\nu \cdot A\nu}A\nu\,, \tag{2.7}$$

$$U^h \rightharpoonup 0, \qquad J^h \rightharpoonup j + J_f \tag{2.8}$$

in $L^2(Q; \mathbb{R}^n)$. Finally we have $j^h - J^h = J_f$ and

$$\begin{aligned} g^h - G^h &= \nabla u^h - \nabla U^h = \nabla u^h - \varphi_h \nabla(u^h - u) - (u^h - u)\nabla\varphi_h \\ &= (1 - \varphi_h)\nabla u^h + \varphi_h \nabla u - (u^h - u)\nabla\varphi_h \to g\,. \end{aligned}$$

Here, the convergence follows from the following facts: $(1 - \varphi_h) \to 0$ strongly in $L^2(Q)$ implies convergence to 0 for the first term. The pointwise convergence $\varphi_h \nabla u \to \nabla u$ with the uniform bound $|\varphi_h \nabla u| \le |\nabla u|$ implies strong convergence of the second term to $g = \nabla u$. The last term $(u^h - u)\nabla\varphi_h$ is uniformly bounded and converges to zero almost everywhere, hence strongly to 0.

Altogether, we obtain that $\big((g^h, j^h), (G^h, J^h)\big) \to \big((g, j), (G, J)\big)$ in the sense of data convergence and conclude that $(g, j) \in \mathcal{D}^{\mathrm{relax}}_*$. □

We next show that also the interior of the cone C will be reached by suitable iterated laminate constructions.

Lemma 2.4 (Iterated laminates)**.** *The cone C of* (1.11) *is contained in $\mathcal{D}_*^{\mathrm{relax}}$,*

$$C \subset \mathcal{D}_*^{\mathrm{relax}} . \tag{2.9}$$

Proof. In view of Lemma 2.3, it remains to show that the interior of the cone is contained in $\mathcal{D}_*^{\mathrm{relax}}$. Let therefore $p_C = (g_C, j_C) \in C \setminus \partial C$ be arbitrary; our aim is to show $p_C \in \mathcal{D}_*^{\mathrm{relax}}$. This is done by constructing sequences $(g^h, j^h) \in \mathcal{D}$ and $(G^h, J^h) \in \mathcal{E}$ as before. In this proof, however, we have to use iterated laminates.

Step 1: Preparations. Let $p_C = (g_C, j_C) \in \mathring{C}$ be a point in the interior of the cone. We show in Lemma A.1 of the appendix that we can write p_C as a convex combination as follows: There exist two points $p_A = (g_A, j_A) \in \mathcal{D}_{\mathrm{loc}}^A$ and $p_L = (g_L, j_L) \in \partial_{\mathrm{lat}} C$ and a parameter $\lambda \in (0, 1)$ such that

$$p_C = \lambda p_L + (1 - \lambda) p_A \tag{2.10}$$

and such that, additionally,

$$(j_A - j_L) \cdot (g_A - g_L) = 0 . \tag{2.11}$$

As in the proof of Lemma 2.3 we exploit Lemma 1.5: We can express the point $p_L \in \partial_{\mathrm{lat}} C$ as a convex combination with some vector $\nu \in \mathbb{R}^n \setminus \{0\}$:

$$(g_L, j_L) = p_L = b\, p_b + (1 - b)\, p_a = b\, (g_b, j_b) + (1 - b)\, (g_a, j_a)$$

with

$$g_a = \frac{\nu_1}{\nu \cdot A\nu} \nu , \quad j_a = A g_a , \quad g_b = 0 , \quad j_b = e_1 .$$

The iterated laminate is constructed as a coarse laminate with layers of width $\sqrt{h}$ and a fine laminate with layers of order h. Every second layer of the coarse mesh uses $p_A = (g_A, j_A)$. The fine laminate uses (g_a, j_a) and (g_b, j_b). The two functions in the fine layer produce, in average, $p_L = (g_L, j_L)$. The mixture of the coarse layers with values p_A and p_L provide the desired values p_C. For a sketch see Figure 3.

Step 2: Construction of the approximating sequence. From now on, the points p_C, p_A, p_L, p_a, p_b, and the volume fractions λ and b are fixed. In addition to ν, we introduce the normal vector

$$\theta := (g_A - g_L)/\|g_A - g_L\| . \tag{2.12}$$

For every $k \in \mathbb{Z}$, the coarse layers L_k^h and M_k^h are defined as

$$L_k^h := \left\{ x \in Q \,\middle|\, x \cdot \theta \in k\sqrt{h} + \left[0, \lambda\sqrt{h}\right) \right\} ,$$
$$M_k^h := \left\{ x \in Q \,\middle|\, x \cdot \theta \in k\sqrt{h} + \left[\lambda\sqrt{h}, \sqrt{h}\right) \right\} .$$

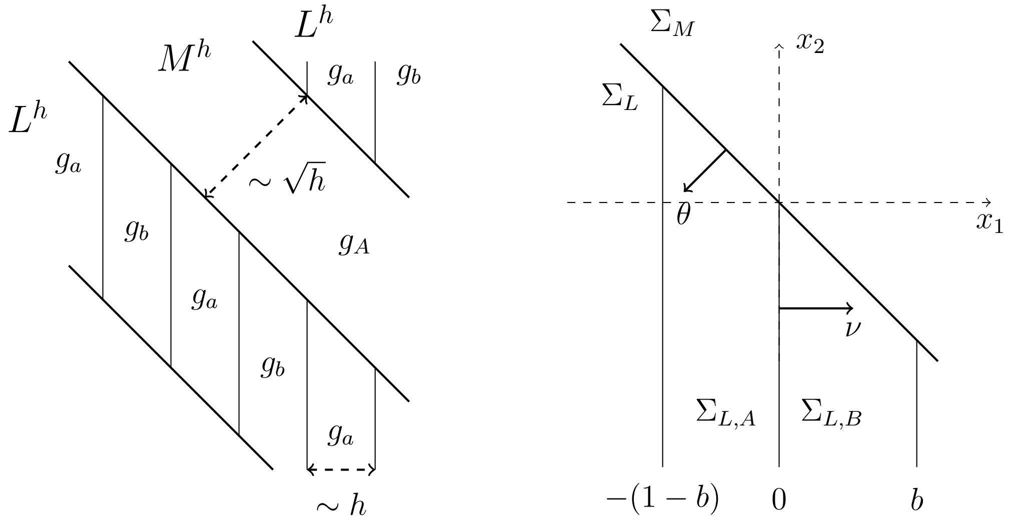

Figure 3: Left: A sketch of the iterated laminate. Right: The local geometry.

The unions are denoted as $L^h := \bigcup_{k\in\mathbb{Z}} L^h_k$ and $M^h := \bigcup_{k\in\mathbb{Z}} M^h_k$.

The iterated laminate is based on a subdivision of every layer L^h_k. We set

$$L^h_{k,b} := \left\{ x \in L^h_k \,\middle|\, x\cdot\nu \in [0, bh) + h\mathbb{Z} \right\} ,$$
$$L^h_{k,a} := \left\{ x \in L^h_k \,\middle|\, x\cdot\nu \in [bh, h) + h\mathbb{Z} \right\} .$$

The unions are denoted as $L^h_b := \bigcup_{k\in\mathbb{Z}} L^h_{k,b}$ and $L^h_a := \bigcup_{k\in\mathbb{Z}} L^h_{k,a}$.

We define the fields g^h and j^h as

$$g^h(x) := \begin{cases} g_A & \text{for } x \in M^h\,, \\ g_a & \text{for } x \in L^h_a\,, \\ g_b & \text{for } x \in L^h_b\,, \end{cases} \qquad j^h(x) := \begin{cases} j_A & \text{for } x \in M^h\,, \\ j_a & \text{for } x \in L^h_a\,, \\ j_b & \text{for } x \in L^h_b\,. \end{cases} \tag{2.13}$$

We next define a function $u^h : \mathbb{R}^n \to \mathbb{R}$ that is piecewise affine and which has piecewise the gradient g^h. In order to construct u^h we introduce the points $x_k := k\sqrt{h}\,\theta \in \mathbb{R}^n$ for $k \in \mathbb{Z}$. The point x_k is chosen such that, if x_k happens to be in Q, it is a point in $\partial L^h_k \cap \partial M^h_{k-1}$. By construction, the weak limit of g^h is g_C. We therefore set $u^h(x_k) := g_C \cdot x_k$. Accordingly, in the layer M^h_{k-1}, we set $u^h(x) := g_C \cdot x_k + g_A \cdot (x - x_k)$. In the layer L^h_k, we define u^h as the unique continuous function with $u^h(x_k) = g_C \cdot x_k$, with the gradient g_a in $L^h_{k,a}$ and the gradient g_b in $L^h_{k,b}$. A continuous function u^h exists in L^h_k since $(g_a - g_b) \parallel \nu$.

As in the proof of Lemma 2.3, we can use a cutoff function φ^k_h in the layer L^h_k to construct $U^h_k : L^h_k \to \mathbb{R}$ with bounded gradient such that

$$U^h_k(x) = g_C \cdot x_k + g_L \cdot (x - x_k) \quad \text{for } x \in \partial L^h_k\,,$$
$$U^h_k(x) = u^h(x) \quad \text{for } x \in L^h_k \text{ with } \operatorname{dist}(x, \partial L^h_k) > h\,.$$

The function U^h can be defined on all of Q by setting

$$U^h(x) := \begin{cases} U_k^h(x) & \text{for } x \in L_k^h\,, \\ u^h(x) & \text{for } x \in M_k^h\,. \end{cases}$$

By construction, the function U^h is continuous. This property follows by inserting the vector $\lambda\sqrt{h}\,\theta$, the normal vector of layer L_k^h, where U^h has the averaged gradient g_L, and the vector $(1-\lambda)\sqrt{h}\,\theta$ in normal direction of layer M_k^h, where U^h has the gradient g_A:

$$g_L \cdot \lambda\sqrt{h}\,\theta + g_A \cdot (1-\lambda)\sqrt{h}\,\theta = (\lambda g_L + (1-\lambda) g_A) \cdot \sqrt{h}\,\theta = g_C \cdot \sqrt{h}\,\theta\,.$$

This is consistent with the choice of $U^h(x_{k+1})$.

Furthermore, the function U^h has a bounded gradient. This can be seen as in the proof of Lemma 2.3: In the layer L_k^h, the difference between $u^h(x)$ and $g_C \cdot x_k + g_L \cdot (x - x_k)$ is of order h (uniformly in x) since g_L is the average slope of u^h and u^h oscillates at order h. The gradient of the cutoff function φ_h^k is of order h^{-1}.

The gradient of U^h coincides with g^h except for a set with a volume bounded by $C\sqrt{h}$: the strips of width h in the layers L_k^h, and there are $O(1/\sqrt{h})$ such layers. With the choice $G^h := \nabla U^h$, this guarantees the strong convergence $\|g^h - G^h\|_{L^2(Q)} \to 0$.

We do not perform here the modification of U^h at the boundary ∂Q. We restrict ourselves to the observation that the weak limit of the sequence U^h is the function $U : \mathbb{R}^n \to \mathbb{R}$, $U(x) = g_C \cdot x$. Moreover, there holds $\|U^h - U\|_{L^\infty} \le C\sqrt{h}$ for some constant $C > 0$, which is independent of h. This fact allows to use the cutoff argument of Lemma 2.3 at ∂Q.

The construction is complete up to the choice of the sequence J^h, which we postpone to Step 3. At this point, we have found the following functions: (g^h, j^h) are functions that are compatible with the data set, G^h is a gradient (after the modification at ∂Q, it is the gradient of an $H_0^1(Q)$-function), and $g^h - G^h$ converges strongly in $L^2(Q)$. All functions converge weakly in $L^2(Q)$ with

$$\begin{aligned} g^h &\rightharpoonup \lambda g_L + (1-\lambda) g_A = g_C\,, \\ j^h &\rightharpoonup \lambda j_L + (1-\lambda) j_A = j_C\,. \end{aligned}$$

If an appropriate sequence J^h can be constructed (with the right divergence and such that the difference to j^h is strongly convergent), this shows that $p_C = (g_C, j_C)$ is in the relaxed data set $\mathcal{D}_*^{\text{relax}}$.

Step 3: The divergence of the approximation. Let us calculate the divergence of j^h. In M^h, the flux is constant and hence $\nabla \cdot j^h = \nabla \cdot j_A = 0$ in M^h. In L^h, the construction uses the fluxes j_a and j_b which satisfy $(j_a - j_b) \cdot \nu = (Ag_a - e_1) \cdot \nu = \left(\frac{\nu_1}{\nu \cdot A\nu} A\nu - e_1\right) \cdot \nu = 0$. This shows that j^h satisfies $\nabla \cdot j^h = 0$ in L^h.

Along ∂L^h, the function j^h has the jumps

$$[j^h]\cdot\theta = (j_A - e_1)\cdot\theta \quad \text{on} \quad \partial M^h \cap \partial L_b^h\,, \tag{2.14}$$

$$[j^h]\cdot\theta = \left(j_A - \frac{\nu_1}{\nu\cdot A\nu}A\nu\right)\cdot\theta \quad \text{on} \quad \partial M^h \cap \partial L_a^h\,. \tag{2.15}$$

Important for the following construction is that the total flux through two subsequent pieces of ∂M^h vanishes:

$$\begin{aligned} b\,(j_A - e_1)\cdot\theta + (1-b)\,&\left(j_A - \frac{\nu_1}{\nu\cdot A\nu}A\nu\right)\cdot\theta \\ &= j_A\cdot\theta - \left(be_1 + (1-b)\,\frac{\nu_1}{\nu\cdot A\nu}A\nu\right)\cdot\theta \\ &= (j_A - j_L)\cdot\theta = 0 \end{aligned} \tag{2.16}$$

by (2.7) and (2.11).

After a rescaling by h and a shift into the origin, the local geometry is as follows:

$$\Sigma_M := \{x\in\mathbb{R}^n\,|\,x\cdot\theta<0\}\,,\qquad \Sigma_L := \{x\in\mathbb{R}^n\,|\,x\cdot\theta>0\}\,,$$
$$\Sigma_{L,B} := \{x\in\Sigma_L\,|\,0<x\cdot\nu<b\}\,,\quad \Sigma_{L,A} := \{x\in\Sigma_L\,|\,b-1<x\cdot\nu<0\}\,,$$

compare the right part of Figure 3. We emphasize that only three regions of unit dimensions are considered.

We claim that there exists a bounded vector field $p:\Sigma_L\to\mathbb{R}^n$ with support in $\{x\in\Sigma_L\,|\,x\cdot\theta<1\}$ and with the properties

$$\nabla\cdot p = 0 \ \text{ in } \Sigma_L\,, \tag{2.17}$$

$$p\cdot\theta = j_A - e_1 \ \text{ on } \partial\Sigma_{L,B}\cap\partial\Sigma_L\,, \tag{2.18}$$

$$p\cdot\theta = j_A - \frac{\nu_1}{\nu\cdot A\nu}A\nu \ \text{ on } \partial\Sigma_{L,A}\cap\partial\Sigma_L\,, \tag{2.19}$$

$$p\cdot\nu = 0 \quad \text{on } \partial\left(\overline{\Sigma_{L,A}}\cup\overline{\Sigma_{L,B}}\right)\setminus\partial\Sigma_L\,. \tag{2.20}$$

The divergence in the first line is understood in the sense of distributions. The function can be constructed in $\mathbb{R}^2$ as follows: We use an ansatz with a rotated gradient, $p := \nabla^\perp\Phi = (-\partial_2\Phi, \partial_1\Phi)$ with a smooth function Φ that is piecewise affine on the boundary $\partial\left(\overline{\Sigma_{L,A}}\cup\overline{\Sigma_{L,B}}\right)$. The fact that the total flux vanishes by (2.16) implies that Φ can be chosen such that it vanishes on $\partial\left(\overline{\Sigma_{L,A}}\cup\overline{\Sigma_{L,B}}\right)\setminus\partial\Sigma_L$. This allows, in particular, to choose a compactly supported function Φ. The rotated gradient p has all the desired properties. In higher dimension, the two-dimensional function can be extended as a constant function in the remaining directions.

Rescaling p as $p^h(x) := p(x/h)$ and extending the function p^h first periodically with period h in all directions perpendicular to θ, then extending the result periodically with period $\sqrt{h}$ in direction θ, we obtain a function p^h that has the same distributional divergence as j^h, see (2.14) and (2.15).

We construct $J^h(x) := j^h - p^h$. This choice assures $\nabla \cdot J^h = 0$. Furthermore, the strong convergence $j^h - J^h = p^h \to 0$ in $L^2(Q)$ is a consequence of the boundedness of p together with the fact that $p^h \neq 0$ holds only on a set with volume fraction of order $h/\sqrt{h} = \sqrt{h}$.

This concludes the proof for $f = 0$. If a function J^h with $\nabla \cdot J^h = f \neq 0$ has to be constructed, it suffices to add an h-independent function J_f as in the proof of Lemma 2.3. □

3 Necessary conditions for relaxed data points

The goal of this section is to prove the inclusion

$$\mathcal{D}_*^{\text{relax}} \subset C \cup \mathcal{D}_{\text{loc}}^A , \tag{3.1}$$

which is the inclusion in the claim of Theorem 1.2 that is not yet shown. In order to show (3.1), it suffices to fix an arbitrary pair of vectors $(g, j) \in \mathcal{D}_*^{\text{relax}} \subset \mathbb{R}^n \times \mathbb{R}^n$ and to show $(g, j) \in C \cup \mathcal{D}_{\text{loc}}^A$.

By Definition 1.1, the condition $(g, j) \in \mathcal{D}_*^{\text{relax}}$ means that there exist sequences (g^h, j^h) and (G^h, J^h) and a limit $(G, J) \in \mathcal{E}$ such that

$$\begin{aligned} g^h \rightharpoonup g\,, \quad j^h \rightharpoonup j\,, \quad G^h \rightharpoonup G\,, \quad J^h \rightharpoonup J \qquad &\text{in } L^2(Q; \mathbb{R}^n), \\ g^h - G^h \to g - G\,, \qquad j^h - J^h \to j - J \qquad &\text{in } L^2(Q; \mathbb{R}^n)\,, \end{aligned}$$

as $h \to 0$. The pairs (G^h, J^h) are in $\mathcal{E}$, i.e.: $G^h = \nabla U^h$ is the gradient of some $U^h \in H_0^1(Q)$ and $\nabla \cdot J^h = f$. The pairs (g^h, j^h) are in the data set $\mathcal{D}$ of (1.7).

3.1 Calculations for $A = \text{id}$ and $n = 2$

In this subsection, we obtain (3.1) in a simple case, namely $A = \text{id}$ and $n = 2$. The general case is treated in the next subsection and does not use any of the intermediate results of this section, which is included only in order to illustrate the approach in a simple setting.

For a sequence (g^h, j^h) we denote by $B^h \subset Q$ those points $x \in Q$ for which $(g^h(x), j^h(x))$ is in $\mathcal{D}_{\text{loc}}^B$ of (1.5),

$$B^h := \left\{x \in Q \,|\, (g^h(x), j^h(x)) \in \mathcal{D}_{\text{loc}}^B\right\} = \left\{x \in Q \,|\, g^h(x) = 0,\, j^h(x) = e_1\right\} . \tag{3.2}$$

The complement is denoted as $A^h := Q \setminus B^h$. Because of the bounds $0 \leq |B^h| \leq |Q|$, we can select a subsequence $h \to 0$ (not relabelled) and a limit $b \in [0, |Q|]$ such that

$$b = \lim_{h\to 0} \frac{|B^h|}{|Q|} . \tag{3.3}$$

Averages. We can calculate, using the weak convergence of g^h, the property $g^h(x) = 0$ for $x \in B^h$, then $g^h(x) = j^h(x)$ for $x \in A^h$, then $g^h(x) = e_1$ for $x \in B^h$, and finally the weak convergence of j^h:

$$g\,|Q| \leftarrow \int_Q g^h = \int_{A^h} g^h = \int_{A^h} j^h = \int_Q j^h - \int_{B^h} e_1 \to (j - be_1)\,|Q|\,. \tag{3.4}$$

We conclude $j = g + be_1$. With reference to Figure 1, we see that the point (g_1, j_1) is above the diagonal.

Div-curl lemma. The convergence properties allow to calculate integrals over the product $g^h \cdot j^h$. In the subsequent calculation, we use the standard div-curl lemma in $L^2(Q)$ for the product $G^h \cdot J^h$, and the strong convergence of differences in the other terms. In the limit $h \to 0$, we obtain

$$\begin{aligned}\int_Q g^h \cdot j^h &= \int_Q (G^h + (g^h - G^h)) \cdot (J^h + (j^h - J^h))\\ &\to \int_Q G \cdot J + \int_Q G \cdot (j - J) + \int_Q (g - G) \cdot j = \int_Q g \cdot j\,.\end{aligned}$$

Calculation 1 with the div-curl lemma. We calculate with the div-curl lemma, exploiting $g^h(x) = 0$ for $x \in B^h$ and $g^h(x) = j^h(x)$ for $x \in A^h$:

$$\int_Q g \cdot j \leftarrow \int_Q g^h \cdot j^h = \int_{A^h} g^h \cdot j^h = \int_{A^h} |g^h|^2 = \int_Q |g^h|^2\,. \tag{3.5}$$

Forming the limes inferior (and recalling that g and j are constant functions), we obtain $|g|^2 \le g \cdot j$. Hence, because of $j_2 = g_2$, the relation $g_1^2 \le g_1\, j_1 = g_1^2 + bg_1$. This implies, in particular,

$$b > 0 \quad \Rightarrow \quad g_1 \ge 0\,. \tag{3.6}$$

Referring to Figure 1, we see that (g_1, j_1) is to the right of the vertical axis.

Calculation 2 with the div-curl lemma. We now exploit the div-curl lemma slightly differently:

$$\int_Q g \cdot j \leftarrow \int_Q g^h \cdot j^h = \int_{A^h} g^h \cdot j^h = \int_{A^h} |j^h|^2 = \int_Q |j^h|^2 - \int_{B^h} 1\,. \tag{3.7}$$

Forming the limes inferior, we obtain $|j|^2 \le b + g \cdot j$. Inserting $j_2 = g_2$ and $j_1 = g_1 + b$ we obtain $j_1^2 \le b + (j_1 - b)\, j_1$, which provides $b(1 - j_1) \ge 0$. We have found

$$b > 0 \quad \Rightarrow \quad j_1 \le 1\,. \tag{3.8}$$

In Figure 1, the point (g_1, j_1) is below the horizontal line $j_1 = 1$.

The cone conditions. The above considerations do not imply any conditions for the second components, g_2 and j_2. With the next calculation, we do not only find conditions for g_2, but we additionally reproduce most of the above findings.

Assuming $b \neq 1$, we use the shorthand notation $\beta := (1-b)^{-1}$ and calculate

$$\begin{aligned}
0 &\leq \int_{A^h} |g^h - \beta g|^2 \\
&= \int_{A^h} \left\{ g^h \cdot g^h - 2\beta g^h \cdot g + \beta^2 |g|^2 \right\} \\
&= \int_Q \left\{ g^h \cdot j^h - 2\beta g^h \cdot g + \beta^2 |g|^2 \right\} - \beta^2 |g|^2 |B^h| \\
&\to |Q| \left(g \cdot j - 2\beta g \cdot g + \beta^2 |g|^2 - b\beta^2 |g|^2 \right) .
\end{aligned}$$

Dividing by $|Q|$, this yields

$$\begin{aligned}
0 &\leq g \cdot j - 2\beta g \cdot g + \beta^2 |g|^2 - b\beta^2 |g|^2 \\
&= g \cdot (g + b e_1) + |g|^2 (-2\beta + \beta^2 - b\beta^2) \\
&= b g_1 + |g|^2 \left(1 - 2\beta + \beta^2 - b\beta^2 \right) \\
&= b g_1 + |g|^2 ((1-\beta)^2 - b\beta^2) \, .
\end{aligned}$$

Using $(1-\beta)^2 = (1 - \frac{1}{1-b})^2 = b^2 \beta^2$ and $(b-1)\beta = 1$ we find

$$0 \leq b g_1 + |g|^2 (b^2 \beta^2 - b\beta^2) = b g_1 - b\beta |g|^2 \, . \tag{3.9}$$

Inequality (3.9) yields the desired restrictions on the pair (g, j). We distinguish three cases.

In the case $b = 0$ the relation $j = g + b e_1 = g$ implies $(g, j) \in \mathcal{D}^A_{\mathrm{loc}}$.

In the case $b = 1$ there holds $g \leftharpoonup g^h = g^h \mathbf{1}_{A^h} \rightharpoonup 0$ by strong convergence $\mathbf{1}_{A^h} \to 0$. Hence, in this case, $g = 0$ and $j = g + b e_1 = e_1$. This yields $(g, j) \in \mathcal{D}^B_{\mathrm{loc}}$.

In the case $b \in (0, 1)$ we conclude with (3.9): $|g|^2 \leq (1-b) g_1$ can be written as

$$\left(g_1 - \frac{1-b}{2} \right)^2 + g_2^2 \leq \left(\frac{1-b}{2} \right)^2 .$$

This is the defining relation of the cone C, compare (1.20). Claim (3.1) is shown for $A = \mathrm{id}$ and $n = 2$.

3.2 The general case

In this subsection, we treat the case of a general matrix A. Moreover, we show a result on $\mathcal{D}^{\mathrm{relax}}$, and not only a result on $\mathcal{D}^{\mathrm{relax}}_*$. The proof of Theorem 1.2 is complete with relation (3.10) of the subsequent proposition. The proposition provides additionally relation (3.12), which is slightly stronger.

Proposition 3.1. *There holds*

$$\mathcal{D}_*^{\mathrm{relax}} \subset C \cup \mathcal{D}_{\mathrm{loc}}^A \,. \tag{3.10}$$

Moreover, with $\mathcal{D}^{\mathrm{relax}}$ *of Definition 1.1 and with*

$$\mathcal{D}_{\mathrm{fct}}^{C\cup\mathcal{D}_{\mathrm{loc}}^A} := \left\{(g,j) \in L^2(Q;\mathbb{R}^n)^2 \,|\, (g,j)(x) \in C \cup \mathcal{D}_{\mathrm{loc}}^A \text{ for a.e. } x \in Q\right\}, \tag{3.11}$$

there holds

$$\mathcal{D}^{\mathrm{relax}} \subset \mathcal{D}_{\mathrm{fct}}^{C\cup\mathcal{D}_{\mathrm{loc}}^A} \,. \tag{3.12}$$

Proof. We note that (3.12) implies (3.10). Indeed, let $(g,j) \in \mathcal{D}_*^{\mathrm{relax}}$ be a pair of vectors in $\mathbb{R}^n \times \mathbb{R}^n$. Once more we identify the vectors with constant functions on Q. The constant functions are in $\mathcal{D}^{\mathrm{relax}}$ by definition of $\mathcal{D}_*^{\mathrm{relax}}$, see (1.9). Relation (3.12) implies $(g,j) \in \mathcal{D}_{\mathrm{fct}}^{C\cup\mathcal{D}_{\mathrm{loc}}^A}$. Since the functions are constant, there holds $(g,j) \in C \cup \mathcal{D}_{\mathrm{loc}}^A$. This shows (3.10).

Step 1: Preparation. In order to prove (3.12), we fix a pair $(g,j) \in L^2(Q;\mathbb{R}^n)^2$ in the relaxed data set $\mathcal{D}^{\mathrm{relax}}$, which means that there exist sequences (g^h, j^h) in $\mathcal{D}$ and (G^h, J^h) in $\mathcal{E}$ with data convergence such that (g^h, j^h) weakly converges to (g,j). Our aim is to show $(g(x), j(x)) \in C \cup \mathcal{D}_{\mathrm{loc}}^A$ for almost every $x \in Q$.

The approximating sequences (g^h, j^h) in $\mathcal{D}$ and (G^h, J^h) in $\mathcal{E}$ with limit $(G,J) \in \mathcal{E}$ satisfy, as $h \to 0$,

$$\begin{aligned} &g^h \rightharpoonup g\,, \quad j^h \rightharpoonup j\,, \quad G^h \rightharpoonup G\,, \quad J^h \rightharpoonup J \quad \text{ in } L^2(Q;\mathbb{R}^n),\\ &g^h - G^h \to g - G\,, \qquad j^h - J^h \to j - J \quad \text{ in } L^2(Q;\mathbb{R}^n)\,. \end{aligned}$$

We denote by $B^h \subset Q$ those points $x \in Q$ for which $(g^h(x), j^h(x))$ is in $\mathcal{D}_{\mathrm{loc}}^B$,

$$B^h := \left\{x \in Q \,|\, (g^h(x), j^h(x)) \in \mathcal{D}_{\mathrm{loc}}^B\right\} = \left\{x \in Q \,|\, g^h(x) = 0,\, j^h(x) = e_1\right\}. \tag{3.13}$$

The complement is denoted as $A^h := Q \backslash B^h$. Because of the bound $0 \le |B^h| \le |Q|$, we can select a subsequence (not relabeled) and a limit $b \in L^\infty(Q)$ such that

$$\mathbf{1}_{B^h} \to b \quad \text{weakly-* in } L^\infty(Q) \text{ as } h \to 0, \qquad 0 \le b(x) \le 1 \text{ for a.e. } x \in Q\,. \tag{3.14}$$

As a consequence, $\mathbf{1}_{A^h} \to (1-b)$ weakly-* in $L^\infty(Q)$.

Step 2: Localization. For any $\varphi \in L^2(Q;\mathbb{R}^n)$ we can calculate, using the weak convergence of g^h, the property $g^h(x) = 0$ for $x \in B^h$, then $Ag^h(x) = j^h(x)$ for $x \in A^h$, then $g^h(x) = e_1$ for $x \in B^h$, and finally the weak convergence of j^h:

$$\begin{aligned} \int_Q \varphi \cdot Ag \leftarrow \int_Q \varphi \cdot Ag^h = \int_{A^h} \varphi \cdot Ag^h = \int_{A^h} \varphi \cdot j^h \\ = \int_Q \varphi \cdot j^h - \int_{B^h} \varphi \cdot e_1 \to \int_Q \varphi \cdot (j - be_1)\,. \end{aligned} \tag{3.15}$$

This shows

$$Ag = j - be_1 \quad \text{ in } Q\,. \tag{3.16}$$

In particular, we find $(g,j)(x) \in \mathcal{D}^A_{\rm loc}$ for almost all $x \in \{x \in Q \,|\, b(x) = 0\}$.

Step 3: Div-curl lemma. The data convergence properties allow to calculate the distributional limit of the product $g^h \cdot j^h$. In the subsequent calculation, we use the standard div-curl lemma in $L^2(Q)$ for the product $G^h \cdot J^h$, and the strong convergence of differences in the other terms. In the limit $h \to 0$, we obtain for any $\varphi \in C^\infty_c(Q)$

$$\begin{aligned}\int_Q \varphi\, g^h \cdot j^h &= \int_Q \varphi\, (G^h + (g^h - G^h)) \cdot (J^h + (j^h - J^h))\\ &\to \int_Q \varphi\, G \cdot J + \int_Q \varphi\, G \cdot (j - J) + \int_Q \varphi\, (g - G) \cdot j = \int_Q \varphi\, g \cdot j\,.\end{aligned}$$

Step 4: The cone condition. We choose $\varepsilon > 0$ and set $\beta_\varepsilon := (1 - b + \varepsilon)^{-1}$. For arbitrary $\varphi \in C^\infty_c(Q)$ with $\varphi \geq 0$ we can calculate, exploiting the positivity and symmetry of A,

$$\begin{aligned}0 &\leq \int_{A^h} \varphi\, (g^h - \beta_\varepsilon g) \cdot A(g^h - \beta_\varepsilon g)\\ &= \int_{A^h} \varphi\, \left[g^h \cdot A g^h - 2\beta_\varepsilon g^h \cdot Ag + \beta_\varepsilon^2 g \cdot Ag\right]\\ &= \int_Q \varphi\, \left[g^h \cdot j^h - 2\beta_\varepsilon g^h \cdot Ag\right] + \int_Q \varphi\, \mathbf{1}_{A^h} \beta_\varepsilon^2 g \cdot Ag\,. \qquad (3.17)\end{aligned}$$

Using the div-curl lemma, the strong convergence of differences and the weak-* convergence of $\mathbf{1}_{A^h}$ we deduce

$$\begin{aligned}0 &\leq \int_Q \varphi\, [g \cdot j - 2\beta_\varepsilon g \cdot Ag] + \int_Q \varphi\, (1-b)\beta_\varepsilon^2 g \cdot Ag\\ &= \int_Q \varphi\, \left[b g_1 + g \cdot Ag \left(1 - 2\beta_\varepsilon + (1-b)\beta_\varepsilon^2\right)\right], \qquad (3.18)\end{aligned}$$

where we have used that $j = b e_1 + Ag$. Since φ was arbitrary, almost everywhere in Q holds

$$0 \leq b g_1 + g \cdot Ag \left(1 - 2\beta_\varepsilon + (1-b)\beta_\varepsilon^2\right). \qquad (3.19)$$

Evaluating this inequality in $\{b = 1\} = \{\beta_\varepsilon = \varepsilon^{-1}\}$, we find $g \cdot Ag \leq \frac{\varepsilon}{2-\varepsilon} g_1$ in this set. Since $\varepsilon > 0$ was arbitrary, we find $g = 0$ and $j = e_1$ almost everywhere in $\{b = 1\}$. In particular, $(g,j)(x) \in \mathcal{D}^B_{\rm loc} \subset C$ for almost all $x \in \{b = 1\}$.

We next consider the set $\{0 < b < 1\}$. In this set, for $\varepsilon \to 0$, there holds $\beta_\varepsilon \to \frac{1}{1-b}$. Relation (3.19) implies

$$0 \leq g_1 - \frac{1}{1-b} g \cdot Ag$$

almost everywhere in $\{0 < b < 1\}$. This is one of the defining relations of the cone C, compare (1.10). Combined with (3.16), we obtain that $(g,j) \in C$ almost everywhere in $\{0 < b < 1\}$.

Finally, in $\{b = 0\}$, relation (3.16) yields $(g,j) \in \mathcal{D}^A_{\rm loc}$. This provides (3.12) and concludes the proof of the proposition. □

A Appendix

We show here a property of the cone C of (1.10) and (1.11): Every inner point of the cone can be written as a convex combination of two points on the boundary; the results is nontrivial, since the additional requirement (2.11) has to be satisfied. The result was used in the construction of iterated laminates in Lemma 2.4.

Lemma A.1. *Let $p_C = (g_C, j_C) \in \mathring{C}$ be given. Then there exist two points $p_A \in \mathcal{D}^A_{\mathrm{loc}}$ and $p_L \in \partial_{\mathrm{lat}} C$ and a parameter $\lambda \in (0,1)$ such that* (2.10) *and* (2.11) *hold.*

Proof. Since $p_C \in \mathring{C}$ is an inner point of the cone, there exists $b \in (0,1)$ such that $j_C = be_1 + Ag_C$ and

$$g_C \cdot Ag_C < (1-b) g_C \cdot e_1 \,. \tag{A.1}$$

We set $\nu := \alpha g_C$ with $\alpha := (g_C \cdot Ag_C)^{-1/2}$. The choice of α implies $\nu \cdot A\nu = 1$. Given ν, we set

$$g_A := g_C + b\nu_1 \nu = \Big(\frac{1}{\alpha} + b\nu_1\Big)\nu \,, \qquad j_A := Ag_A \,.$$

This choice guarantees $(g_A, j_A) \in \mathcal{D}^A_{\mathrm{loc}}$. Next, for some $b_L \in [0,1]$ to be determined below, we define

$$g_L := \frac{b_L}{b}(g_C - g_A) + g_A = \Big((b - b_L)\nu_1 + \frac{1}{\alpha}\Big)\nu \,, \tag{A.2}$$

$$j_L := b_L e_1 + Ag_L \,. \tag{A.3}$$

The condition $(g_L, j_L) \in \partial_{\mathrm{lat}} C$ is equivalent to the condition

$$\begin{aligned}
0 &\overset{!}{=} g_L \cdot Ag_L - (1-b_L) g_L \cdot e_1 \\
&= (b-b_L)^2 \nu_1^2 + \frac{2}{\alpha}(b-b_L)\nu_1 + \frac{1}{\alpha^2} - (1-b_L)\Big((b-b_L)\nu_1 + \frac{1}{\alpha}\Big)\nu_1 \\
&= (b-b_L)\Big[(b-b_L)\nu_1^2 + \frac{2}{\alpha}\nu_1\Big] + \frac{1}{\alpha^2} - (b-b_L)\Big((b-b_L)\nu_1 + \frac{1}{\alpha}\Big)\nu_1 \\
&\qquad - (1-b)(b-b_L)\nu_1^2 - (1-b)\frac{1}{\alpha}\nu_1 \\
&= (b-b_L)\Big[-(1-b)\nu_1^2 + \frac{1}{\alpha}\nu_1\Big] + \frac{1}{\alpha^2} - (1-b)\frac{1}{\alpha}\nu_1 \\
&= (b-b_L)\Big[-(1-b)\alpha^2 (g_C \cdot e_1)^2 + (g_C \cdot e_1)\Big] + \frac{1}{\alpha^2} - (1-b) g_C \cdot e_1 \\
&= (b-b_L)\alpha^2 (g_C \cdot e_1)\Big[-(1-b) g_C \cdot e_1 + \frac{1}{\alpha^2}\Big] + \frac{1}{\alpha^2} - (1-b) g_C \cdot e_1 \\
&= \Big((b-b_L)\alpha^2 (g_C \cdot e_1) + 1\Big)\Big[\frac{1}{\alpha^2} - (1-b) g_C \cdot e_1\Big] \,.
\end{aligned}$$

We note that the expression on the right hand side is negative for $b_L = b$ by (A.1). On the other hand, for $b_L = 1$, the expression on the right hand side is a product of two identical terms and hence nonnegative. This implies that there exists a value $b_L \in (b, 1]$ such that the expression vanishes. For this parameter b_L, the above condition is satisfied and hence $(g_L, j_L) \in \partial_{\mathrm{lat}} C$.

We set $\lambda := \frac{b}{b_L} \in (0, 1)$. With this choice, by definition of g_L in (A.2), we obtain $g_L = \frac{1}{\lambda}(g_C - g_A) + g_A$ and therefore

$$g_C = \lambda g_L + (1 - \lambda) g_A \,. \tag{A.4}$$

Regarding the component j, we find

$$\begin{aligned} \lambda j_L + (1-\lambda) j_A &= \lambda (b_L e_1 + A g_L) + (1-\lambda) A g_A \\ &= \lambda b_L e_1 + A(\lambda g_L + (1-\lambda) g_A) = b e_1 + A g_C = j_C \,. \end{aligned}$$

Together with (A.4), this shows (2.10).

Finally, the definitions of g_A, g_L, j_A, and j_L imply $g_A - g_L = \frac{b_L}{b}(g_A - g_C) = b_L \nu_1 \nu$ and hence

$$\begin{aligned} (g_A - g_L) \cdot (j_A - j_L) &= (g_A - g_L) \cdot (A g_A - b_L e_1 - A g_L) \\ &= (g_A - g_L) \cdot A(g_A - g_L) - b_L e_1 \cdot (g_A - g_L) \\ &= b_L^2 \nu_1^2 \nu \cdot A \nu - b_L e_1 \cdot (b_L \nu_1 \nu) \; = \; 0 \,. \end{aligned}$$

This shows (2.11) and completes the proof of the lemma. □

References

[1] G. Allaire. *Shape optimization by the homogenization method*, volume 146 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2002.

[2] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63(4):337–403, 1976/77.

[3] J. M. Ball and R. D. James. Fine phase mixtures as minimizers of energy. *Arch. Rational Mech. Anal.*, 100(1):13–52, 1987.

[4] A. Cherkaev. *Variational methods for structural optimization*, volume 140 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2000.

[5] S. Conti and G. Dolzmann. On the theory of relaxation in nonlinear elasticity with constraints on the determinant. *Arch. Ration. Mech. Anal.*, 217(2):413–437, 2015.

[6] S. Conti, S. Müller, and M. Ortiz. Data-driven problems in elasticity. *Arch. Ration. Mech. Anal.*, 229(1):79–123, 2018.

[7] B. Dacorogna. *Weak continuity and weak lower semicontinuity of nonlinear functionals*, volume 922 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1982.

[8] G. Dal Maso. *An introduction to Γ-convergence*, volume 8 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1993.

[9] R. Eggersmann, T. Kirchdoerfer, S. Reese, L. Stainier, and M. Ortiz. Model-free data-driven inelasticity. *Comput. Methods Appl. Mech. Engrg.*, 350:81–99, 2019.

[10] I. Fonseca and S. Müller. $\mathcal{A}$-quasiconvexity, lower semicontinuity, and Young measures. *SIAM J. Math. Anal.*, 30(6):1355–1390, 1999.

[11] G. A. Francfort and G. W. Milton. Sets of conductivity and elasticity tensors stable under lamination. *Comm. Pure Appl. Math.*, 47(3):257–279, 1994.

[12] G. A. Francfort and F. Murat. Homogenization and optimal bounds in linear elasticity. *Arch. Rational Mech. Anal.*, 94(4):307–334, 1986.

[13] Z. Hashin and S. Shtrikman. A variational approach to the theory of the elastic behaviour of multiphase materials. *J. Mech. Phys. Solids*, 11:127–140, 1963.

[14] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian.

[15] Y. Kanno. Simple heuristic for data-driven computational elasticity with material data involving noise and outliers: a local robust regression approach. *Jpn. J. Ind. Appl. Math.*, 35(3):1085–1101, 2018.

[16] T. Kirchdoerfer and M. Ortiz. Data-driven computational mechanics. *Comput. Methods Appl. Mech. Engrg.*, 304:81–101, 2016.

[17] R. V. Kohn and G. W. Milton. On bounding the effective conductivity of anisotropic composites. In *Homogenization and effective moduli of materials and media (Minneapolis, Minn., 1984/1985)*, volume 1 of *IMA Vol. Math. Appl.*, pages 97–125. Springer, New York, 1986.

[18] R. V. Kohn and G. Strang. Optimal design and relaxation of variational problems. I. *Comm. Pure Appl. Math.*, 39(1):113–137, 1986.

[19] R. V. Kohn and G. Strang. Optimal design and relaxation of variational problems. II. *Comm. Pure Appl. Math.*, 39(2):139–182, 1986.

[20] S. Müller. Variational models for microstructure and phase transitions. In *Calculus of variations and geometric evolution problems (Cetraro, 1996)*, volume 1713 of *Lecture Notes in Math.*, pages 85–210. Springer, Berlin, 1999.

[21] F. Murat. Compacité par compensation. II. In *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978)*, pages 245–256. Pitagora, Bologna, 1979.