Existence result for Maxwell's equations in half-waveguides

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February 19, 2025

Abstract

Maxwell's equations are considered in a half-waveguide $\Omega_+ := \mathbb{R}_+ \times S \subset \mathbb{R}^3$ where $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain in $\mathbb{R}^2$. The electric permittivity ε and the magnetic permeability μ are assumed to be strictly positive and periodic outside a compact set. A standard radiation condition accompanies the equations. We give a result on existence and uniqueness in the form of a Fredholm alternative: When there is no bound state, i.e., no non-trivial solution of the homogeneous problem, then there is a unique solution for every right-hand side.

1 Introduction

We consider the time-harmonic Maxwell system in the following form: With a prescribed frequency $\omega > 0$ and the two coefficients $\mu = \mu(x)$ (permeability) and $\varepsilon = \varepsilon(x)$ (permittivity) we investigate

$$\operatorname{curl} E = \mathrm{i}\omega\mu\, H + f_h \tag{1.1}$$

$$\operatorname{curl} H = -\mathrm{i}\omega\varepsilon\, E + f_e \tag{1.2}$$

in a waveguide $\Omega_+ := \mathbb{R}_+ \times S \subset \mathbb{R}^3$ with perfectly conducting boundary, $S \subset \mathbb{R}^2$ a bounded Lipschitz domain. We use right-hand sides $f_e, f_h \in L^2(\Omega_+)$ and write $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ for the independent variable, compare Figure 1 for a two-dimensional sketch.

We show an existence and uniqueness result for this system when ε and μ are x_1-periodic outside a compact subset of Ω_+. When the homogeneous system has no non-trivial solution, then, for every right-hand side (f_e, f_h), the system (1.1)–(1.2) has a solution in Ω_+. The result is formulated in Theorem 1.2 below. The more technical assumptions in this result are: (i) Decay of the right-hand side and a condition on the divergence of f_e. (ii) Non-degeneracy of the frequency ω. (iii) Non-existence of edge-resonances, i.e., non-trivial solutions of the corresponding system in Ω_-. We discuss the conditions (i)–(iii) within the text.

An overview over the literature on the subject is provided below. Here, we want to describe our results briefly and in a broad perspective. The existence problem for periodic waveguides, for the Helmholtz equation and for the Maxwell system, was studied in many contributions, some corner-stones are: Helmholtz in full waveguide in [12] and later in

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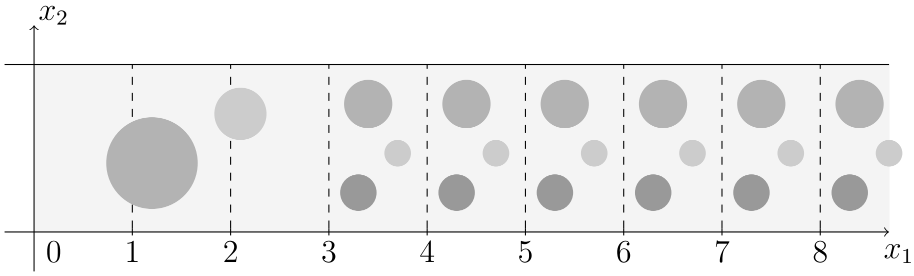

Figure 1: A symbolic sketch of the waveguide geometry, we recall that the domain Ω_+ has three dimensions. The coefficient ε is indicated by different levels of gray. To the right of some value R_0 (we have chosen $R_0 = 3$ in the image), the coefficient is 1-periodic in x_1-direction.

[19], Helmholtz in semi-infinite waveguide in [16], Maxwell in full waveguide in [22]. This text yields the existence for Maxwell in a semi-infinite waveguide.

We use the methods that were introduced in [30] for the Helmholtz equation. They consist in the following steps: (1) A truncated problem is formulated. This problem contains the condition that, in a "radiation box" $W_R = (R, R+1) \times S$, the solution is a linear combination of outgoing waves. (2) One shows that the truncated problem has a unique solution, see Proposition 3.9 for the conditional existence and Lemma 4.6 for the uniqueness. (3) Limits of the truncated solutions are studied. It is shown that, when the solution sequence (for $R \to \infty$) is bounded in an appropriate norm, the local limit of the sequence is the desired solution of the problem in the unbounded domain, including the radiation condition, see Proposition 4.4. (4) A contradiction argument yields that the solution sequence is indeed bounded in the norm of interest, see Section 4.3.

We see two advantages of the above-described method. The first regards the needed analytical tools: We do neither need Floquet-Bloch transformations nor any representations of operators with line integrals in the complex plane. The second advantage regards the applicability as a numerical method: The truncated problem is represented with a sesquilinear form on a Hilbert space of functions, the functions are defined on a bounded domain. This problem can easily be discretized and solved numerically. We recall that existence and uniqueness of this problem is provided in this work, see (2) above.

For a complete description of our method, we should mention that we do use the fact $Y = B$, that all bounded solutions are quasiperiodic solutions, see Theorem 2.13. This fact was derived in [22] with the Floquet-Bloch transform, it uses only an analysis of periodic coefficients in all of Ω. Still – since the derivation of $Y = B$ is based on the Floquet-Bloch transform, we cannot say that our method is entirely independent of this transformation. Nevertheless, in the work at hand, dealing with coefficients that are periodic outside a compact set, we do not need any transformation or representation tools.

The method of choice is an energy-based method, it relies on Hilbert space structures and testing procedures. It is based on observations regarding energy flux in Maxwell's system and decompositions into left-going and right-going modes. The method is very flexible. Indeed, with only notational changes, the method allows to treat also a full waveguide $\Omega = \mathbb{R} \times S$ with two different media at the left and at the right (as in [30]). In the same way, one can also treat a domain with finitely many unbounded components when each of the unbounded components is described by a periodic medium, compare Figure 2.

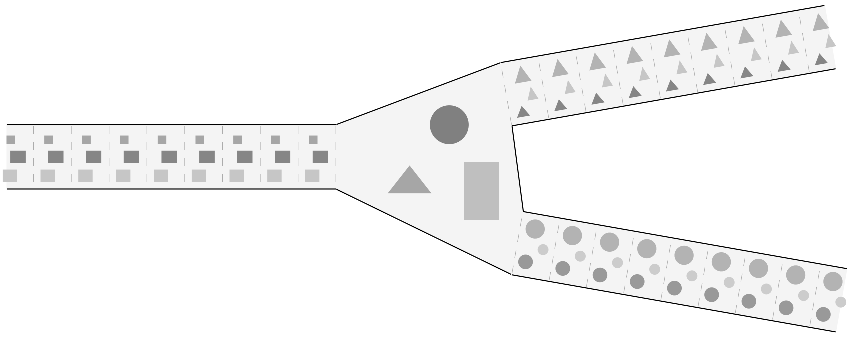

Figure 2: Sketch of a possible waveguide geometry with three unbounded half-waveguide components. In each of the unbounded components, outside a compact set, the coefficients are periodic in the unbounded direction.

Setting, solution concept and main result

Our aim is to give an existence result for system (1.1)–(1.2). Let us specify the geometry, the coefficients and the weak solution concept, and present the main existence and uniqueness result.

Geometry and coefficients. We recall that the half-waveguide is $\Omega_+ = \mathbb{R}_+ \times S \subset \mathbb{R}^3$ with $S \subset \mathbb{R}^2$ a bounded Lipschitz domain, and that the independent variable is $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We write e_1, e_2, e_3 for the unit vectors in $\mathbb{R}^3$ such that $x = \sum_{j=1}^3 x_j e_j$. For $z \in \mathbb{C}$, the complex conjugate is denoted by $\bar{z}$, this operation can also be applied to a $\mathbb{C}$-valued or a $\mathbb{C}^3$-valued function. The right-hand side of the system is given by functions $f_e, f_h : \Omega_+ \to \mathbb{C}^3$, for notational convenience we assume that they have bounded support in Ω_+ (or, equivalently: for some $M > 0$, there holds $f_e(x) = f_h(x) = 0$ for every x with $|x| > M$). We conjecture that the assumption $x \mapsto (1 + |x_1|^2) f_{e,h}(x) \in L^2(\Omega_+)$ would be sufficient; this assumption is sufficient in the Helmholtz setting, see [21]. Here, we additionally demand $\operatorname{div} f_e \in L^2(\Omega_+)$.

The exterior normal vector to Ω_+ is denoted by $\nu : \partial\Omega_+ \to \mathbb{R}^3$, $\nu = \nu(x)$. We complete system (1.1)–(1.2) with the boundary conditions that model a perfectly conducting boundary, $E \times \nu = 0$ on $\partial\Omega_+$. Occasionally, we will also use the half-waveguide to the left, $\Omega_- := \mathbb{R}_- \times S \subset \mathbb{R}^3$, and the doubly unbounded waveguide $\Omega := \mathbb{R} \times S$.

We consider coefficients $\varepsilon, \mu \in L^\infty(\Omega_+, \mathbb{R})$ with a positive lower bound: There exist $0 < \lambda < \Lambda < \infty$ with $\varepsilon(x), \mu(x) \in [\lambda, \Lambda]$ for almost every $x \in \Omega_+$. We assume that the coefficients are periodic in x_1-direction at the far right: For two functions $\varepsilon_{\mathrm{per}}, \mu_{\mathrm{per}} \in L^\infty(\Omega, \mathbb{R})$ with the periodicity $\varepsilon_{\mathrm{per}}(x + e_1) = \varepsilon_{\mathrm{per}}(x)$ and $\mu_{\mathrm{per}}(x + e_1) = \mu_{\mathrm{per}}(x)$ for almost every $x \in \Omega$, and for some number $R_0 > 0$ holds $\varepsilon(x) = \varepsilon_{\mathrm{per}}(x)$ and $\mu(x) = \mu_{\mathrm{per}}(x)$ for every $x \in \Omega_+$ with $x_1 > R_0$.

Weak form of the equations. With appropriate function spaces, we can introduce a weak formulation of the equations and present the main result of this work.

Function spaces. For an open set $\Sigma \subset \mathbb{R}^3$, we use spaces of functions with bounded distributional curl as follows:

$$H(\operatorname{curl}, \Sigma) := \left\{ u \in L^2(\Sigma, \mathbb{C}^3) \;\middle|\; \operatorname{curl} u \in L^2(\Sigma, \mathbb{C}^3) \right\}, \tag{1.3}$$

$$H_0(\operatorname{curl},\Sigma) := \left\{ u \in H(\operatorname{curl},\Sigma) \,\middle|\, \int_\Sigma u \cdot \operatorname{curl}\varphi = \int_\Sigma \operatorname{curl} u \cdot \varphi \quad \forall\, \varphi \in H(\operatorname{curl},\Sigma) \right\}. \tag{1.4}$$

On both spaces we use the norm that is defined by

$$\|u\|^2_{H(\operatorname{curl},\Sigma)} := \|u\|^2_{L^2(\Sigma)} + \|\operatorname{curl} u\|^2_{L^2(\Sigma)}. \tag{1.5}$$

Additionally, we need spaces of locally integrable functions. Deviating slightly from standard notations, we define $H_{\mathrm{loc}}(\operatorname{curl},\Omega_+)$ as the space of functions u for which u and $\operatorname{curl} u$ are square integrable over *bounded* subsets: With $\Omega_R := (0,R)\times S$, we set

$$H_{\mathrm{loc}}(\operatorname{curl},\Omega_+) := \left\{ u\colon \Omega_+ \to \mathbb{C}^3 \,\middle|\, \forall\, R>0 : u|_{\Omega_R} \in L^2(\Omega_R,\mathbb{C}^3)\,,\ (\operatorname{curl} u)|_{\Omega_R} \in L^2(\Omega_R,\mathbb{C}^3) \right\}.$$

Finally, we define $H_{\mathrm{loc},0}(\operatorname{curl},\Omega_+)$ as the space of functions $u \in H_{\mathrm{loc}}(\operatorname{curl},\Omega_+)$ with vanishing tangential boundary values,

$$H_{\mathrm{loc},0}(\operatorname{curl},\Omega_+) := \left\{ u \in H_{\mathrm{loc}}(\operatorname{curl},\Omega_+) \,\middle|\, \int_{\Omega_+} u \cdot \operatorname{curl}\varphi = \int_{\Omega_+} \operatorname{curl} u \cdot \varphi \ \forall\, \varphi \in C^1_c(\overline{\Omega}_+,\mathbb{C}^3) \right\}.$$

Spaces such as $H_{\mathrm{loc}}(\operatorname{curl},\Omega_-)$, $H_{\mathrm{loc}}(\operatorname{curl},\Omega)$, $H_{\mathrm{loc},0}(\operatorname{curl},\Omega_-)$ and $H_{\mathrm{loc},0}(\operatorname{curl},\Omega)$ are defined analogously. The strong formulation of Maxwell's equations is: Find $E \in H_{\mathrm{loc},0}(\operatorname{curl},\Omega_+)$ and $H \in H_{\mathrm{loc}}(\operatorname{curl},\Omega_+)$ such that (1.1)–(1.2) are satisfied in $L^2(\Omega_+)$.

The equations can now be formulated in a variational form with $u := E$ as the only unknown.

Definition 1.1 (Weak formulation of the Maxwell system)**.** A function $u \in H_{\mathrm{loc},0}(\operatorname{curl},\Omega_+)$ is called a *weak solution to the Maxwell system* (1.1)–(1.2) iff u satisfies, for every $\Phi \in H_0(\operatorname{curl},\Omega_+)$ with bounded support,

$$\int_{\Omega_+} \left(\frac{1}{\mu} \operatorname{curl} u \cdot \operatorname{curl}\bar{\Phi} - \omega^2 \varepsilon\, u \cdot \bar{\Phi} \right) = \int_{\Omega_+} \left(\frac{1}{\mu} f_h \cdot \operatorname{curl}\bar{\Phi} + \mathrm{i}\omega f_e \cdot \bar{\Phi} \right). \tag{1.6}$$

Regarding the equivalence of the solution concepts: Let $u \in H_{\mathrm{loc},0}(\operatorname{curl},\Omega_+)$ be a solution of (1.6). Setting

$$E := u \qquad \text{and} \qquad H := \frac{(\operatorname{curl} E - f_h)}{\mathrm{i}\omega\mu}\,,$$

the function E lies in the desired function space and equation (1.1) is satisfied by definition of H. Replacing in (1.6) the term $\frac{1}{\mu}(\operatorname{curl} u - f_h)$ by $\mathrm{i}\omega H$, we find

$$\int_{\Omega_+} \left(i\omega H \cdot \operatorname{curl}\bar{\Phi} - \omega^2 \varepsilon\, u \cdot \bar{\Phi} \right) = \int_{\Omega_+} \left(\mathrm{i}\omega f_e \cdot \bar{\Phi} \right).$$

The property $\Phi \in H_0(\operatorname{curl},\Omega_+)$ allows to integrate by parts and provides (1.2). This, in turn, yields also the property $H \in H_{\mathrm{loc}}(\operatorname{curl},\Omega_+)$. We have found the desired solution (E,H).

Radiation condition. The Maxwell equations in a waveguide must be accompanied by a radiation condition. Essentially, one demands that solutions "look like outgoing waves at the far right". Appropriate radiation conditions are standard, they are used, e.g., in [22] or [30]. We introduce a convenient form of the radiation conditions in Definition 2.14.

Let us present here a loose description of the radiation condition. When we consider the homogeneous Maxwell system in the two-sided waveguide $\Omega := \mathbb{R} \times S$, we seek for $u \in H_{\mathrm{loc},0}(\mathrm{curl}, \Omega)$ such that, for every $\Phi \in H_0(\mathrm{curl}, \Omega)$ with bounded support,

$$\int_\Omega \left(\frac{1}{\mu} \operatorname{curl} u \cdot \operatorname{curl} \bar{\Phi} - \omega^2 \varepsilon\, u \cdot \bar{\Phi} \right) = 0\,. \tag{1.7}$$

When the standard non-degeneracy Assumption 2.9 is satisfied, the space of bounded solutions u to problem (1.7) is finite dimensional and spanned by quasiperiodic functions $(\phi_k)_{k \le K}$. Half of these basis functions are transporting energy to the left (we write "left-going" for short), the other half is transporting energy to the right ("right-going"). The radiation condition of Definition 2.14 demands that, in cylindrical sets of the form $W_R = (R, R+1) \times S$ with large $R > 0$, the projection of u to left-going waves is small.

Main result. Our main result is the following statement on existence and uniqueness of solutions to the radiation problem. Since we cannot exclude the existence of bound states in a half-waveguide, our result has the character of a Fredholm alternative.

Theorem 1.2 (Existence and uniqueness result). *Let the geometry with $\Omega_+ = \mathbb{R}_+ \times S$ be as above, let the data ε, μ, f_e and f_h be as above and let Assumption 2.9 on (1.7) be satisfied. We furthermore assume that the radiation problems on the domains Ω_+ and Ω_- with $f_e = 0 = f_h$ possess only the trivial solution. Then the radiation problem to (1.6) has a unique weak solution u for every right hand side $f_e, f_h \in L^2(\Omega_+)$ with bounded support and with $\operatorname{div} f_e \in L^2(\Omega_+)$.*

The theorem is shown in Section 4. Let us comment the assumptions in the theorem. Assumption 2.9 is a fundamental ingredient in the analysis of waveguides, it essentially demands that every quasiperiodic homogeneous solution transports energy either left or right. Simple scalar examples show that such an assumption is necessary in order to obtain existence of solutions.

The assumption that the homogeneous radiation problem on the domain Ω_+ has only the trivial solution: This is part of the formulation as a Fredholm alternative. Finally: We must also assume that the homogeneous radiation problem on the domain Ω_- has only the trivial solution. The opposite case is that there are non-trivial edge-resonance solutions. We give a precise definition of edge-resonances in appendix B and clearify the assumption of the theorem that "the radiation problem on the domain Ω_- with $f_e = 0 = f_h$ possesses only the trivial solution". We include comments on how edge resonances might be treated. Our belief is that the existence proof should also work in the case when edge-resonances exist, but it seems that this case requires major modifications of the proof.

Literature

Maxwell's equations are originally formulated in a time-dependent setting; when one is interested in a fixed frequency $\omega > 0$ and in solutions of the form $E(x)e^{-i\omega t}$ and $H(x)e^{-i\omega t}$, the equations reduce to system (1.1)–(1.2). Some more background and an overview over mathematical methods for this system can be found, e.g., in [18]. In the case of constant coefficients ε and μ, when the domain is of the form $\mathbb{R}^3 \setminus U$ for some bounded closed set U, the radiation condition for the system is the Silver-Müller radiation condition. Much of the classical works are related to radiation conditions in different domains, for constant coefficients.

When the problem involves periodic coefficients, a classical tool for the analysis is the Floquet-Bloch transform. Regarding this method we refer to [23, 26] and the references therein. The technique is also fully described in [21].

Let us now describe the *Helmholtz problem* (with periodic coefficients and adequate radiation conditions). This problem is very similar to the Maxwell system; indeed, in special geometries and for certain fields, the Maxwell system simplifies to the Helmholtz equation. A fundamental contribution is [12]. In that work, the existence and uniqueness for the periodic waveguide was derived. A more functional analytic approach for the same problem can be found in [19, 20], simplified proofs and strengthened results in [21].

The Helmholtz equation in a semi-infinite waveguide is examined in [16]. We already mentioned that a new method was introduced in [30], it allows the study of the Helmholtz equation with energy methods and without a Floquet-Bloch transform. It was shown in [10] that the corresponding truncated domain scheme can be used numerically. A radiation condition in terms of Bloch waves is covered in [8, 25]. Another approach to truncated domain problems is to use clever Dirichlet-to-Neumann boundary operators. For such techniques we refer to [11, 13]. We want to mention that, apart from the fact that a numerical solution is always an approximation, the Dirichlet-to-Neumann operators allow to formulate problems on bounded domains that are equivalent to the problems on the entire domain. This is not the case in our truncation scheme.

As indicated, *Maxwell's equations* require similar analytical and numerical tools. In a waveguide of the form $\Omega = \mathbb{R} \times S$ with x_1-periodic coefficients, the problem is solved in [22]. This work contains also an analysis in the case that the coefficients are perturbed in a compact subdomain. The method of choice in [22] is to apply a Floquet-Bloch transform and to use a functional analytic approach.

Leaving the setting of closed waveguides, a classical contribution is [7], where the problem is reduced to a two-dimensional problem with a periodic boundary condition. We note that the open waveguide imposes additional problems and refer to [17] for a treatment in the case of the Helmholtz equation. The interaction of waves with a locally perturbed biperiodic structure is considered in [3].

A basis for analytical results are regularity results, we refer to [1, 9]. Related are compactness results, which are of particular importance, see [4, 28, 33]. We mention also the interesting approach to scattering problems in [2]. We cannot cover here the broad field of numerical methods and mention only [14, 15, 31] for different approaches. We believe that Maxwell's equations in periodic media gained in attention because of the astonishing limit equations that can be obtained by homogenization, in particular, negative index materials. For the analysis of such meta-materials, we refer to [5, 6, 24, 27].

2 Preparations and tools

In this section, we introduce essential objects for the analysis of the homogeneous Maxwell system (1.7). We characterize quasiperiodic and bounded solutions to the homogeneous

Figure 3: Cut-off function $\vartheta_{\rho,r}$ for $\rho, r \in \mathbb{Z}$.

Maxwell system, following [22]. The statements of [22] are formulated for $u = H$, but they remain valid for the unknown $u = E$, since E can always be obtained from H, and vice versa.

We introduce a hermitian form Q to measure the energy flux of solutions u. With Q, we find a finite orthogonal basis of the space of quasiperiodic solutions to (1.7). The basis functions allow to formulate the radiation condition.

2.1 Energy flux and the sesquilinear form Q

For an arbitrary value of $r \in \mathbb{R}$, we use the cylindrical set $W_r := (r, r+1) \times S$. For numbers $\rho, r \in \mathbb{Z}$ with $\rho < r$, we also introduce the cylinders $\Omega_{\rho,r} := (\rho, r) \times S$, they can be used as truncated waveguides. A piecewise affine cut-off function $\vartheta_{\rho,r} \colon \mathbb{R} \to \mathbb{R}$ is defined as follows, compare Figure 3: $\vartheta_{\rho,r}(x_1) = 0$ for $x_1 < \rho$ and $x_1 > r+1$ and $\vartheta_{\rho,r}(x_1) = 1$ for $\rho + 1 \le x_1 \le r$. In the two remaining intervals we choose $\vartheta_{\rho,r}$ affine and continuous: $\vartheta_{\rho,r}(x_1) = x_1 - \rho$ for $\rho \le x_1 < \rho + 1$ and $\vartheta_{\rho,r}(x_1) = -x_1 + (r+1)$ for $r < x_1 \le r+1$. We idenitfy $\vartheta_{\rho,r}$ with a cut-off function on Ω by setting $\vartheta_{\rho,r}(x) := \vartheta_{\rho,r}(x_1)$ for $x \in \Omega$.

To introduce the energy-related forms, we use the special cylindrical domain $W_0 = (0,1) \times S$. We consider the sesquilinear form $Q^\circ \colon H(\mathrm{curl}, W_0) \times H(\mathrm{curl}, W_0) \to \mathbb{C}$ and the associated quadratic form $\mathcal{Q} \colon H(\mathrm{curl}, W_0) \to \mathbb{C}$:

$$Q^\circ(u, \Psi) := \int_{W_0} \left(\frac{1}{\mu} \bar{\Psi} \times \mathrm{curl}\, u \right) \cdot e_1 \qquad \text{and} \qquad \mathcal{Q}(u) := Q^\circ(u, u)\,. \tag{2.1}$$

We also introduce the following hermitian variant of Q°:

$$Q(u, \Psi) := \frac{\mathrm{i}}{2} \left(\overline{Q^\circ(\Psi, u)} - Q^\circ(u, \Psi) \right) . \tag{2.2}$$

The definition assures $Q(u, \Psi) = \overline{Q(\Psi, u)}$ for all $u, \Psi \in H(\mathrm{curl}, W_0)$, hence Q is indeed hermitian. We note that the sesquilinear forms Q and Q° can be reconstructed from the quadratic form $\mathcal{Q}$ with the polarization identity, a fact that we will not exploit here. The relation between Q and $\mathcal{Q}$ is given by $Q(u, u) = \mathrm{Im}(\mathcal{Q}(u))$ for all $u \in H(\mathrm{curl}, W_0)$.

We will evaluate energy fluxes also in positions $r \neq 0$. To make precise statements, we use the following notations.

Remark 2.1 (Notation regarding evaluations in different positions)**.** *When the arguments of Q° are functions $u, \Psi \in H(\mathrm{curl}, W_r)$ for $r \in \mathbb{Z}$, we use*

$$Q^\circ(u, \Psi) := \int_{W_r} \left(\frac{1}{\mu} \bar{\Psi} \times \mathrm{curl}\, u \right) \cdot e_1 \,.$$

When the arguments of Q° *are functions on the entire domain, e.g.,* $u, \Psi \in H_{\mathrm{loc}}(\mathrm{curl}, \Omega)$*, we evaluate the integral for* $r = 0$*, i.e., we use* (2.1)*.*

Another tool to study a function at an arbitrary position $r \in \mathbb{Z}$ *is the shift operator, defined as follows:*

$$S_r \colon L^2(W_r) \to L^2(W_0),\ \ u(\cdot) \mapsto u(\cdot + r e_1)\,. \tag{2.3}$$

The shift S_r *is an isometry between* $L^2(W_r)$ *and* $L^2(W_0)$*. It is also an isometry between* $H(\mathrm{curl}, W_r)$ *with* $H(\mathrm{curl}, W_0)$*. We identify such spaces with the help of the shift operator.*

Lemma 2.2 (Flux equality)**.** *Let* $u, v \in H_{\mathrm{loc},0}(\mathrm{curl}, \Omega)$ *be two solutions to* (1.7)*. Then, for* $\rho, r \in \mathbb{Z}$ *with* $\rho + 1 \le r$*, the following equality holds:*

$$\mathrm{Im}\left(\mathcal{Q}(u|_{W_\rho})\right) = \mathrm{Im}\left(\mathcal{Q}(u|_{W_r})\right)\,. \tag{2.4}$$

Furthermore, the hermitian form Q *satisfies*

$$Q(u|_{W_\rho}, v|_{W_\rho}) = Q(u|_{W_r}, v|_{W_r})\,. \tag{2.5}$$

Proof. Equality (2.5) implies equality (2.4) because of $Q(u,u) = \mathrm{Im}\left(\mathcal{Q}(u)\right)$. We therefore show (2.5). Let $\rho, r \in \mathbb{Z}$ with $\rho + 1 \le r$ be fixed and $\vartheta_{\rho,r}$ be the corresponding cut-off function. Using $\Phi = v\vartheta_{\rho,r}$ as a test-function in (1.7) for u allows to calculate

$$\begin{aligned}
0 &= \int\limits_{\Omega_{\rho,r+1}} \left(\frac{1}{\mu}\,\mathrm{curl}\, u \cdot \mathrm{curl}(\bar{v}\vartheta_{\rho,r})\right) - \int\limits_{\Omega_{\rho,r+1}} \omega^2 \varepsilon\, u \cdot \bar{v}\vartheta_{\rho,r} \\
&= \int\limits_{\Omega_{\rho,r+1}} \left(\frac{1}{\mu}\,\mathrm{curl}\, u \cdot \left(\nabla\vartheta_{\rho,r} \times \bar{v} + (\mathrm{curl}\,\bar{v})\,\vartheta_{\rho,r}\right)\right) - \int\limits_{\Omega_{\rho,r+1}} \omega^2 \varepsilon\, u \cdot \bar{v}\vartheta_{\rho,r} \\
&= \int\limits_{\Omega_{\rho,r+1}} \left(\frac{1}{\mu}\,\mathrm{curl}\, u \cdot \mathrm{curl}\,\bar{v}\right)\vartheta_{\rho,r} - \int\limits_{\Omega_{\rho,r+1}} \omega^2 \varepsilon\, u \cdot \bar{v}\vartheta_{\rho,r} \\
&\qquad - \int\limits_{W_r} \frac{1}{\mu}\,\mathrm{curl}\, u \cdot (e_1 \times \bar{v}) + \int\limits_{W_\rho} \frac{1}{\mu}\,\mathrm{curl}\, u \cdot (e_1 \times \bar{v}) \\
&= \int\limits_{\Omega_{\rho,r+1}} \left(\frac{1}{\mu}\,\mathrm{curl}\, u \cdot \mathrm{curl}\,\bar{v}\right)\vartheta_{\rho,r} - \int\limits_{\Omega_{\rho,r+1}} \left(\omega^2 \varepsilon\, u \cdot \bar{v}\vartheta_{\rho,r}\right) \\
&\qquad - Q^\circ(u|_{W_r}, v|_{W_r}) + Q^\circ(u|_{W_\rho}, v|_{W_\rho})\,.
\end{aligned}$$

Using $u\vartheta_{\rho,r}$ as a test-function in (1.7) for v and performing a complex conjugation yields

$$0 = \int\limits_{\Omega_{\rho,r+1}} \left(\frac{1}{\mu}\,\mathrm{curl}\, u \cdot \mathrm{curl}\,\bar{v}\right)\vartheta_{\rho,r} - \int\limits_{\Omega_{\rho,r+1}} \left(\omega^2 \varepsilon\, u \cdot \bar{v}\vartheta_{\rho,r}\right) - \overline{Q^\circ(v|_{W_r}, u|_{W_r})} + \overline{Q^\circ(v|_{W_\rho}, u|_{W_\rho})}\,.$$

Subtracting the two results, the first two integrals cancel and we obtain

$$Q^\circ(u|_{W_r}, v|_{W_r}) - \overline{Q^\circ(v|_{W_r}, u|_{W_r})} = Q^\circ(u|_{W_\rho}, v|_{W_\rho}) - \overline{Q^\circ(v|_{W_\rho}, u|_{W_\rho})}\,,$$

which gives (2.5). □

Remark 2.3 (Energy flux through boundaries)**.** *Let $u \in H_{\mathrm{loc},0}(\mathrm{curl},\Omega)$ be a solution to* (1.7)*. For $\rho, r \in \mathbb{Z}$ with $\rho+1 \le r$ we set $\Sigma := \Omega_{\rho,r}$ and formally use $u\mathbb{1}_\Sigma$ as a test-function in* (1.7)*. With the boundaries $\Gamma_\rho = \{\rho\} \times S$ and $\Gamma_r = \{r\} \times S$, an integration by parts leads to*

$$\mathrm{Im} \int_{\Gamma_\rho} \frac{1}{\mu}(\bar{u} \times \mathrm{curl}\, u) \cdot e_1 = \mathrm{Im} \int_{\Gamma_r} \frac{1}{\mu}(\bar{u} \times \mathrm{curl}\, u) \cdot e_1 \tag{2.6}$$

in the sense of traces. This is the classical relation expressing that the energy flux is independent of the position. It also implies that the volume integral $\mathrm{Im}\,(\mathcal{Q}(u|_{W_r}))$ of (2.4) *coincides with the expression of* (2.6)*.*

2.2 Quasiperiodicity and propagating modes

Definition 2.4 (Quasiperiodicity)**.** A function $u\colon \Omega \to \mathbb{C}^3$ is called *quasiperiodic* iff there exists a real number $\alpha \in (-\pi, \pi]$ such that $u(x + e_1) = \exp(\mathrm{i}\alpha)u(x)$ holds for all $x \in \Omega$. The number α is called the quasimomentum or the *quasimoment.*

Remark 2.5 (Periodic and α-quasiperiodic functions)**.** *Let $U\colon \Omega \to \mathbb{C}^3$ be a function. Then $[x \mapsto U(x)]$ is α-quasiperiodic in x_1, iff $[x \mapsto U(x)\exp(-\mathrm{i}\alpha x_1)]$ is 1-periodic in x_1.*

The following remark includes a warning regarding the shifts. The shifted version of a quasiperiodic function coincides with the original function only up to a complex pre-factor.

Remark 2.6 (Shift of quasiperiodic function)**.** *Let $\alpha \in (-\pi, \pi]$ be a quasimoment, u an α-quasiperiodic function, $R \in \mathbb{Z}$ a position and $S_R\colon L^2(W_R) \to L^2(W_0)$ the shift operator of* (2.3)*. Then*

$$S_R(u|_{W_R}) = \exp(\mathrm{i}R\alpha)\, u|_{W_0}\,.$$

Definition 2.7 (The spaces of periodic and α-quasiperiodic functions; [22] Chapter 2.1)**.** For a position $R \in \mathbb{Z}$ and a quasimoment $\alpha \in (-\pi, \pi]$ we define the spaces

$$\begin{aligned}
H_{\mathrm{per,loc}}(\mathrm{curl},\Omega) &:= \{u \in H_{\mathrm{loc}}(\mathrm{curl},\Omega) \mid u \text{ is 1-periodic in } x_1\}\,,\\
H_{\mathrm{per}}(\mathrm{curl},W_R) &:= \{u|_{W_R} \mid u \in H_{\mathrm{per,loc}}(\mathrm{curl},\Omega)\}\,,\\
H_{\mathrm{per},0}(\mathrm{curl},W_R) &:= \{u|_{W_R} \mid u \in H_{\mathrm{per,loc}}(\mathrm{curl},\Omega) \cap H_{\mathrm{loc},0}(\mathrm{curl},\Omega)\}\,,\\
H_{0,\alpha}(\mathrm{curl},W_R) &:= \{u|_{W_R} \mid [x \mapsto u(x)\exp(-\mathrm{i}\alpha\, x_1)] \in H_{\mathrm{per,loc}}(\mathrm{curl},\Omega) \cap H_{\mathrm{loc},0}(\mathrm{curl},\Omega)\}\,.
\end{aligned}$$

On the space of *α-quasiperiodic functions* $H_{0,\alpha}(\mathrm{curl}, W_R)$ we use the inner product

$$\langle u, \phi\rangle_{H_{0,\alpha}(\mathrm{curl},W_R)} := \langle \mathrm{curl}\, u, \mathrm{curl}\,\phi\rangle_{L^2(W_R)} + \langle u, \phi\rangle_{L^2(W_R)}\,. \tag{2.7}$$

We always identify an element $u \in H_{0,\alpha}(\mathrm{curl}, W_0)$ with its α-quasiperiodic extension $u\colon \Omega \to \mathbb{C}^3$. The space $H_{0,\alpha}(\mathrm{curl}, W_0)$ can, in this sense, be regarded as a subspace of $H_{\mathrm{loc}}(\mathrm{curl}, \Omega)$.

Definition 2.8 (Quasiperiodic propagating modes and critical α-values)**.** For a quasimoment $\alpha \in (-\pi, \pi]$ we define *the space of α-quasiperiodic propagating modes* as

$$Y^\alpha := \{u \in H_{0,\alpha}(\mathrm{curl},\Omega) \mid u \text{ solves (1.7)}\}\,. \tag{2.8}$$

We define the set of *critical α-values* as

$$\mathcal{A}_* := \{\alpha \in (-\pi,\pi] \mid Y^\alpha \neq \{0\}\}\,. \tag{2.9}$$

The following assumption on the frequency is crucial for the further analysis. It is also used, e.g., in [13, 22, 30]. The work [22] contains a simple scalar example that shows that an existence result cannot be expected without such a condition.

Assumption 2.9 (Non-degenerancy of Q)**.** *For every $\alpha \in \mathcal{A}_*$ and $0 \neq \varphi \in Y^\alpha$, the map $Q(\cdot, \varphi)\colon Y^\alpha \to \mathbb{C}$ is a non-trivial form.*

Theorem 2.10 (The set of critival α-values; [22] Theorem 3.3)**.** *When Assumption 2.9 is satisfied, the following holds: The set of critical values $\mathcal{A}_*$ is finite. Accordingly, for a number $J \in \mathbb{N}_0$ (we allow $J = 0$ for an empty set $\mathcal{A}_*$) and values $(\alpha_j)_{0\le j\le J}$ with $\alpha_j \in (-\pi, \pi]$ holds*

$$\mathcal{A}_* = \{\alpha_j \mid 0 < j \le J\} . \tag{2.10}$$

Furthermore, the spaces Y^{α_j} are finite dimensional.

Using this Theorem, it is standard to construct spaces and basisfunctions; for the construction see e.g. Subsection 3.5 and Section 4 of [22], Subsection 2.2 of [30] or Subsection 2.3 of [13]. The space of all quasiperiodic homogeneous solutions is

$$Y := \bigoplus_{j=1}^{J} Y^{\alpha_j} \subset H_{\mathrm{loc},0}(\mathrm{curl}, \Omega) . \tag{2.11}$$

The space Y has a finite basis $\mathcal{B} := \{\phi_1^+, \dots, \phi_N^+, \phi_1^-, \dots, \phi_N^-\}$, where $m_j := \dim Y^{\alpha_j}$ and $2N := \sum_{j=1}^{J} m_j$. For every basisfunction $\phi_k^\pm$ exists $\alpha_k^\pm \in (-\pi, \pi]$ such that $\phi_k^\pm$ is $\alpha_k^\pm$-quasiperiodic for all $k = 1, \dots, N$. Furthermore, there holds $\phi_k^- = \overline{\phi_k^+}$ and $\alpha_k^- = -\alpha_k^+$ for all $k = 1, \dots, N$; in the critical case $\alpha_k^+ = \pi$ holds $\alpha_k^- = \pi$, both quasimoments encode odd functions. Finally, there holds

$$Q(\phi_k^+, \phi_k^+) > 0 \qquad \text{and} \qquad Q(\phi_k^-, \phi_k^-) < 0 \quad \forall\, k \in \{1, \dots, N\} \tag{2.12}$$

and the orthogonality

$$Q(u, v) = 0 \tag{2.13}$$

for all $u, v \in \mathcal{B}$ with $u \neq v$. For the basis $\mathcal{B}$ we introduce two subspaces of $H_{\mathrm{loc},0}(\mathrm{curl}, \Omega)$:

$$Y^+ := \operatorname{span}\left\{\phi_1^+, \dots, \phi_N^+\right\} \qquad \text{and} \qquad Y^- := \operatorname{span}\left\{\phi_1^-, \dots, \phi_N^-\right\} , \tag{2.14}$$

the *right-going modes* Y^+ and the *left-going modes* Y^-.

An immediate consequence of (2.12)–(2.13) and the fact that the space Y is finite dimensional is the following Corollary.

Corollary 2.11 (Sign of the sesquilinear form and regularity)**.** *There exist positive constants $\gamma_+, \gamma_- > 0$ such that, for every $R \in \mathbb{Z}$:*

$$\operatorname{Im}(\mathcal{Q}(u^+|_{W_R})) \ge \gamma_+ \|u^+|_{W_R}\|^2_{L^2(W_R)} \quad \forall\, u^+ \in Y^+ , \tag{2.15}$$

$$-\operatorname{Im}(\mathcal{Q}(u^-|_{W_R})) \ge \gamma_- \|u^-|_{W_R}\|^2_{L^2(W_R)} \quad \forall\, u^- \in Y^- . \tag{2.16}$$

For a constant $C_1 = C_1(\lambda, \omega) > 0$, the following regularity estimate holds:

$$\|u|_{W_R}\|_{H(\mathrm{curl}, W_R)} \le C_1 \|u|_{W_R}\|_{L^2(W_R)} \quad \forall\, u \in Y . \tag{2.17}$$

2.3 Bounded solutions and radiation condition

Definition 2.12 (The space of bounded homogeneous solutions)**.** For a function $u\colon \Omega \to \mathbb{C}^3$ we define the sL-norm as the supremum over L^2-norms:

$$\|u\|_{sL} := \sup_{l\in\mathbb{Z}} \|u|_{W_l}\|_{L^2(W_l)} \,. \tag{2.18}$$

The *space of bounded homogeneous solutions* is defined as

$$B := \{u \in H_{\mathrm{loc},0}(\mathrm{curl},\Omega) \mid u \text{ solves } (1.7)\,,\ \|u\|_{sL} < \infty\} \tag{2.19}$$

For functions on the positive half-waveguide, $u\colon \Omega_+ \to \mathbb{C}^3$, we use the analogous notation, $\|u\|_{sL} := \sup_{l\in\mathbb{N}} \|u|_{W_l}\|_{L^2(W_r)}$.

A result of [22] is the following:

Theorem 2.13 (Characterization of bounded homogeneous solutions; [22] Theorem 4.1)**.** *When Assumption 2.9 holds, the spaces Y of (2.11) and B of (2.19) coincide:*

$$Y = B\,. \tag{2.20}$$

To define a radiation condition, it is useful to define the following three subspaces of $H(\mathrm{curl}, W_0)$:

$$Y^+_{W_0} := \mathrm{span}\left\{\phi^+_1|_{W_0},\dots,\phi^+_N|_{W_0}\right\}\,,\quad Y^-_{W_0} := \mathrm{span}\left\{\phi^-_1|_{W_0},\dots,\phi^-_N|_{W_0}\right\}$$

and $Y_{W_0} := Y^+_{W_0} \oplus Y^-_{W_0}$. A general element $u \in Y_{W_0}$ is a linear combination of the above basis functions: For appropriately chosen factors $\lambda^+_k, \lambda^-_k \in \mathbb{C}$, $k = 1,\dots,N$, holds

$$u = \sum_{k=1}^N \left(\lambda^+_k \phi^+_k|_{W_0} + \lambda^-_k \phi^-_k|_{W_0}\right)\,.$$

The projections $\Pi_{Y,+}$ and $\Pi_{Y,-}$ are defined for such an element u as

$$\Pi_{Y,+}(u) = \sum_{k=1}^N \lambda^+_k \phi^+_k|_{W_0} \quad\text{and}\quad \Pi_{Y,-}(u) = \sum_{k=1}^N \lambda^-_k \phi^-_k|_{W_0}\,.$$

We denote the $L^2(W_0)$-orthogonal projection onto Y_{W_0} by $\Pi_Y\colon L^2(W_0) \to L^2(W_0)$. As concatenations, we finally introduce the two projections $\Pi_+ := \Pi_{Y,+} \circ \Pi_Y\colon L^2(W_0) \to L^2(W_0)$ onto $Y^+_{W_0}$ and $\Pi_- := \Pi_{Y,-} \circ \Pi_Y\colon L^2(W_0) \to L^2(W_0)$ onto $Y^-_{W_0}$. For $R \in \mathbb{Z}$ we also use the shifted projections $\Pi^R_Y := \Pi_Y \circ S_R$ and

$$\Pi^R_\pm := \Pi_\pm \circ S_R\colon L^2(W_R) \to L^2(W_0)\,. \tag{2.21}$$

Definition 2.14 (Radiation condition)**.** Let Assumption 2.9 be satisfied. We say that a function $u\colon \Omega_+ \to \mathbb{C}^3$ with $\|u\|_{sL} < \infty$ satisfies the *radiation condition* iff

$$\Pi^R_-(u|_{W_R}) \to 0 \qquad \text{for } R \to \infty \tag{2.22}$$

holds in $L^2(W_0)$. We say that a function $u \in H_{\mathrm{loc},0}(\mathrm{curl},\Omega_+)$ solves the *radiation problem* iff u solves (1.6) and (2.22).

As observed in [22] Subsection 3.7 and [30] Lemma 3.2, the above radiation condition is equivalent to classical radiation conditions.

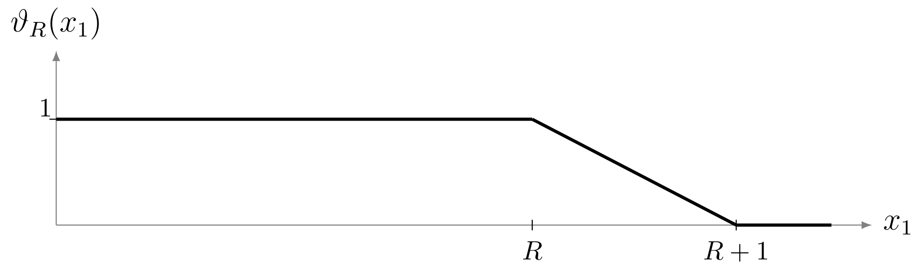

Figure 4: Cut-off function ϑ_R.

3 The truncated problem

In this section, we introduce the truncated problem and show the existence of a solution with a limiting absorption principle. Throughout this section, we demand that Assumption 2.9 is satisfied.

3.1 Definitions and first observations

We use truncated domains of the form $\Omega_R := (0,R)\times S$ with a natural number $R\in\mathbb{N}$ and use $\Gamma_r := \{r\}\times S$ for integers $r\in\mathbb{Z}$ to describe interfaces. We define a cut-off function ϑ_R similar to the one of Subsection 2.1, compare Figure 4,

$$\vartheta_R\colon \mathbb{R}_+\to\mathbb{R},\ \vartheta_R(x_1) := \begin{cases} 1 & \text{for } 0\le x_1\le R\,,\\ -x_1+(R+1) & \text{for } R<x_1<R+1\,,\\ 0 & \text{for } x_1\ge R+1\,, \end{cases} \tag{3.1}$$

and identify it with a cut-off function on Ω_+.

Definition 3.1 (Solution space)**.** For $R\in\mathbb{N}$ we define the space of functions on Ω_{R+1} that consist of a right-going wave in W_R:

$$V_R := \left\{ u\in H(\operatorname{curl},\Omega_{R+1}) \;\middle|\; S_R(u|_{W_R})\in Y^+_{W_0}\,,\ \int_{\Omega_{R+1}} u\cdot\operatorname{curl}\varphi = \int_{\Omega_{R+1}}\operatorname{curl}u\cdot\varphi \quad \forall\varphi \right\}, \tag{3.2}$$

where the test-functions φ are taken from the space $C_c^\infty\big([0,R+1)\times\overline{S}\big)$. The integral condition is a weak formulation of the boundary condition $u\times\nu=0$ on the boundaries $\{0\}\times S$ and $(0,R+1)\times\partial S$.

We now perform a calculation that motivates the subsequent definition of a sesquilinear form. For an arbitrary element $\Psi\in V_R$, we use $\Phi=\Psi\vartheta_R$ as a test-function in (1.6); here and elsewhere, we identify a function with its trivial extension to the entire domain. The left-hand side of (1.6) provides, using Q° of (2.1),

$$\begin{aligned} &\int_{\Omega_+}\frac{1}{\mu}\operatorname{curl}u\cdot\operatorname{curl}(\bar\Psi\vartheta_R)-\omega^2\varepsilon\,u\cdot\bar\Psi\vartheta_R\\ &\quad=\int_{\Omega_{R+1}}\left[\frac{1}{\mu}\operatorname{curl}u\cdot\operatorname{curl}\bar\Psi-\omega^2\varepsilon\,u\cdot\bar\Psi\right]\vartheta_R-\int_{W_R}\frac{1}{\mu}\operatorname{curl}u\cdot(e_1\times\bar\Psi)\end{aligned}$$

$$= \int_{\Omega_{R+1}} \left[\frac{1}{\mu}\operatorname{curl} u \cdot \operatorname{curl}\bar{\Psi} - \omega^2 \varepsilon\, u \cdot \bar{\Psi}\right] \vartheta_R - Q^\circ(u|_{W_R}, \Psi|_{W_R})\,.$$

This expression is used to define the sesquilinear form β_R, see Definition 3.2. The right-hand side of (1.6) with $\Phi = \Psi\vartheta_R$ defines the linear form F_R. We always assume that $R > 0$ is chosen large enough such that the support of f_e and f_h is contained in Ω_R.

Definition 3.2 (The sesquilinear form)**.** With ϑ_R of (3.1) we define the sesquilinear form $\beta_R \colon V_R \times V_R \to \mathbb{C}$ as

$$\beta_R(u,\Psi) := \int_{\Omega_{R+1}} \left[\frac{1}{\mu}\operatorname{curl} u \cdot \operatorname{curl}\bar{\Psi} - \omega^2 \varepsilon\, u \cdot \bar{\Psi}\right] \vartheta_R - Q^\circ(u|_{W_R}, \Psi|_{W_R})\,, \tag{3.3}$$

and the linear form $F_R \colon V_R \to \mathbb{C}$ as

$$F_R(\Psi) := \int_{\Omega_R} \left(\frac{1}{\mu} f_h \cdot \operatorname{curl}\bar{\Psi} + \mathrm{i}\omega f_e \cdot \bar{\Psi}\right)\,. \tag{3.4}$$

We can now formulate the truncated problem. It is defined in such a way that it is an approximate problem to the radiation problem of Subsection 2.3.

Definition 3.3 (Truncated problem)**.** Let $R \in \mathbb{N}$ be a number such that $f_e, f_h \in L^2(\Omega_+)$ have support in Ω_R. We say that a function u solves the *truncated problem,* if $u \in V_R$ and

$$\beta_R(u,\Psi) = F_R(\Psi) \quad \forall\, \Psi \in V_R\,. \tag{3.5}$$

Remark 3.4 (Properties of solutions to the truncated problem)**.** *Let $R \in \mathbb{N}$ be a parameter and let $u \in V_R$ be a solution to the truncated problem* (3.5)*. Then $u|_{\Omega_R} \in H(\operatorname{curl},\Omega_R)$ satisfies the Maxwell equation in Ω_R:*

$$\int_{\Omega_R} \left(\frac{1}{\mu}\operatorname{curl} u \cdot \operatorname{curl}\bar{\Psi} - \omega^2 \varepsilon\, u \cdot \bar{\Psi}\right) = \int_{\Omega_R} \left(\frac{1}{\mu} f_h \cdot \operatorname{curl}\bar{\Psi} + \mathrm{i}\omega f_e \cdot \bar{\Psi}\right) \quad \forall\, \Psi \in H_0(\operatorname{curl},\Omega_R)\,.$$

Furthermore, a solution $u \in V_R$ to the truncated problem for $f_e = 0 = f_h$ satisfies $u|_{W_R} =$ 0.

The first claim is achieved by using $\Psi \in H_0(\operatorname{curl},\Omega_R)$, extended by 0 *on W_R, as a test-function in* (3.5)*. The second claim is achieved by using the solution $u \in V_R$ as a test-function in* (3.5)*, taking the imaginary part and using the sign property of* (2.15).

A compact embedding. For an open set $\Sigma \subset \mathbb{R}^3$, we use the space of functions with bounded distributional divergence as follows:

$$H(\operatorname{div},\Sigma) := \left\{ u \in L^2(\Sigma,\mathbb{C}^3) \;\middle|\; \operatorname{div} u \in L^2(\Sigma,\mathbb{C}) \right\}\,. \tag{3.6}$$

On this space we use the norm that is defined by

$$\|u\|^2_{H(\operatorname{div},\Sigma)} := \|u\|^2_{L^2(\Sigma)} + \|\operatorname{div} u\|^2_{L^2(\Sigma)}\,. \tag{3.7}$$

Furthermore, we introduce the space

$$\varepsilon^{-1} H(\operatorname{div},\Sigma) := \left\{ u \colon \Sigma \to \mathbb{C}^3 \;\middle|\; \varepsilon\, u \in H(\operatorname{div},\Sigma) \right\}\,. \tag{3.8}$$

The following compactness result is a well-known fact for bounded Lipschitz domains $\Sigma \subset \mathbb{R}^3$. For similar results we refer to [4, 28, 29, 33].

Theorem 3.5 (Compact embedding; [32] Theorem 2.1 and 2.2)**.** *Let $\Sigma \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Furthermore, let $\varepsilon \in L^\infty(\Sigma, \mathbb{R})$ be with a positive lower bound. Then the embedding*

$$H_0(\mathrm{curl}, \Sigma) \cap \varepsilon^{-1} H(\mathrm{div}, \Sigma) \hookrightarrow L^2(\Sigma, \mathbb{C}^3) \tag{3.9}$$

is compact.

3.2 Existence results for the sesquilinear form with absorption

Here, we keep the parameter $R \in \mathbb{N}$ fixed, and analyze, for a small parameter $\delta > 0$, the sesquilinear form with positive absorption δ. We introduce the form $\beta_R^\delta \colon V_R \times V_R \to \mathbb{C}$ as

$$\beta_R^\delta(u, \Psi) := \int\limits_{\Omega_{R+1}} \left(\frac{1}{\mu} \mathrm{curl}\, u \cdot \mathrm{curl}\, \bar{\Psi} \right) \vartheta_R - \int\limits_{\Omega_{R+1}} \left\{ (\omega^2 + \mathrm{i}\delta) \varepsilon\, u \cdot \bar{\Psi} \vartheta_R \right\} - Q^\circ(u|_{W_R}, \Psi|_{W_R}) \,. \tag{3.10}$$

We will see that the form β_R^δ is coercive for positive δ.

Theorem 3.6 (Coercivity estimates)**.** *For $R \in \mathbb{N}$ and $\delta > 0$ we consider the sesquilinear form β_R^δ of (3.10). Let $\lambda, \Lambda > 0$ be the lower and upper bounds for $\varepsilon, \mu \in L^\infty(\Omega_+, \mathbb{R})$, $\gamma_+ > 0$ the constant of Corollary 2.11 and $C_1 > 0$ the constant of inequality (2.17). We define $c_\delta := \min\{\delta\lambda, \gamma_+\} > 0$ and $C_2 := 2\max\{\omega^2\Lambda, C_1/\lambda\} > 0$. Then, for every $u \in V_R$, there hold the inequalities*

$$-\,\mathrm{Im}\, \beta_R^\delta(u, u) \ge c_\delta \|u\|^2_{L^2(\Omega_{R+1})} \,, \tag{3.11}$$

$$\mathrm{Re}\, \beta_R^\delta(u, u) \ge \frac{1}{\Lambda} \|\mathrm{curl}\, u\|^2_{L^2(\Omega_R)} - C_2 \|u\|^2_{L^2(\Omega_{R+1})} \,. \tag{3.12}$$

With the two numbers $C_3 := (1 + C_1^2)\Lambda^{-1} + C_2 > 0$ and $\xi := (1 + \mathrm{i}C_3/c_\delta)$, the sesquilinear form $\xi\, \beta_R^\delta$ is coercive on V_R in the sense that

$$\mathrm{Re}[\xi\, \beta_R^\delta(u, u)] \ge \frac{1}{\Lambda} \|u\|^2_{H(\mathrm{curl}, \Omega_{R+1})} \qquad \text{for all } u \in V_R \,. \tag{3.13}$$

We note that (3.13) implies the inequality

$$|\beta_R^\delta(u, u)| \ge \frac{1}{\Lambda|\xi|} \|u\|^2_{H(\mathrm{curl}, \Omega_{R+1})} \,, \tag{3.14}$$

which is sometimes used to define coercivity of a sesquilinear form.

Proof. For $u \in V_R$, the imaginary part satisfies

$$\begin{aligned} -\,\mathrm{Im}\, \beta_R^\delta(u, u) &= -\,\mathrm{Im} \int\limits_{\Omega_{R+1}} \frac{1}{\mu} |\mathrm{curl}\, u|^2 \vartheta_R + \mathrm{Im} \int\limits_{\Omega_{R+1}} (\omega^2 + \mathrm{i}\delta)\varepsilon\, |u|^2 \vartheta_R + \mathrm{Im}\, \mathcal{Q}(u|_{W_R}) \\ &= \int\limits_{\Omega_{R+1}} \delta\varepsilon\, |u|^2 \vartheta_R + \mathrm{Im}\, \mathcal{Q}(u|_{W_R}) \ge \delta\lambda \|u\|^2_{L^2(\Omega_R)} + \gamma_+ \|u\|^2_{L^2(W_R)} \\ &\ge c_\delta \|u\|^2_{L^2(\Omega_{R+1})} \,. \end{aligned}$$

This shows (3.11). The real part satisfies

$$\begin{aligned}
\operatorname{Re}\beta_R^\delta(u,u) &= \operatorname{Re}\int\limits_{\Omega_{R+1}} \frac{1}{\mu}|\operatorname{curl} u|^2\vartheta_R - \operatorname{Re}\int\limits_{\Omega_{R+1}} (\omega^2+\mathrm{i}\delta)\varepsilon\,|u|^2\vartheta_R - \operatorname{Re}\mathcal{Q}(u|_{W_R})\\
&= \int\limits_{\Omega_{R+1}} \frac{1}{\mu}|\operatorname{curl} u|^2\vartheta_R - \int\limits_{\Omega_{R+1}} \omega^2\varepsilon\,|u|^2\vartheta_R - \operatorname{Re}\mathcal{Q}(u|_{W_R})\\
&\geq \frac{1}{\Lambda}\|\operatorname{curl} u\|^2_{L^2(\Omega_R)} - \omega^2\Lambda\|u\|^2_{L^2(\Omega_{R+1})} - \frac{1}{\lambda}\|u\|_{H(\operatorname{curl},W_R)}\|u\|_{L^2(W_R)}\\
&\geq \frac{1}{\Lambda}\|\operatorname{curl} u\|^2_{L^2(\Omega_R)} - \omega^2\Lambda\|u\|^2_{L^2(\Omega_{R+1})} - \frac{C_1}{\lambda}\|u\|^2_{L^2(W_R)}\\
&\geq \frac{1}{\Lambda}\|\operatorname{curl} u\|^2_{L^2(\Omega_R)} - C_2\|u\|^2_{L^2(\Omega_{R+1})}\,,
\end{aligned}$$

where we have used the definition of Q and $\mathcal{Q}$ of (2.1) in the first inequality and (2.17) in the second inequality. This shows (3.12). For the coercivity of β_R^δ we calculate:

$$\begin{aligned}
\frac{1}{\Lambda}\|u\|^2_{H(\operatorname{curl},\Omega_{R+1})} &= \frac{1}{\Lambda}\left(\|u\|^2_{L^2(\Omega_{R+1})} + \|\operatorname{curl} u\|^2_{L^2(\Omega_{R+1})}\right)\\
&\overset{(3.12)}{\leq} \frac{1}{\Lambda}\|u\|^2_{L^2(\Omega_{R+1})} + \operatorname{Re}\beta_R^\delta(u,u) + C_2\|u\|^2_{L^2(\Omega_{R+1})} + \frac{1}{\Lambda}\|u\|^2_{H(\operatorname{curl},W_R)}\\
&\overset{(2.17)}{\leq} \frac{1}{\Lambda}\|u\|^2_{L^2(\Omega_{R+1})} + \operatorname{Re}\beta_R^\delta(u,u) + C_2\|u\|^2_{L^2(\Omega_{R+1})} + \frac{C_1^2}{\Lambda}\|u\|^2_{L^2(W_R)}\\
&\leq \operatorname{Re}\beta_R^\delta(u,u) + \left(\frac{1}{\Lambda} + C_2 + \frac{C_1^2}{\Lambda}\right)\|u\|^2_{L^2(\Omega_{R+1})}\\
&\overset{(3.11)}{\leq} \operatorname{Re}\beta_R^\delta(u,u) - \frac{C_3}{c_\delta}\operatorname{Im}\beta_R^\delta(u,u)\\
&= \operatorname{Re}\left[\left(1+\frac{C_3}{c_\delta}\mathrm{i}\right)\beta_R^\delta(u,u)\right] = \operatorname{Re}[\xi\,\beta_R^\delta(u,u)]\,.
\end{aligned}$$

This shows (3.13). □

Corollary 3.7 (Solvability of the truncated problem with absorption)**.** *For every $\delta > 0$ and every $F_R \in V_R'$ there exists an unique solution $u^\delta \in V_R$ of the truncated problem with absorption:*

$$\beta_R^\delta(u^\delta,\Psi) = F_R(\Psi) \qquad \textit{for all } \Psi \in V_R\,. \tag{3.15}$$

There exists a constant $C(\delta) > 0$ such that

$$\|u\|_{H(\operatorname{curl},\Omega_{R+1})} \leq C(\delta)\|F_R\|_{V_R'}\,. \tag{3.16}$$

Proof. Let $\delta > 0$ be arbitrary. The sesquilinear form β_R^δ is continuous and coercive and the linear form F_R is continuous. Theorem 3.6 allows to apply the Lax-Milgram theorem. We find that there exists an unique element $u^\delta \in V_R$ and a constant $C(\delta) > 0$ such that (3.15) and (3.16) are satisfied. □

Lemma 3.8 (Compactness of bounded solutions to the truncated problem)**.** *Let $f_e, f_h \in L^2(\Omega_+)$ have support in Ω_R, we assume $\operatorname{div} f_e \in L^2(\Omega_+)$. For a sequence $\delta \searrow 0$, let u^δ be a sequence of solutions to the truncated problems with absorption. We assume the uniform bound $\|u^\delta\|_{L^2(\Omega_{R+1})} \le C_0$ for a constant $C_0 > 0$. Then there exists a subsequence $\delta \to 0$ (not relabeled) and a function u such that $u^\delta \to u$ in $L^2(\Omega_{R+1})$. Furthermore, the sequence u^δ is bounded in $H(\operatorname{curl}, \Omega_{R+1})$ and, accordingly, there holds $u^\delta \rightharpoonup u$ in $H(\operatorname{curl}, \Omega_{R+1})$.*

Proof. Our aim is to show that u^δ is bounded in $H(\operatorname{curl}, \Omega_{R+1})$ and in $\varepsilon^{-1}H(\operatorname{div}, \Omega_R)$. Taking the real part of $\beta_R^\delta(u^\delta, u^\delta) = F_R(u^\delta)$ and using (3.12) and (2.17), we find

$$
\begin{aligned}
\|\operatorname{curl} u^\delta\|^2_{L^2(\Omega_{R+1})} &= \|\operatorname{curl} u^\delta\|^2_{L^2(\Omega_R)} + \|\operatorname{curl} u^\delta\|^2_{L^2(W_R)} \\
&\le \Lambda\left(\operatorname{Re}\beta_R^\delta(u^\delta, u^\delta) + C_2\|u^\delta\|^2_{L^2(\Omega_{R+1})}\right) + C_1^2\|u^\delta\|^2_{L^2(W_R)} \\
&\le \Lambda \operatorname{Re}\beta_R^\delta(u^\delta, u^\delta) + (C_1^2 + \Lambda C_2)C_0^2 \\
&\overset{(3.5)}{=} \Lambda \operatorname{Re}\left(\int_{\Omega_R}\left\{\frac{1}{\mu} f_h \cdot \operatorname{curl}\overline{u^\delta} + \mathrm{i}\omega f_e \cdot \overline{u^\delta}\right\}\right) + (C_1^2 + \Lambda C_2)C_0^2 \\
&\le \underbrace{\frac{\Lambda}{\lambda}\|f_h\|_{L^2(\Omega_R)}}_{=:\hat{C}<\infty} \|\operatorname{curl} u^\delta\|_{L^2(\Omega_R)} + \underbrace{\Lambda\omega\|f_e\|_{L^2(\Omega_R)}C_0 + (C_1^2 + \Lambda C_2)C_0^2}_{=:C<\infty} \\
&\le \frac{\hat{C}^2}{2} + \frac{1}{2}\|\operatorname{curl} u^\delta\|^2_{L^2(\Omega_{R+1})} + C\,.
\end{aligned}
$$

This implies $\|\operatorname{curl} u^\delta\|^2_{L^2(\Omega_{R+1})} \le \hat{C}^2 + 2C$. The sequence u^δ is not only bounded in $L^2(\Omega_{R+1})$, but also in $H(\operatorname{curl}, \Omega_{R+1})$.

For the estimates for $\operatorname{div}\left(\varepsilon\, u^\delta\right)$ we proceed as follows: Let $\varphi \in C_c^\infty(\Omega_R)$ be extended by 0 onto W_R. Using $\Psi := \nabla\varphi$ in (3.15) shows that

$$
\operatorname{div}\left(\varepsilon\, u^\delta\right) = -\frac{\mathrm{i}\omega}{\omega^2 + \mathrm{i}\delta}\operatorname{div} f_e \qquad \text{holds in } L^2(\Omega_R)\,. \tag{3.17}
$$

Based on the boundedness of $\operatorname{curl} u^\delta$ and $\operatorname{div}(\varepsilon\, u^\delta)$, the compactness statement of Theorem 3.5 will provide compactness of the sequence u^δ.

Formally, we cannot apply Theorem 3.5 directly, since u^δ is not vanishing at the right boundary. We argue as follows: Because of $S_R(u^\delta|_{W_R}) \in Y^+_{W_0}$, there exist $\lambda_1^\delta, \dots, \lambda_N^\delta \in \mathbb{C}$ such that for the quasimoments α_k^+ of the basisfunctions ϕ_k^+ holds

$$
S_R(u^\delta|_{W_R}) = \sum_{k=1}^N \lambda_k^\delta\, \phi_k^+|_{W_0} \qquad \text{and} \qquad u^\delta|_{W_R} = \sum_{k=1}^N \underbrace{\lambda_k^\delta \exp(-\mathrm{i}R\alpha_k^+)}_{=:\mu_k^\delta}\, \phi_k^+|_{W_R}\,.
$$

Let $\theta \in C_c^\infty\big((0, R+1], \mathbb{R}\big)$ be a cut-off function with $\theta(x_1) = 1$ for $R \le x_1 \le R+1$. As before, we identify ϕ_k^+ with its quasiperiodic extension onto Ω_{R+1} and define the functions

$$
v^\delta := u^\delta - \sum_{k=1}^N \mu_k^\delta\, \phi_k^+ \theta
$$

for every $\delta > 0$. Then $v^\delta|_{\Omega_R}$ is a bounded sequence in $H_0(\operatorname{curl}, \Omega_R) \cap \varepsilon^{-1} H(\operatorname{div}, \Omega_R)$. The compactness statement of Theorem 3.5 and the Bolzano–Weierstrass theorem assure that there exists a subsequence $\delta \to 0$ (not relabeled), a limit function v and limits μ_k such that $v^\delta|_{\Omega_R} \to v$ in $L^2(\Omega_R)$ and $\mu_k^\delta \to \mu_k$ in $\mathbb{C}$ for $k = 1, \dots, N$ along this subsequence. This assures that u^δ converges strongly in $L^2(\Omega_{R+1})$ for $\delta \to 0$. □

3.3 The vanishing absorption limit $\delta \to 0$

The following result provides a Fredholm alternative for the truncated problem of Definition 3.3. Solutions are obtained in the limit $\delta \to 0$.

Proposition 3.9 (Existence for the truncated problem)**.** *Let the setting be that of Theorem 1.2 and, in particular, let Assumption 2.9 be satisfied. Let $f_e, f_h \in L^2(\Omega_+)$ have support in Ω_R, we assume* $\operatorname{div} f_e \in L^2(\Omega_+)$*. If the truncated problem of Definition 3.3 has at most one solution, then there exists a solution to the truncated problem.*

Proof. Let $R \in \mathbb{N}$ be fixed and let $\delta \searrow 0$ be a sequence. Furthermore, let u^δ be a solution of (3.15) for every δ. We set

$$N_\delta := \|u^\delta\|_{L^2(\Omega_{R+1})} . \tag{3.18}$$

Case 1: There exists a subsequence $\delta \to 0$ such that N_δ is bounded along this subsequence. We consider the subsequence $\delta \to 0$ along which N_δ is bounded. The corresponding sequence u^δ is then bounded in $L^2(\Omega_{R+1})$ and, by Lemma 3.8, also bounded in $H(\operatorname{curl}, \Omega_{R+1})$. We find a further subsequence $\delta \to 0$ and a limit function $u \in H(\operatorname{curl}, \Omega_{R+1})$ such that $u^\delta \rightharpoonup u$ in $H(\operatorname{curl}, \Omega_{R+1})$. We have to show $u \in V_R$. Since $Y^+_{W_0}$ is a finite dimensional space and every $S_R(u^\delta|_{W_R})$ is in $Y^+_{W_0}$, we conclude $S_R(u|_{W_R}) \in Y^+_{W_0}$. Since $u^\delta \rightharpoonup u$ in $H(\operatorname{curl}, \Omega_{R+1})$ for $\delta \to 0$, we have, for arbitrary $\varphi \in C_c^\infty\big([0, R+1) \times \overline{S}\big)$:

$$\int_{\Omega_{R+1}} u \cdot \operatorname{curl} \varphi \leftarrow \int_{\Omega_{R+1}} u^\delta \cdot \operatorname{curl} \varphi = \int_{\Omega_{R+1}} \operatorname{curl} u^\delta \cdot \varphi \to \int_{\Omega_{R+1}} \operatorname{curl} u \cdot \varphi .$$

This implies $u \in V_R$. We now take the limit $\delta \to 0$ in the relation $\beta_R^\delta(u^\delta, \Psi) = F_R(\Psi)$. We obtain, for arbitrary $\Psi \in V_R$, as $\delta \to 0$:

$$F_R(\Psi) = \beta_R^\delta(u^\delta, \Psi) = \beta_R(u^\delta, \Psi) - \delta \mathrm{i} \int_{\Omega_{R+1}} \varepsilon\, u^\delta \cdot \bar{\Psi} \vartheta_R \to \beta_R(u, \Psi) .$$

This shows that u is a solution of the truncated problem (3.5).

Case 2: $N_\delta \to \infty$. We consider the normalized functions $v^\delta := N_\delta^{-1} u^\delta$. The sequence v^δ has all the properties of u^δ of Case 1. Lemma 3.8 assures that there exists a limit function v such that $v^\delta \to v$ in $L^2(\Omega_{R+1})$. For every $\delta > 0$, the function v^δ solves the truncated problem (3.15) with $f_{e,\delta} := N_\delta^{-1} f_e$ and $f_{h,\delta} := N_\delta^{-1} f_h$. There holds $f_{e,\delta}, f_{h,\delta} \to 0$ for $\delta \to 0$ in $L^2(\Omega_{R+1})$. Therefore, the limit function v solves the homogeneous truncated problem $\beta_R(v, \Psi) = 0$ for all $\Psi \in V_R$. Our assumption on the truncated problem assures $v = 0$. Since we have the strong L^2-convergence $v^\delta \to v = 0$, this is a contradiction to the normalization $\|v^\delta\|^2_{L^2(\Omega_{R+1})} = 1$. We conclude that Case 2 can actually not occur. □

4 Existence on unbounded domains

Throughout this section, we demand that the assumptions of Theorem 1.2 are satisfied, in particular: The non-degeneracy of Assumption 2.9 holds. The homogeneous radiation problems on Ω_+ and Ω_- have only the trivial solution (Definition 1.1 with $f_e = 0$ and $f_h = 0$ together with the radiation condition of Definition 2.14 or, in the case of Ω_-, the modified radiation condition $\Pi_+^R(u|_{W_R}) \to 0$ for $R \to -\infty$). More information on the assumption on Ω_- is given in Appendix B on edge-resonances.

We consider the truncated problems for a fixed sequence $R = R_k \to \infty$ as $\mathbb{N} \ni k \to \infty$. Let $u = u_k$ be the corresponding sequence of solutions to truncated problems. The right hand side $f = (f_e, f_h) \in H(\mathrm{div}, \Omega_+) \times L^2(\Omega_+)$ with $\mathrm{supp}(f) \subset \Omega_M$ for some $M > 0$ is kept fixed, $R_k > M$ is assumed along the sequence. The idea of this section is quite simple: The limit function $u = \lim_k u_k$ solves the *radiation problem* of Subsection 2.3, i.e. u solves (1.6) and the radiation condition of Definition of 2.14.

4.1 Preparations for the proof of the main theorem

Following the sketched program, we have to show that a limit of the solution sequence $(u_k)_k$ is a solution of the Maxwell equations. Under an appropriate boundedness assumption, this is done in the next lemma.

We note that the sL-norm is a natural norm, since we expect that solutions look like a combination of quasiperiodic functions on large domains, there is no decay of solutions. Accordingly, limits can only be taken locally. We have to conclude locally the strong compactness of the solution sequence from a div-curl compactness argument.

Lemma 4.1 (Local limits for solution sequences)**.** *Let the setting be that of Theorem 1.2. For a sequence $R_k \to \infty$, let $u_k \colon \Omega_{R_k+1} \to \mathbb{C}^3$ be a sequence of solutions to the truncated problems with right-hand sides $f_h \in L^2(\Omega_+)$ and $f_e \in H(\mathrm{div}, \Omega_+)$ with support in Ω_M for some $M > 0$. We assume that the sequence*

$$\|u_k\|_{sL} = \sup \left\{ \|u_k|_{W_l}\|_{L^2(W_l)} \;\middle|\; l \in \mathbb{N},\ l \le R_k \right\} \tag{4.1}$$

is bounded. Then there exists a subsequence $k \to \infty$ (not relabeled) and a local limit $u \in H_{\mathrm{loc},0}(\mathrm{curl}, \Omega_+)$ such that $u_k \to u$ in $L^2(\Omega_{r_0})$ and $u_k \rightharpoonup u$ in $H(\mathrm{curl}, \Omega_{r_0})$ for every $r_0 \in \mathbb{N}$. Moreover $u_k \to u$ in $H(\mathrm{curl}, \Omega_{M+1,r_0})$. The limit u is a weak solution in the sense of Definition 1.1.

The statement remains true when we consider solutions u_k to strongly L^2-convergent right-hand sides $f_h^k \to f_h$ and $f_h^k \to f_h$ such that every f_h^k and every f_e^k has its support contained in a fixed set Ω_M.

Proof. We consider a fixed value of $r_0 > 0$ and, without loss of generality, only indices k such that $R_k \ge r_0$ holds. The boundedness (4.1) assures that u_k is a bounded sequence in $L^2(\Omega_{r_0+1})$ and we find a subsequence and a weak L^2-limit u. We start by observing that, since u_k is a solution to the Maxwell system, the L^2-regularity transfers to an $H(\mathrm{curl})$-regularity:

$$\|\mathrm{curl}(u_k)\|^2_{L^2(\Omega_{r_0})} \le C \left(\|u_k\|^2_{L^2(\Omega_{r_0+1})} + \|f_e\|^2_{L^2(\Omega_+)} + \|f_h\|^2_{L^2(\Omega_+)} \right)$$

for some constant C that is independent of k, compare Lemma C.1.Thus $\|\mathrm{curl}(u_k)\|_{L^2(\Omega_{r_0})}$ is bounded.

Our goal is to obtain compactness. We use a cut-off function $\theta = \theta_{r_0}$ with support in Ω_{r_0+1} and with $\theta(x) = 1$ for $0 \le x_1 \le r_0$. Then $u_k\theta$ is a bounded sequence in $H_0(\mathrm{curl}, \Omega_{r_0+1})$.

For the estimates for $\mathrm{div}(\varepsilon\, u_k\theta)$ we argue as follows: Let $\varphi_k \in C_c^\infty(\Omega_{R_k})$ be extended by 0 onto W_{R_k}. Using $\Psi_k := \nabla\varphi_k$ in (3.5) shows that

$$\mathrm{div}(\varepsilon\, u_k) = -\frac{\mathrm{i}}{\omega}\,\mathrm{div}\, f_e \qquad \text{holds in } L^2(\Omega_{R_k})\,. \tag{4.2}$$

We conclude that

$$\mathrm{div}(\varepsilon\, u_k\theta) = \nabla\theta\cdot(\varepsilon\, u_k) + \theta\,\mathrm{div}(\varepsilon\, u_k) = \nabla\theta\cdot(\varepsilon\, u_k) - \frac{\mathrm{i}\theta}{\omega}\,\mathrm{div}\, f_e$$

is bounded in $L^2(\Omega_{r_0+1})$. This shows that the sequence $u_k\theta$ is bounded in $H_0(\mathrm{curl}, \Omega_{r_0+1}) \cap \varepsilon^{-1}H(\mathrm{div}, \Omega_{r_0+1})$. Theorem 3.5 implies the strong convergence $u_k\theta \to u\theta$ in $L^2(\Omega_{r_0+1})$. This assures $u_k \to u$ in $L^2(\Omega_{r_0})$, since $\theta(x) = 1$ for $0 \le x_1 \le r_0$.

To show the convergence $u_k \to u$ in $H(\mathrm{curl}, \Omega_{M+1,r_0})$ we argue as follows. Let $\Theta \in C_c^\infty(\Omega_+, \mathbb{R})$ be a cut-off function with $\Theta(x) = 1$ for $M+1 \le x_1 \le r_0$ and $\Theta(x) = 0$ for $x_1 \ge r_0 + 1$ and $x_1 \le M$. We insert $\Psi = (u_k - u|_{\Omega_{R_k+1}})\Theta$ into $\beta_{R_k}(u, \Psi) = F_{R_k}(\Psi)$. Then $F_{R_k}(\Psi) = 0$ since f_e, f_h have support in Ω_M. The same calculation for $\beta_{R_k}(u, \Psi)$ as in Lemma C.1 assures that there exists a constant $C = C(\omega, \lambda, \Lambda, \nabla\Theta) > 0$ such that

$$\|\,\mathrm{curl}\, u_k - \mathrm{curl}\, u\|^2_{L^2(\Omega_{M+1,r_0})} \le \|\Theta(\mathrm{curl}\, u_k - \mathrm{curl}\, u)\|^2_{L^2(\Omega_{M,r_0+1})} \le C\|u_k - u\|^2_{L^2(\Omega_{M,r_0+1})} \to 0$$

for $k \to \infty$. This assures that $\mathrm{curl}\, u_k \to \mathrm{curl}\, u$ in $L^2(\Omega_{M+1,r_0})$ and in particular $u_k \to u$ in $H(\mathrm{curl}, \Omega_{M+1,r_0})$.

Every function u_k satisfies the weak formulation (1.6). We can take the limit $k \to \infty$ and obtain (1.6) for u. This shows that u is a weak solution. □

The above lemma can provide the solution of the limit problem. There remain three problems: 1. Solutions u_k to the truncated problems must exist (this is shown in Section 4.4). 2. We have to show the bound on the sequence $(u_k)_k$. 3. We have to show that the limit u satisfies the radiation condition. For points 2. and 3., the subsequent observation will turn out to be very helpful.

Lemma 4.2 (Solutions on large domains are locally close to propagating waves)**.** *Let $\eta > 0$ be an arbitrary error quantifier and $C_0 > 0$ an arbitrary number. Then there exists a distance variable $r_0 := r_0(\eta, C_0) \in \mathbb{N}$ with the following property: Let $R \in \mathbb{N}$ be a number and let $u_R \in V_R$ be a solution to the truncated problem with f_e, f_h supported in Ω_M, the bound* $\sup\left\{\|u_R|_{W_l}\|_{L^2(W_l)} \;\middle|\; l \in \mathbb{N},\ l \le R\right\} \le C_0$*. Then, for every position $r \in \mathbb{N}$ satisfying $M + r_0 < r < R - r_0$, there holds*

$$\|S_r(u_R|_{W_r}) - \Pi_Y^r(u_R|_{W_r})\|_{H(\mathrm{curl},W_0)} \le \eta\,. \tag{4.3}$$

Proof. We argue by contradiction. Let $\eta > 0$ be fixed. If there is no r_0 with the desired properties, then there are sequences $r \to \infty$ and $R \to \infty$ with $R - r \to \infty$ and a sequence of functions u_R which satisfy the boundedness and solution properties, but not (4.3). The boundedness of the sequence u_R allows to use Lemma 4.1. Since $r \to \infty$ and $R - r \to \infty$ we find a local limit $\tilde{u} \in H_{\mathrm{loc},0}(\mathrm{curl}, \Omega)$ of the shifted functions: $S_r(u_R) \to \tilde{u}$ locally in space $H(\mathrm{curl}, \Omega_{-m,m})$ for every $m > 0$. The strong convergence implies that the limit satisfies

$$\|\tilde{u}|_{W_0} - \Pi_Y(\tilde{u}|_{W_0})\|_{H(\mathrm{curl},W_0)} \geq \eta\,.$$

The boundedness and solution properties imply that the local limit $\tilde{u}$ is an element of B, and, hence, by Theorem 2.13, an element of Y. Then the projection acts trivially and we find $\tilde{u}|_{W_0} = \Pi_Y(\tilde{u}|_{W_0})$, a contradiction. □

The final preparatory lemma is very easy to believe: When we take the difference of a solution to the truncated problem and of a right-going wave (i.e., an element of Y^+ as defined in (2.14)), then this difference also satisfies the flux equality. This is helpful since we can subtract the right-going wave part of $u|_{W_\rho}$ from u; then the rest w is near to a left-going wave in W_ρ, but a right-going wave in W_R. The flux equality then allows to conclude that both waves must be small in norm.

Lemma 4.3 (Flux equality "looking right"). *Let $f_e \in H(\mathrm{div}, \Omega_+)$ and $f_h \in L^2(\Omega_+)$ have support in Ω_M for some $M > 0$ and let u be a solution to the truncated problem of Definition 3.3 to $\mathbb{N} \ni R > M$. Furthermore, let $\phi \in Y^+$ be a right-going wave. Then, for every $\rho \in \mathbb{N}$, $M < \rho < R$, the difference $w := u - \phi|_{\Omega_{R+1}}$ satisfies the flux equality*

$$\mathrm{Im}\, \mathcal{Q}(w|_{W_\rho}) = \mathrm{Im}\, \mathcal{Q}(w|_{W_R}) \geq 0\,. \tag{4.4}$$

Proof. By Definition 3.3, the truncated solution u satisfies, for every $\Psi \in V_R$,

$$F_R(\Psi) = \beta_R(u, \Psi)\,.$$

Let $\vartheta_{\rho,R+1}$ be the cut-off function defined in Subsection 2.1. Note that $\vartheta_{\rho,R+1}(x_1) = 1$ for all $x_1 \in [\rho + 1, R + 1]$. We define $\Psi := w\vartheta_{\rho,R+1} = (u - \phi)\vartheta_{\rho,R+1}$. In the cylinder W_R holds $\vartheta_{\rho,R+1} \equiv 1$ and $S_R(u|_{W_R}), S_R(\phi|_{W_R}) \in Y^+_{W_0}$ such that $S_R(\Psi|_{W_R}) \in Y^+_{W_0}$. We conclude that $\Psi = w\vartheta_{\rho,R+1}$ is an element in V_R and therefore can be used as a test-function: $\beta_R(u, \Psi) = F_R(\Psi)$. The right-hand side vanishes, since f_e and f_h have support in Ω_M for $M < \rho$. As in Lemma 2.2, we can insert $w\vartheta_{\rho,R}$ as a test-function for ϕ, which satisfies (1.7). Subtracting the two expressions and taking the imaginary part shows the flux equality (4.4). The property $\mathrm{Im}\, \mathcal{Q}(w|_{W_R}) \geq 0$ follows from Corollary 2.11 since $S_R(u|_{W_R}) \in Y^+_{W_0}$ and $\phi \in Y^+$. □

4.2 Verification of the radiation condition

Let us recall the outline of the proof of our main theorem: We want to show that every limit of solutions to the truncated problems is a solution to the problem on the unbounded domain. We have already shown in Lemma 4.1 that the limit is in the right function space and that it is a solution. We now show that the limit satisfies the radiation condition of Definition 2.14.

Proposition 4.4 (Radiation condition for the limit). *Let the setting be that of Theorem 1.2. For a sequence $R_k \to \infty$, let u_k be a sequence of solutions to the truncated problems. We assume that the sequence*

$$\|u_k\|_{sL} = \sup\left\{ \|u_k|_{W_l}\|_{L^2(W_l)} \;\middle|\; l \in \mathbb{N},\ l \leq R_k \right\}$$

is bounded. Let $u \in H_{\mathrm{loc},0}(\mathrm{curl}, \Omega_+)$ be locally the weak $H(\mathrm{curl})$-limit of the solutions u_k as in Lemma 4.1. Then u satisfies the radiation condition of Definition 2.14.

Proof. As solutions to the truncated problem, the functions u_k satisfy $u_k \in V_{R_k}$, $u_k \in H(\mathrm{curl}, \Omega_{R_k+1})$, and $\beta_{R_k}(u_k, \Psi) = F_{R_k}(\Psi)$ for every $\Psi \in V_{R_k}$. The limit function u, which exists by Lemma 4.1, solves the Maxwell equation (1.6) with right hand sides f_e, f_h. Our goal is to verify the radiation condition for u.

We fix a sequence $r_m \to \infty$ as $m \to \infty$. Along this sequence, we want to verify the radiation condition for u. Given the sequence $(r_m)_m$, we can choose a subsequence of indices $(k_m)_{m\in\mathbb{N}}$ with $k_m \to \infty$ as $m \to \infty$ satisfying the following two properties:

a) There holds $R_{k_m} \to \infty$ and $R_{k_m} - r_m \to \infty$ as $m \to \infty$ along the subsequence k_m.

b) There holds $\|(u_{k_m} - u)|_{W_{r_m}}\|_{L^2(W_{r_m})} \to 0$ as $m \to \infty$ along the sequences k_m and r_m.

Indeed, for every fixed m and, hence, fixed r_m, there is convergence in W_{r_m}. Therefore k_m can be chosen accordingly for the smallness in b); we can actually also achieve a fixed rate, e.g., that the error is smaller than $1/m$.

Our aim is to show that the left-going part of the limit function u in the cylinder W_{r_m} is small. Using the triangle inequality, we calculate

$$\begin{aligned}\left\|\Pi_-^{r_m}(u|_{W_{r_m}})\right\|_{L^2(W_0)} &= \left\|\Pi_-^{r_m}(u_{k_m}|_{W_{r_m}}) + \Pi_-^{r_m}(u|_{W_{r_m}}) - \Pi_-^{r_m}(u_{k_m}|_{W_{r_m}})\right\|_{L^2(W_0)} \\ &\le \left\|\Pi_-^{r_m}(u_{k_m}|_{W_{r_m}})\right\|_{L^2(W_0)} + \left\|\Pi_-^{r_m}(u|_{W_{r_m}}) - \Pi_-^{r_m}(u_{k_m}|_{W_{r_m}})\right\|_{L^2(W_0)} \\ &= \left\|\Pi_-^{r_m}(u_{k_m}|_{W_{r_m}})\right\|_{L^2(W_0)} + \left\|\Pi_-\left(S_{r_m}((u_{k_m} - u)|_{W_{r_m}})\right)\right\|_{L^2(W_0)} . \end{aligned} \tag{4.5}$$

The second summand is small by the choice of the subsequence and the boundedness of the projection. It remains to show smallness of the first summand on the right-hand side as $k_m \to \infty$. We omit the subscript m and want show that $\Pi_-^r(u_k|_{W_r}) \to 0$ in $L^2(W_0)$ for $k \to \infty$. To this end, let $0 < \eta \le 1$ be an arbitrary error quantifier.

We use the projection of $u_k|_{W_r}$ on right-going waves; for some function $\Phi = \sum_{j=1}^N \lambda_j^+ \phi_j^+ \in Y^+$ (i.e., for some complex coefficients λ_j^+) there holds $\Pi_+^r\left((u_k - \Phi)|_{W_r}\right) = 0$. We subtract this right-going wave and define $v_k := u_k - \Phi|_{\Omega_{R_k+1}}$. Then v_k satisfies $\Pi_+^r\left(v_k|_{W_r}\right) = 0$ and $\Pi_-^r\left(v_k|_{W_r}\right) = \Pi_-^r\left((u_k - \Phi)|_{W_r}\right) = \Pi_-^r\left(u_k|_{W_r}\right)$. Since $r_m \to \infty$ for $m \to \infty$ and f_e, f_h have bounded support, the flux equality of Lemma 4.3 assures for large m:

$$\mathrm{Im}\, \mathcal{Q}\left(v_k|_{W_r}\right) = \mathrm{Im}\, \mathcal{Q}\left(v_k|_{W_{R_k}}\right) \ge 0\,, \tag{4.6}$$

where we have used $S_{R_k}(v_k|_{W_{R_k}}) \in Y_{W_0}^+$ to conclude the non-negativity of the flux in W_{R_k}.

We now study the function $v_k|_{W_r}$ and in particular the flux $\mathrm{Im}\, \mathcal{Q}(v_k|_{W_r})$. The boundedness of the sequence

$$\sup\left\{\|u_k|_{W_l}\|_{L^2(W_l)} \;\middle|\; l \in \mathbb{N},\ l \le R_k\right\}$$

allows to use the inequality of Lemma 4.2. For sufficiently large k holds

$$\|S_r\left(v_k|_{W_r}\right) - \Pi_-^r\left(v_k|_{W_r}\right)\|_{H(\mathrm{curl},W_0)} = \|S_r\left(v_k|_{W_r}\right) - \Pi_Y^r\left(v_k|_{W_r}\right)\|_{H(\mathrm{curl},W_0)} \le \eta\,,$$

hence $v_k|_{W_r}$ is close to a left-going wave.

This information is sufficient to conclude that, actually, $v_k|_{W_r}$ is small: Inequality (4.6) implies for $v_k|_{W_r}$ a right-going energy flux, but $v_k|_{W_r}$ is essentially left-going wave. We formalize the argument with the subsequent calculation. In the first line, we use (4.6) and re-write the argument in different forms. In the second line, we use the integral definition of $\mathcal{Q}$ in (2.1), the lower bound $\lambda > 0$ of μ, and the η-smallness of the term in squared brackets. In the third inequality we use the factor $\gamma_- > 0$ of Corollary 2.11, the constant C_1 of (2.17) and $\eta \leq 1$.

$$\begin{aligned}
0 &\leq \operatorname{Im} \mathcal{Q}(v_k|_{W_r}) = \operatorname{Im} \mathcal{Q}\big(S_r(v_k|_{W_r})\big) = \operatorname{Im} \mathcal{Q}\Big(\Pi^r_-(v_k|_{W_r}) + [S_r(v_k|_{W_r}) - \Pi^r_-(v_k|_{W_r})]\Big)\\
&\leq \operatorname{Im} \mathcal{Q}\big(\Pi^r_-(v_k|_{W_r})\big) + \frac{1}{\lambda}\left(\eta\|\Pi^r_-(v_k|_{W_r})\|_{H(\mathrm{curl},W_0)} + \eta^2\right)\\
&\leq -\gamma_-\|\Pi^r_-(v_k|_{W_r})\|^2_{L^2(W_0)} + \frac{\eta}{\lambda}\left(C_1\|\Pi^r_-(v_k|_{W_r})\|_{L^2(W_0)} + 1\right)\\
&\leq -\gamma_-\|\Pi^r_-(v_k|_{W_r})\|^2_{L^2(W_0)} + \frac{C}{\lambda}\eta\,.
\end{aligned} \tag{4.7}$$

The constant C depends on the bounds for $S_r(u_k)$ and Φ on W_0. We have obtained the smallness

$$\|\Pi^r_-(u_k|_{W_r})\|^2_{L^2(W_0)} = \|\Pi^r_-(v_k|_{W_r})\|^2_{L^2(W_0)} \leq \frac{C}{\lambda\gamma_-}\eta\,.$$

Since $0 < \eta \leq 1$ was arbitrary, this is the convergence $\Pi^r_-(u_k|_{W_r}) \to 0$ in $L^2(W_0)$ for $k \to \infty$. We obtain the smallness of the left-hand side of (4.5) for large m and therefore the radiation condition for u. □

4.3 Boundedness of the solution sequence

The aim of this section is to prove our main result, Theorem 1.2. We continue to assume that the assumptions of that theorem hold, that $\mathbb{N} \ni R_k \to \infty$ is a sequence, that $u_k\colon \Omega_{R_k+1} \to \mathbb{C}^3$ is a corresponding sequence of solutions to the truncated problems of Definition 3.3. The right-hand sides are $f_h \in L^2(\Omega_+)$ and $f_e \in H(\mathrm{div}, \Omega_+)$, both supported in Ω_M for some $M > 0$ and there holds $R_k > M$ for all k.

Proof of Theorem 1.2. In this section, we conclude the proof of the theorem under the assumption that solutions $(u_k)_k$ of the truncated problems exist; this assumption is verified in Section 4.4.

We consider the sequence of norms,

$$N_k := \sup\left\{\|u_k\|_{L^2(W_\rho)} \,\middle|\, \rho \in \mathbb{N},\ \rho \leq R_k\right\}\,. \tag{4.8}$$

In the case that N_k is bounded along a subsequence, the corresponding sequence u_k has a further subsequence along which there holds: u_k has locally a strong limit u which satisfies the original problem, see Lemma 4.1. This limit also satisfies the radiation condition, see Proposition 4.4. This shows the existence claim of Theorem 1.2.

With this central observation, it remains to analyze the case $N_k \to \infty$. We will actually see that this case cannot occur. Our approach is to normalize the solution. For the rest of this section we consider

$$N_k \to \infty\,, \qquad v_k := N_k^{-1} u_k\,. \tag{4.9}$$

The normalized sequence consists of solutions to the scaled right-hand sides $N_k^{-1} f_e$ and $N_k^{-1} f_h$, which converge to 0. Lemma 4.1 provides a subsequence and a limit v with $v_k \to v$

locally, and the limit v solves the homogeneous limit equation: For every $\Phi \in H_0(\mathrm{curl}, \Omega_+)$ with bounded support holds

$$\int_{\Omega_+} \left(\frac{1}{\mu} \operatorname{curl} v \cdot \operatorname{curl} \bar{\Phi} - \omega^2 \varepsilon \, v \cdot \bar{\Phi} \right) = 0\,. \tag{4.10}$$

Furthermore, the radiation condition is satisfied. By our assumption in Theorem 1.2, this problem has only the trivial solution, and we therefore obtain $v = 0$. This fact will allow us to find a contradiction: The sequence $(v_k)_k$ was normalized, and the limit is trivial. The remaining argument is still non-trivial, since v is only locally the limit of $(v_k)_k$.

Because of the normalization, we find a position $\rho = \rho(k) \in \mathbb{N}$ with the property $\|S_\rho(v_k|_{W_\rho})\|_{L^2(W_0)} = 1$. Regarding the sequence $\rho(k)$, we will distinguish three cases, all of them will lead to a contradiction.

In the analysis of the three cases, another variant of a flux inequality will turn out to be useful.

Lemma 4.5 (Flux inequality "looking left")**.** *For a constant $C > 0$, for arbitrary $\rho \leq R_k$, there holds*

$$\left|\operatorname{Im} \mathcal{Q}(v_k|_{W_\rho})\right| \leq C N_k^{-1}\,. \tag{4.11}$$

Proof. We distinguish two cases. For $\rho < R_k$ we use $\Psi := v_k \vartheta_R$ as a test-function in the equation for v_k. For $\rho = R_k$ the solution v_k itself is used as a test-funktion. In both cases the test-function is admissible and one finds

$$\begin{aligned} \int_{\Omega_{R_k}} \left[\frac{1}{\mu} |\operatorname{curl} v_k|^2 - \omega^2 \varepsilon |v_k|^2 \right] \vartheta_\rho - Q^\circ(v_k|_{W_\rho}, v_k|_{W_\rho}) \\ = N_k^{-1} \int_{\Omega_{R_k}} \frac{1}{\mu} f_h \cdot \operatorname{curl}(\overline{v_k} \vartheta_\rho) + \mathrm{i}\omega f_e \cdot \overline{v_k} \vartheta_\rho. \end{aligned} \tag{4.12}$$

Since the supports of f_e, f_h are contained in the bounded set Ω_M and since v_k and $\operatorname{curl} v_k$ are locally bounded, the right hand side of (4.12) is of small,

$$\left| N_k^{-1} \int_{\Omega_{R_k}} \frac{1}{\mu} f_h \cdot \operatorname{curl}(\overline{v_k} \vartheta_\rho) + \mathrm{i}\omega f_e \cdot \overline{v_k} \vartheta_\rho \right| \leq C N_k^{-1}.$$

Exploiting this smallness and taking the imaginary part in (4.12) we conclude the claimed result. □

We can now turn to the analysis of the three cases.

Case 1: Large values near left boundary. We consider the case that, along a subsequence $k \to \infty$, the sequence $\rho(k)$ is bounded. In this case, we find a number $\rho_0 \in \mathbb{N}$ and a further subsequence with $\rho(k) = \rho_0$ along the subsequence. The local convergence yields $v_k \to v = 0$ in $L^2(W_{\rho_0})$. This is in contradiction with the choice of $\rho(k)$. We conclude that Case 1 cannot occur.

Case 2: Large values near right boundary. We consider the case that, along a subsequence $k \to \infty$, the sequence $R_k - \rho(k)$ is bounded. Choosing a further subsequence and an appropriate number $D_0 \in \mathbb{N}$, we can assume that $R_k - \rho(k) = D_0$ for all k along a further subsequence. We consider the shifted version of the sequence by defining $w_k(\cdot) := v_k(\cdot + R_k e_1)$. Local boundedness of v_k implies the local boundedness of w_k and allows to select a subsequnce such that, for a limit function $w \in L^2_{\text{loc}}(\Omega_{-\infty,1})$, there holds $w_k \rightharpoonup w$ in $L^2(\Omega_{-r_0,1})$ for every $r_0 > 0$. The function w satisfies $\|w\|_{sL} \le 1$. As in Lemma 4.1, the convergence is actually strong in $L^2(\Omega_{-r_0,1})$ and the limit function w is a solution of the homogeneous problem. The strong convergence and the choice of $\rho(k)$ assures $\|w|_{W_{-D_0}}\|_{L^2(W_{-D_0})} = 1$. In particular, w is not vanishing.

The function w satisfies also the radiation condition on the left. This follows as in Proposition 4.4; indeed, Lemma 4.5 together with Lemma 4.3 imply that the right-going part of w in every test-volume W_{-r} must be small for large $r > 0$. We give more details in the appendix, see Lemma A.1.

The result is that the limit is an edge-resonance solution: A homogeneous solution on Ω_- satisfying the radiation condition on the left. By assumption, there is only the trivial solution to this problem. Since we have seen that w is non-trivial, we have found the desired contradiction. Case 2 cannot occur.

Case 3: Large values in a sequence of interior points. In this remaining case holds $\rho(k) \to \infty$ and $R_k - \rho(k) \to \infty$. The principle idea is: In every segment W_ρ with large ρ holds:

- $v_k|_{W_\rho}$ looks like a sum of a right-going and a left-going wave by (4.3)
- $v_k|_{W_\rho}$ looks like a right-going wave by (4.4)
- This right-going wave is small by (4.11): $v_k|_{W_\rho} \approx 0$

This will lead to a contradiction when ρ is chosen such that $v_k|_{W_\rho}$ has norm 1.

We choose the error quantifier $\eta = 1/2$. For some index $k_0 > 0$, (4.3) provides, for every $k \ge k_0$:

$$\left\|S_\rho(v_k|_{W_\rho}) - \Pi^\rho_Y(v_k|_{W_\rho})\right\|_{H(\text{curl},W_0)} \le \eta\,. \tag{4.13}$$

As in Lemma 4.3, we subtract from v_k a right-going wave. We consider $w_k := v_k - \phi_k|_{\Omega_{R+1}}$ such that $\Pi^\rho_+(w_k|_{W_\rho}) = 0$. Lemma 4.3 provides the flux equality

$$\operatorname{Im} \mathcal{Q}(w_k|_{W_\rho}) = \operatorname{Im} \mathcal{Q}(w_k|_{W_R}) \ge 0\,. \tag{4.14}$$

With this information, we can perform the same calculation as in (4.7), with r replaced by ρ and v_k replaced by w_k. The result is

$$0 \le \operatorname{Im} \mathcal{Q}(w_k|_{W_\rho}) \le -\gamma_- \|\Pi^\rho_-(w_k|_{W_\rho})\|^2_{L^2(W_0)} + \frac{C}{\lambda}\eta\,.$$

This shows smallness of $\Pi^\rho_-(w_k|_{W_\rho})$. We obtain the smallness of $\Pi^\rho_-(v_k|_{W_\rho}) = \Pi^\rho_-(w_k|_{W_\rho})$. By (4.13), $v_k|_{W_\rho}$ is approximately a right-going wave.

Then, (4.11) implies that this right-going wave has a small norm. Altogether, we find that $v_k|_{W_\rho}$ has a small norm. This is in contradiction to the choice of ρ. Case 3 cannot occur. □

4.4 Uniqueness of solutions to the truncated problems

We still have to verify that solutions $(u_k)_k$ of the truncated problems exist. By Proposition 3.9, this follows from uniqueness for these problems.

We will actually not show that, for every $R > 0$, the truncated problem has a unique solution. But we can show that this is true for every sufficiently large $R > 0$.

Lemma 4.6 (Uniqueness for truncated problems)**.** *There exists a lower bound R_0 such that, for every $R \geq R_0$, the truncated problem has a unique solution.*

Proof. By Proposition 3.9, we only have to show uniqueness. For a contradiction argument we assume that there is a sequence $R_k \to \infty$ such that the corresponding truncated problems with vanishing right-hand sides have non-trivial solutions u_k. Without loss of generality, we assume that the sequence $(u_k)_k$ is normalized, the number N_k of (4.8) is 1 for every k.

The sequence $v_k = u_k$ has all the properties of the sequence v_k of the last subsection for $f_e = 0$ and $f_h = 0$. The proof of the last subsection shows that all the three cases lead to a contradiction. □

A Radiation condition in Ω_-

Lemma A.1. *We study the limit function w that appears in the proof of Theorem 1.2, Case 2. The function w satisfies the following radiation condition to the left:*

$$\|\Pi_+^{-r}(w|_{W_{-r}})\|_{L^2(W_0)} \to 0 \qquad \text{for } r \to \infty\,. \tag{A.1}$$

Proof. We fix a sequence $r \to \infty$ and show the radiation condition along this sequence.

We recall that w is the local limit of the functions w_k; the latter are obtained as shifts of the solutions u_k to truncated problems with size R_k. We argue as in the proof of Proposition 4.4. In a first step we choose a sequence of indices $k_r \to \infty$ as $r \to \infty$ such that w_{k_r} is close to w on W_{-r}. It is actually sufficient to achieve that the difference

$$\left\|\Pi_+^{-r}(w|_{W_{-r}}) - \Pi_+^{-r}(w_{k_r}|_{W_{-r}})\right\|_{L^2(W_0)}$$

is small. We now subtract the right-going part of $w_{k_r}|_{W_{-r}}$ and exploit that, by Lemma 4.2, the result is essentially a left-going wave. Since there are no energy sources in the domain of relevance, the flux equality of Lemma 4.3 provides the smallness of $\Pi_-^{-r}(w_{k_r}|_{W_{-r}})$ (just as in the proof of Proposition 4.4). We emphasize that the smallness of $\Pi_+^{-r}(w_{k_r}|_{W_{-r}})$ can not be concluded in the same way, since the flux equality Lemma 4.3 allows only subtracting right-going waves.

To prove that $\Pi_+^{-r}(w_{k_r}|_{W_{-r}}) \to 0$, one again uses Lemma 4.2 which shows that (the shift) of $w_{k_r}|_{W_{-r}}$ is close to its projection onto propagating waves,

$$\|S_{-r}(w_{k_r}|_{W_{-r}}) - \Pi_Y^{-r}(w_{k_r}|_{W_{-r}})\|_{H(\mathrm{curl},W_0)} \to 0 \quad \text{as } r \to \infty\,. \tag{A.2}$$

By the flux inequality Lemma 4.5, the flux of $w_{k_r}|_{W_{-r}}$ is of order $N_{k_r}^{-1}$ and thus vanishes in the limit $r \to \infty$. By (A.2) the same is true for the flux of $\Pi_Y^{-r}(w_{k_r}|_{W_{-r}})$. Since $\Pi_Y^{-r} = \Pi_+^{-r} + \Pi_-^{-r}$ and the left-going part $\Pi_-^{-r}(w_{k_r}|_{W_{-r}})$ has already been shown to be small, we conclude that also the right-going part $\Pi_+^{-r}(w_{k_r}|_{W_{-r}})$ vanishes in the limit. This completes the proof. □

B Edge resonances

We have assumed that, for two 1-periodic (in direction e_1) functions $\varepsilon_{\mathrm{per}}, \mu_{\mathrm{per}} \in L^\infty(\Omega, \mathbb{R})$, there holds, for some number $R_0 > 0$, $\varepsilon(x) = \varepsilon_{\mathrm{per}}(x)$ and $\mu(x) = \mu_{\mathrm{per}}(x)$ for every $x \in \Omega_+$

with $x_1 > R_0$. One of the assumptions in Theorem 1.2 is that, with the coefficients $\varepsilon = \varepsilon_{\text{per}}$ and $\mu = \mu_{\text{per}}$, the homogeneous problem on Ω_- has only the trivial solution. In other words: The space of edge resonances Z of the subsequent definition is given by $Z = \{0\}$.

Definition B.1 (Edge resonances)**.** We define the space $Z \subset H_{\text{loc},0}(\text{curl}, \Omega_-)$ as the subspace that consists of functions u that satisfy

$$\int_{\Omega_-} \left(\frac{1}{\mu} \operatorname{curl} u \cdot \operatorname{curl} \bar{\Phi} - \omega^2 \varepsilon\, u \cdot \bar{\Phi} \right) = 0 \tag{B.1}$$

for every $\Phi \in H_0(\text{curl}, \Omega_-)$ with bounded support, and the radiation condition (A.1).

We expect the following. Every element of Z is (close to) a linear combination of left-going waves at the far left. On the other hand, no energy is introduced in the equation (vanishing functions f_e and f_h), so also the left-going part must be small. We therefore expect that elements $u \in Z$ are decaying for $x_1 \to -\infty$.

With this observation in mind, we regard elements of Z as eigenvectors of an eigenvalue problem. A Fredholm problem can imply that the space Z is finite dimensional.

Under the assumption that Z is finite dimensional, a possible strategy for an existence proof is to consider, as in the proof of this contribution, the solutions u_k of the truncated problems. In the case $Z \neq \{0\}$, our proof fails to work in the derivation of boundedness of the solution sequence, Case 2. This step of the proof could work if we considered, instead of u_k, a sequence $U_k = u_k - z_k$, where z_k are appropriately chosen elements of Z, possibly multiplied with a cut-off function. Unfortunately, we did not manage to make this idea work, the reason was a missing control on the norms of z_k.

C Caccioppoli estimate of the curl

The classical Caccioppoli inequality allows to bound, for solutions of an elliptic equation, the L^2-norm of the gradient of a solution u on a smaller domain by the L^2-norm of the solution on a larger domain. An analogous estimate holds for the (truncated) Maxwell system.

Lemma C.1 (Bounds on the curl for solutions of the truncated problem)**.** *Let the setting be that of Theorem 1.2. Let $r_0, R \in \mathbb{N}$ with $r_0 < R$ be given. Let $u_R \colon \Omega_{R+1} \to \mathbb{C}^3$ be a solution to the truncated problem of Definition 3.3. Then there exists a constant $C > 0$, independent of R, such that*

$$\|\operatorname{curl}(u_R)\|^2_{L^2(\Omega_{r_0})} \le C \left(\|u_R\|^2_{L^2(\Omega_{r_0+1})} + \|f_e\|^2_{L^2(\Omega_+)} + \|f_h\|^2_{L^2(\Omega_+)} \right). \tag{C.1}$$

Proof. Let $\theta \in C^{0,1}(\Omega_+, \mathbb{R})$ be a cut-off function satisfying $\theta(x) \in [0,1]$ for all $x \in \Omega_+$, $\theta(x) = 1$ for $0 \le x_1 \le r_0$ and $\theta(x) = 0$ for $x_1 \ge r_0 + 1$. Because of $\theta \equiv 1$ on Ω_{r_0}, there holds

$$\|\operatorname{curl}(u_R)\|^2_{L^2(\Omega_{r_0})} \le \|\theta \operatorname{curl}(u_R)\|^2_{L^2(\Omega_{r_0+1})}.$$

The function $\Psi := u_R\theta^2$ is an element of V_R and can therefore be used as a test-function in the truncated problem: $\beta_R(u_R,\Psi) = F_R(\Psi)$. The left-hand side of this equation reads

$$\begin{aligned}\beta_R(u_R,\Psi) &= \int_{\Omega_{R+1}} \left[\frac{1}{\mu}\operatorname{curl}(u_R)\cdot\operatorname{curl}(\bar u_R\theta^2) - \omega^2\varepsilon\, u_R\cdot\bar u_R\theta^2\right]\\ &= \int_{\Omega_{R+1}} \left[\frac{1}{\mu}|\theta\operatorname{curl}(u_R)|^2 + \frac{2}{\mu}\operatorname{curl}(u_R)\cdot(\theta\nabla\theta\times\bar u_R) - \omega^2\varepsilon\,|u_R\theta|^2\right].\end{aligned}$$

The right-hand side reads

$$\begin{aligned}F_R(\Psi) &= \int_{\Omega_R}\left[\frac{1}{\mu}f_h\cdot\operatorname{curl}(\bar u_R\theta^2) + \mathrm{i}\omega f_e\cdot\bar u_R\theta^2\right]\\ &= \int_{\Omega_R}\left[\frac{1}{\mu}f_h\cdot\big(2\theta\nabla\theta\times\bar u_R + \theta^2\operatorname{curl}(\bar u_R)\big) + \mathrm{i}\omega f_e\cdot\bar u_R\theta^2\right].\end{aligned}$$

Comparing left-hand side and right-hand side allows to estimate the term containing the norm of the curl. We obtain

$$\begin{aligned}\frac{1}{\Lambda}\|\theta\operatorname{curl}(u_R)\|^2_{L^2(\Omega_{r_0+1})} &\le \int_{\Omega_{R+1}}\frac{1}{\mu}|\theta\operatorname{curl}(u_R)|^2\\ = \int_{\Omega_{R+1}}&\left[\omega^2\varepsilon\,|u_R\theta|^2 - \frac{2}{\mu}\operatorname{curl}(u_R)\cdot(\theta\nabla\theta\times\bar u_R)\right]\\ &+ \int_{\Omega_R}\left[\frac{1}{\mu}f_h\cdot\big(2\theta\nabla\theta\times\bar u_R + \theta^2\operatorname{curl}(\bar u_R)\big) + \mathrm{i}\omega f_e\cdot\bar u_R\theta^2\right] =:\ I\ +\ II\,.\end{aligned}$$

Since θ is supported on Ω_{r_0+1}, using $\theta(x)\in[0,1]$, the triangle inequality and Young's inequality, we find

$$\begin{aligned}I &\le \omega^2\Lambda\|u_R\|^2_{L^2(\Omega_{r_0+1})} + \frac{1}{3\Lambda}\|\theta\operatorname{curl}(u_R)\|^2_{L^2(\Omega_{r_0+1})} + \frac{3\Lambda}{\lambda^2}\|\nabla\theta\|^2_{L^\infty(W_{r_0})}\|u_R\|^2_{L^2(W_{r_0})}\\ &\le \left(\omega^2\Lambda + \frac{3\Lambda}{\lambda^2}\|\nabla\theta\|^2_{L^\infty(W_{r_0})}\right)\|u_R\|^2_{L^2(\Omega_{r_0+1})} + \frac{1}{3\Lambda}\|\theta\operatorname{curl}(u_R)\|_{L^2(\Omega_{r_0+1})}.\end{aligned}$$

For the second summand, we obtain

$$\begin{aligned}II &\le \frac{1}{\lambda}\left(\|f_h\|^2_{L^2(\Omega_+)} + \|\nabla\theta\|^2_{L^\infty(W_{r_0})}\|u_R\|^2_{L^2(\Omega_{r_0+1})}\right) + \frac{3\Lambda}{\lambda^2}\|f_h\|^2_{L^2(\Omega_+)}\\ &\quad + \frac{1}{3\Lambda}\|\theta\operatorname{curl}(u_R)\|^2_{L^2(\Omega_{r_0+1})} + \frac{\omega}{2}\left(\|f_e\|^2_{L^2(\Omega_+)} + \|u_R\|^2_{L^2(\Omega_{r_0+1})}\right).\end{aligned}$$

Combining the above estimates we conclude that

$$\begin{aligned}\frac{1}{3\Lambda}\|\theta\operatorname{curl}(u_R)\|^2_{L^2(\Omega_{r_0+1})} &\le \left(\omega^2\Lambda + \frac{\omega}{2} + \frac{3\Lambda+\lambda}{\lambda^2}\|\nabla\theta\|^2_{L^\infty(W_{r_0})}\right)\|u_R\|^2_{L^2(\Omega_{r_0+1})}\\ &\quad + \left(\frac{3\Lambda+\lambda}{\lambda^2} + \frac{\omega}{2}\right)\left(\|f_e\|^2_{L^2(\Omega_+)} + \|f_h\|^2_{L^2(\Omega_+)}\right).\end{aligned}$$

Multiplying by 3Λ yields the claimed estimate with a constant C independent of R. □

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