Effective sound absorbing boundary conditions for complex geometries

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Abstract: We analyze a system of equations that describes the propagation of sound waves. We are interested in complex constructions along a part of the boundary of the domain, for example constructions with small chambers that are connected to the domain. We also allow that different flow equations are used in the chambers, e.g., modelling a damping material. In addition to the complex geometry, we assume that the viscosity vanishes in the limit. The limiting system is given by wave equations, we derive these equations and determine the effective boundary conditions. The effective boundary conditions replace the large number of small chambers. We provide examples for sound absorbing constructions and their Dirichlet-to-Neumann boundary conditions.

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1. Introduction

To improve the acoustic properties of a room, one often uses sound absorbing constructions, see Figure 1 for an example. Essentially, the rigid (sound-hard) wall is replaced by a construction with many holes, these holes lead to small chambers, in the chambers some acoustically relevant process takes place. The effect of the whole construction is that sound waves are not perfectly reflected, but some part of the energy is absorbed. The motivation for the article at hand is to understand these constructions mathematically.

We use the compressible Stokes equations to describe air — the medium in which sound propagation takes place. Even though this is a mathematical work, we provide for all physical quantities the order of magnitude. This allows to compare different terms that describe different effects. This analysis allows to check that the chosen system of equations is, in the setting of interest, a good replacement for the full compressible Navier-Stokes equations.

Our analysis also shows that, using only one equation in the whole complex domain (the underlying domain plus the sound-active small chambers) cannot explain any sound absorbing effect. We therefore allow that, in the single chamber, other flow equations are used. The most useful example is a Darcy law that models a sound absorbing material. For this example, we derive the sound absorbing effective boundary condition. Nevertheless, we keep the approach general, any flow model

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can be used in the chamber (or even different flow models in different parts of the chamber).

Mathematically, we describe the problem as follows. The domain of interest is a subset $\Omega \subset \mathbb{R}^d$, where $d \geq 2$ is the dimension. The physics in Ω are described with the following unknowns: density ρ, velocity v, pressure p, all depending on $x \in \Omega$ and a time variable $t \in \mathbb{R}$. We use the compressible Stokes equations:

$$\bar{\rho}\, \partial_t v = -\nabla p + \mu_\varepsilon \Delta v\,, \tag{1.1}$$

$$\partial_t p + \bar{p}\, \nabla \cdot v = 0\,. \tag{1.2}$$

The numbers $\bar{\rho}$, $\bar{p}$ and μ_ε are given physical constants, compare Table 1. The equations are to be solved in Ω or, more precisely, for some $T > 0$, on $\Omega_T := \Omega \times (0, T)$.

Along some part $\Gamma \subset \partial\Omega$ of the boundary (with exterior normal vector ν), there is some construction that, ideally, leads to sound absorption. This construction could be, e.g., small chambers with typical diameter ε that are connected to Ω. We choose here an abstract setting and describe the effect of the construction with a Dirichlet-to-Neumann map N^ε, it maps pressure distributions $p|_\Gamma$ to normal velocity distributions $\nu \cdot v|_\Gamma$. We impose

$$\nu \cdot v|_\Gamma = N^\varepsilon(p|_\Gamma)\,. \tag{1.3}$$

The system is closed by imposing, for some smooth function $g : \partial\Omega \times [0, T] \to \mathbb{R}$ with $g = 0$ on Γ, the boundary condition

$$\nu \cdot v = g \qquad \text{on } \partial\Omega \setminus \Gamma\,. \tag{1.4}$$

Our aim is to analyze, for relevant choices of the maps N^ε, the limiting equations for the above system (1.1)–(1.4). Below, we give more comments on the choice of the physical equations and provide relevant choices of N^ε.

The main mathematical result is an averaging result. We assume that N^ε is given by local operators N^ε_Y through a procedure of rescaling and glueing. Under a convergence assumption on the operators N^ε_Y, we derive the convergence of N^ε in appropriate function spaces.

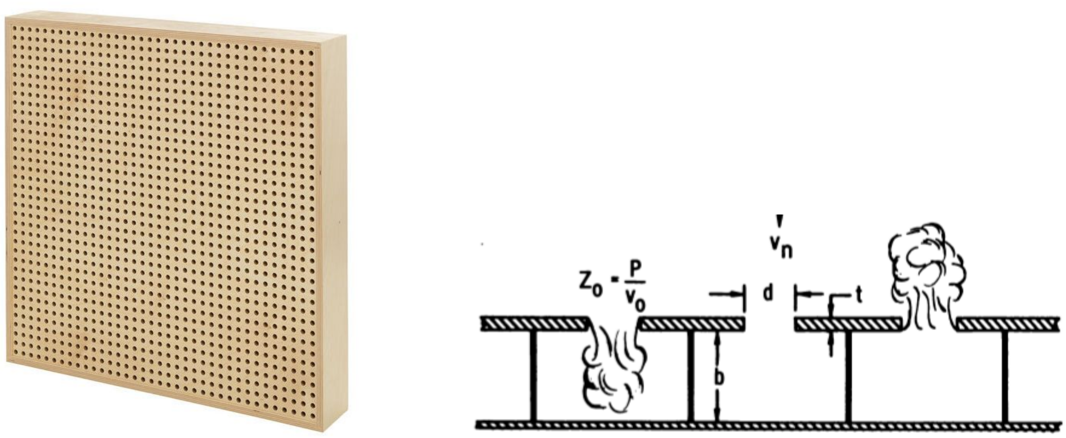

FIGURE 1. Left: A commercial broad-band absorber element. Right: An illustration of effects leading to a nontrivial impedance condition, taken from [7].

Our results allow to analyze sound absorbing structures as follows:

(A) Chambers are specified by choosing a chamber geometry and by choosing flow equations in the chamber.

(B) The chamber data define the operator N_Y^ε. By averaging, one obtains the operator N_*^ε. One calculates the limit N_0 of N_*^ε and, if necessary, also the limit N_1 of $\varepsilon^{-1}(N_*^\varepsilon - N_0)$.

(C) Theorem 1.1 and Theorem 1.2 yield the effective system. A possible sound absorption effect of the complex geometry is typically expressed by N_1.

We illustrate the procedure with the two most relevant settings.

Stokes chambers: (A) We consider an arbitrary geometry of the chambers and consider only compressible Stokes equations in the chamber. We use sound-hard boundary conditions and assume $\varepsilon^{-1}\mu_\varepsilon \to 0$. (B) The limit operators are $N_0 = 0$ and $N_1(p_*) = \beta\,\partial_t p_*$, see (2.6). (C) The effective equations are given by (1.15)–(1.20). The limit system for p_0 is the wave equation with a sound-hard boundary condition (a limit system as if there were no chambers). The limit system for p_1 is the wave equation with the boundary condition $\partial_\nu p_1 = -\bar\rho\alpha\beta\,\partial_t^2 p_0$. By energy conservation (see Section 3), this limit system does not induce sound absorption.

Darcy chambers: (A) For an arbitrary geometry of the chambers we consider a Darcy law in the chamber with flow resistence M_ε, see (2.9). We assume that there exists a nontrivial limit, $M_\varepsilon\varepsilon \to M_* > 0$. (B) The limit operators can be calculated. There holds $N_0 = 0$ and the operator $N_1(p_*)$ is given by (2.11)–(2.12). (C) Theorems 1.1 and 1.2 provide the effective equations: p_0 solves the wave equation with a sound-hard boundary condition (again: as if there were no chambers). The equation for p_1 is now different: p_1 solves the wave equation with the boundary condition $\partial_\nu p_1 = -\bar\rho\alpha\,\partial_t N_1(p_0)$ along Γ. This boundary condition induces sound absorption.

1.1. **Literature.** Our research is related to homogenization. Classical homogenization questions treat the case that ε-scaled structures are distributed in the domain, e.g., periodically. The situation is different here since the small structures are situated along a lower dimensional object, namely part of the boundary of the domain.

Domains with small structures along a manifold. An analysis of such a structure for the Helmholtz equation was performed in [3]. As in our setting, one can derive effective equations that provide a zero-order approximation p_0 and a first order approximation p_1. Another aspect is analyzed in [16], where asymptotic expansions are presented for a Helmholtz problem with a perforation along an interface; the focus here is the approximation near the end-points of the perforation. In the analysis of [15], a more abstract approach is introduced; this approach needs fewer orders in the expansions, allows to study arbitrary dimensions and can cover more general geometries.

Another important contribution of [3] is that effective boundary conditions are investigated, we discuss this in some detail in the comments to Remark 1.3. In [8], the time dependent Maxwell equations were treated with a similar goal. Here, one is interested in effective boundary conditions in the situation that a perfect conductor is coated with a thin layer of dielectric material, see also [4]. The effective description of a thin layer of elastic material is treated in [1].

Homogenization of boundary conditions. Oftentimes, it is interesting to have small structures along the boundary, one then asks for an effective boundary condition. A work of this type is [11]; part of the boundary is covered by a chess board structure of period ε, a Dirichlet condition is imposed on the black and a Neumann condition is imposed on the white squares. The authors ask for effective boundary conditions, much like the "strange term coming from nowhere".

In some sense, our main mathematical result in Proposition 4.7 is similar to such a problem: We ask for the effective boundary condition (or, equivalently: Dirichlet-to-Neumann map), when many Dirichlet-to-Neumann maps on small subsets are combined.

Resonance. A special feature of the small structure can be resonance. In the present work, resonance is not the main subject. Nevertheless, with an appropriate choice of the response map N_Y^ε, the model can have the features of a resonant system.

A mathematical verification that small objects (of size $\varepsilon > 0$) can show a resonant behavior in an equation that has a fixed frequency ω (and, hence, a fixed typical wave-length λ), was provided in [13]. The important feature of the geometry is a three-scale structure. One chooses channels that are thin in comparison to the chambers. Bulk homogenization of such resonators was analyzed in [9], where the effective acoustic properties of the corresponding meta-material were derived. In [5], comparable geometries were distributed along the boundary and a limit model was derived for such a setting that is closely related to the sound absorber geometry. We note that some of these resonances of small structures are addressed in the overview article [14].

Also in [2], the boundary consists of many small resonators. For the Laplace operator in this complex geometry, not only effective boundary conditions, but also an analysis of the spectrum is presented. The acoustic problem is studied in [17] for a thin layer of a complex microstructure; micro-channels of the size of the wave-length in the thin layer lead to Fabry-Pérot resonances and to interesting transmission conditions.

Some more applied and less mathematical literature. A very early study of Helmholtz resonator arrays is presented in [7], we show an illustration of [7] in Figure 1. Our analysis actually suggests that nonlinear terms are not important for standard sound absorbers. An historically influential article is [10], a more modern contribution is [12]; in such works, the authors are interested in effective acoustic impedances. To make the connection to these applications, we introduced and discussed the effective condition of (1.22), we relate this condition to an impedance condition in the time-harmonic case in Remark 2.2. Books like [6] present many sound absorbing structures, we see the importance of the research for applications.

1.2. **On the choice of the model equations.** Without doubt, the compressible Navier-Stokes equations are a good model for air. The fundamental unknowns are density, pressure, and velocity. In the situation of sound waves, linearizations are justified, we will check this below. We therefore introduce density and pressure at rest, denoted as $\bar\rho$ and $\bar p$, and variations of these quantities due to a sound wave, denoted as ρ and p. Since we assume that velocity is only induced by the sound wave, we use $\bar v = 0$.

For air at room temperature and room pressure, a linear gas law is adequate, this leads to $\rho/\bar\rho = p/\bar p$. Our methods cover also linearizations of general gas laws $\bar p + p = P(\bar\rho + \rho)$ for some function $P : (0,\infty) \to \mathbb{R}$, the linearization reads $p = P'(\bar\rho)\rho =: c^2\,\rho$. In fact, also in our setting, we use $c^2 = \bar p/\bar\rho$ and $p = c^2\,\rho$.

We linearize also the momentum equation and the equation for mass conservation in the values $\bar\rho$, $\bar p$, and $\bar v = 0$. This leads to the system (1.1)–(1.2). The coefficient $\mu = \mu_\varepsilon$ denotes the dynamic viscosity, a material constant for air. The subscript ε is introduced for later use. Table 1 lists orders and units for all quantities.

Without further mentioning, we always assume that the initial conditions are given by $v = 0$ and $p = 0$. In our model, sound waves are generated by the nontrivial Neumann condition (1.4), which is imposed on $(\partial\Omega \setminus \Gamma) \times (0, T)$. Additionally, we always impose that tangential components of v vanish along all boundaries. This condition is a natural condition for a system of Stokes type such as (1.1). Later on, we will consider the limit $\mu_\varepsilon \to 0$ of a vanishing viscosity. In the limit equation, we loose the condition of a vanishing tangential velocity.

A system of channels and chambers, possibly containing sound damping material, can be encoded in a Dirichlet-to-Neumann map along a part Γ of the boundary $\partial\Omega$. We use an abstract operator N^ε and write the condition as in (1.3), which must be understood as an equality on $\Gamma \times (0, T)$.

In the linearization, we neglect the inertia term $\bar\rho\,(v \cdot \nabla)v$ and other terms such as $\nabla\rho \cdot v$. We will check below that these terms are indeed small compared to the terms that are linear in the triple of unknowns (ρ, p, v).

Our analysis can be used also for other flow equations in Ω. Thinking in the direction of more complete models, one could, e.g., treat nonlinear laws. On the other hand, thinking of simpler models, it is also possible to start the analysis from a wave equation in Ω.

The methods can also be used to study the time-harmonic case in which all quantities have the time dependence $e^{i\omega t}$. In such a setting, one would probably start from the Helmholtz equation in Ω.

1.3. **Geometries with active interfaces and main results.** We assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$ with

$$\Omega \subset \mathbb{R}^{d-1} \times (-\infty, 0)\,, \quad \Gamma := (0,1)^{d-1} \times \{0\} \subset \partial\Omega\,. \tag{1.5}$$

In particular, the $(d-1)$-dimensional set Γ is flat and $\nu = e_d$ on Γ. We assume that, for some height $h > 0$, the domain $\Omega_h := (0,1)^{d-1} \times (-h, 0)$ is contained in Ω.

The boundary part Γ contains active interfaces. When these are given by small devices, we denote by $\varepsilon > 0$ the typical distance of these devices. In applications, they are the interfaces to channels which, in turn, connect Ω to chambers. The chambers might contain damping material, possibly also an active elements such as a loudspeaker. For a Lipschitz domain $\Gamma_Y \subset (0,1)^{d-1}$ (compactly contained), and the index set $K_\varepsilon := \{k \in \mathbb{Z}^2 \,|\, \varepsilon(k + \Gamma_Y) \subset \Gamma\}$, we use

$$\Gamma_k^\varepsilon := \varepsilon(k + \Gamma_Y)\,, \qquad \Gamma^\varepsilon := \bigcup_{k \in K_\varepsilon} \Gamma_k^\varepsilon \subset \Gamma\ . \tag{1.6}$$

The area fraction of active interfaces is $\alpha := |\Gamma_Y|$, it is a real number with $0 < \alpha < 1$.

We assume that the interfaces are independent of each other (locality). More precisely, we assume that each interface piece Γ_k^ε is described by a Dirichlet-to-Neumann map that relates pressure distributions to velocity distributions: For every $k \in K_\varepsilon$, we are given a map

$$N_k^\varepsilon : H^1(0, T; H^{1/2}(\Gamma_k^\varepsilon)) \to L^2(0, T; H^{-1/2}(\Gamma_k^\varepsilon))\,. \tag{1.7}$$

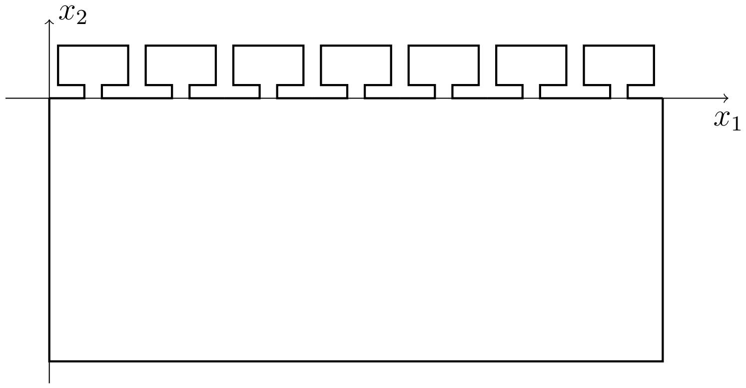

FIGURE 2. The complex domain Ω_ε in two space dimensions. It is given as the union of domain $\Omega \subset \{x|x_2 < 0\}$ and a large set of small chambers. The interfaces $\Omega_\varepsilon \cap \{x|x_2 = 0\}$ are treated as active interfaces that induce a Dirichlet-to-Neumann map.

The family of maps $(N_k^\varepsilon)_{k\in K_\varepsilon}$ defines a Dirichlet-to-Neumann map on Γ,

$$N^\varepsilon : H^1(0,T;H^{1/2}(\Gamma)) \to L^2(0,T;H^{-1/2}(\Gamma)) \tag{1.8}$$

through the following piecewise definition: A single function $p \in H^1(0,T;H^{1/2}(\Gamma))$ defines boundary data $p_k^\varepsilon \in H^1(0,T;H^{1/2}(\Gamma_k^\varepsilon))$ through restriction. The flux fields $w_k^\varepsilon = N_k^\varepsilon(p_k^\varepsilon)$ are, at first, defined on Γ_k^ε, but they can be extended trivially and glued together:

$$w^\varepsilon(x) := \begin{cases} w_k^\varepsilon(x) & \text{for } x \in \Gamma_k^\varepsilon \\ 0 & \text{for } x \in \Gamma \setminus \Gamma^\varepsilon\,. \end{cases} \tag{1.9}$$

Strictly speaking, the action of w^ε to a test-function must be prescribed, and a point-wise definition as in (1.9) is not permitted; we skip here the obvious rigorous formulation of (1.9). The resulting function $w^\varepsilon =: N^\varepsilon(p)$ is an element of $L^2(0,T;H^{-1/2}(\Gamma))$. More on mathematical aspects related to restrictions and extensions in $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ are discussed in the main part of this text.

Our second assumption is that all active boundary parts Γ_k^ε have the same response behavior: Except for their position in space, the maps N_k^ε are identical. This reflects the fact that the same chamber construction is realized behind each interface Γ_k^ε. We make the concept precise with the following assumption: There is a single map N_Y^ε

$$N_Y^\varepsilon : H^1(0,T;H^{1/2}(\Gamma_Y)) \to L^2(0,T;H^{-1/2}(\Gamma_Y))\,, \tag{1.10}$$

such that, for $p \in H^1(0,T;H^{1/2}(\Gamma_k^\varepsilon))$,

$$N_k^\varepsilon(p) = w^\varepsilon \quad \text{with} \quad w^\varepsilon(\varepsilon(k+.)) := N_Y^\varepsilon(p(\varepsilon(k+.))) \tag{1.11}$$

for all $k \in K_\varepsilon$. The map N_Y^ε is usually related to a cell-problem in the chamber. We emphasize that we did not introduce any scaling in (1.11). Typically, prescribing a pressure of order 1 can only create a response in the velocity field of order ε; we do not want to anticipate the order of the response and introduce therefore no scaling in (1.11).

Our next aim is to replace N_Y^ε by a simpler function. Variations along Γ_k^ε are not relevant (we will actually derive this fact), hence we insert only constant functions in N_Y^ε. We define the map

$$(1.12) \qquad N_*^\varepsilon : H^1(0,T;\mathbb{R}) \to L^2(0,T;\mathbb{R}) \,, \quad N_*^\varepsilon(p(.))(t) := \fint_{\Gamma_Y} N_Y^\varepsilon(p(.))(t) \,.$$

Here and below, we use the integral average, defined as $\fint_A f := |A|^{-1} \int_A f$. On the right hand side we identify, for every time instance t, the real number $p(t)$ with a constant function on Γ_Y. Strictly speaking, the last expression cannot be formulated as an integral; mathematically, the integral should be replaced by the application of the $H^{-1/2}(\Gamma_Y)$-element $N_Y^\varepsilon(p(.))(t)$ to the 1-function (the function that is identical to 1 on Γ_Y).

The reduction to functions that depend only on time makes convergences more accessible. It is often possible to find a limit map that encodes the relevant behavior of the active interface,

$$(1.13) \qquad N_*^\varepsilon \to N_0 \text{ as } \varepsilon \to 0 \,, \quad N_0 : H^1(0,T;\mathbb{R}) \to L^2(0,T;\mathbb{R}) \,.$$

Of course, the precise meaning of the convergence needs to be specified, we do that in Assumption 4.5. In the application of interest, typically, a nontrivial behavior can only be observed at first order. We then have $N_0 \equiv 0$ and

$$(1.14) \qquad \varepsilon^{-1} N_*^\varepsilon \to N_1 \text{ as } \varepsilon \to 0 \,, \quad N_1 : H^1(0,T;\mathbb{R}) \to L^2(0,T;\mathbb{R}) \,.$$

Our main results are the following two theorems. The first theorem treats an approximation of order $O(1)$, the second theorem an approximation of order $O(\varepsilon)$.

Theorem 1.1 (Effective boundary condition N_0)**.** *Let the geometry and the active elements be given by Ω, Γ, Γ_Y, $\alpha = |\Gamma_Y|$, N^ε, N_Y^ε, N_*^ε, as described in* (1.5)–(1.12)*. Let $(v^\varepsilon, p^\varepsilon)$ be a solution sequence to* (1.1)–(1.2)*, we assume $(v^\varepsilon, p^\varepsilon) \rightharpoonup (v,p)$ in $H^1(0,T;H^1(\Omega,\mathbb{R}^d)) \times H^1(0,T;H^1(\Omega,\mathbb{R}))$ as $\varepsilon \to 0$. We use $c^2 = \bar{p}/\bar{\rho}$ and assume $\mu_\varepsilon \to 0$. Let the convergence conditions of Assumption 4.5 on N_Y^ε be satisfied.*

Then the limit pressure p solves, in the sense of distributions, the effective wave equation

$$(1.15) \qquad \partial_t^2 p = c^2 \Delta p \qquad \text{in } \Omega \,,$$

$$(1.16) \qquad \partial_\nu p = -\bar{\rho}\alpha \, \partial_t N_0(p) \qquad \text{along } \Gamma \,,$$

together with the Neumann condition

$$(1.17) \qquad \partial_\nu p = -\bar{\rho}\, \partial_t g \qquad \text{on } \partial\Omega \setminus \Gamma \,.$$

Theorem 1.2 (Effective system with N_1)**.** *Let the geometry and the active elements be given by Ω, Γ, Γ_Y, $\alpha = |\Gamma_Y|$, N^ε, N_Y^ε, N_*^ε, as described in* (1.5)–(1.12)*. We use $c^2 = \bar{p}/\bar{\rho}$ and assume $\varepsilon^{-1}\mu_\varepsilon \to 0$. Let $(v^\varepsilon, p^\varepsilon)$ be a solution sequence to* (1.1)–(1.2) *with $(v^\varepsilon, p^\varepsilon) \to (v_0, p_0)$ in $H^1(0,T;H^1(\Omega,\mathbb{R}^{d+1}))$ as $\varepsilon \to 0$. We assume that, additionally, first order corrections have a weak limit:*

$$(1.18) \qquad \frac{1}{\varepsilon}\left[(v^\varepsilon, p^\varepsilon) - (v_0, p_0)\right] \rightharpoonup (v_1, p_1) \qquad \text{in } H^1(0,T;H^1(\Omega,\mathbb{R}^{d+1})) \,.$$

Let the conditions of Assumption 4.8 on N_Y^ε with limits $N_0 = 0$ and N_1 be satisfied.

Then the limit pressure p_0 is the solution of the effective wave equation (1.15)–(1.17) *with $N_0 = 0$. The pressure p_1 solves the effective wave equation*

$$(1.19) \qquad \partial_t^2 p_1 = c^2 \Delta p_1 \qquad \text{in } \Omega \,,$$

$$(1.20) \qquad \partial_\nu p_1 = -\bar{\rho}\alpha \, \partial_t N_1(p_0) \qquad \text{along } \Gamma \,,$$

together with the homogeneous Neumann condition $\partial_\nu p_1 = 0$ on $\partial\Omega \setminus \Gamma$.

We consider the above system with solutions p_0 and p_1 as *the* limit system for the ε-problem. It provides an equation for the zero-order approximation p_0 and another equation (using p_0) for the first-order approximation p_1.

In the applied literature, one is oftentimes interested to have a single unknown $\tilde{p}^\varepsilon$ and to have an effective equation for $\tilde{p}^\varepsilon$. The following remark suggests with (1.22) a simple effective condition. The boundary condition in such an effective system can be regarded as the effective impedance condition and one can ask whether or not the effective condition leads to sound absorption.

Remark 1.3 (Formal effective system with one unknown)**.** *The convergence* (1.18) *of Theorem 1.2 yields that $p_0 + \varepsilon p_1$ is an $o(\varepsilon)$-approximation to the solution p^ε of the original system.*

Formally, the solution $\tilde{p}^\varepsilon$ of the following system should also provide an $o(\varepsilon)$-approximation:

$$\partial_t^2 \tilde{p}^\varepsilon = c^2 \Delta \tilde{p}^\varepsilon \qquad \text{in } \Omega\,, \tag{1.21}$$

$$\partial_\nu \tilde{p}^\varepsilon = -\bar{\rho}\alpha\, \varepsilon\, \partial_t N_1(\tilde{p}^\varepsilon) \qquad \text{along } \Gamma\,, \tag{1.22}$$

with the homogeneous Neumann condition $\partial_\nu \tilde{p}^\varepsilon = -\bar{\rho}\partial_t g$ on $\partial\Omega \setminus \Gamma$.

A rigorous justification of system (1.21)–(1.22) is beyond the scope of this contribution and certainly not available for general N_1. We sketch a formal derivation and give some more comments in Appendix B.

In the time-harmonic case and for some classes of N_1-operators, a verification of (an appropriate variant of) Remark 1.3 is possible, see [3]. Indeed, one has to choose a "non-centered" version of the limit system. Even in the time-harmonic case, a justification is quite involved, see the non-centered scheme in Section 5 of [3] and, in particular, the strong assumptions on the boundary operators in Hypothesis 17 and 19.

2. Active interfaces defined by small chambers

Before we start the analysis of the system and the proof of the two main theorems, we sketch relevant choices for the active interfaces, defined by chambers. Our interest is two-fold: On the one hand, we want to check if our assumptions on the maps are satisfied in relevant examples. On the other hand, we want to show that our theorems can provide interesting information on the effective properties for certain models.

2.1. **Compressible Stokes equations in chambers.** A simple active interface can be obtained from a complex geometry, leaving the equations unchanged. We can describe the underlying model as follows. We assume that $\Sigma_Y \subset \mathbb{R}^d$ is a Lipschitz domain $\Sigma_Y \subset Y := (0,1)^d$ with $\bar{\Sigma}_Y \cap \partial Y = \Gamma_Y$. The domain with a complex boundary is

$$\Omega_\varepsilon := \Omega \cup \Gamma^\varepsilon \cup \bigcup_{k \in K_\varepsilon} \varepsilon(k + \Sigma_Y)\,. \tag{2.1}$$

Imposing that (1.1)–(1.2) should hold for $v^\varepsilon : \Omega_\varepsilon \to \mathbb{R}^d$ and $p^\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ gives a relevant description of a sound-active geometry.

Here, we describe this model with active interfaces. The rescaling of v^ε and p^ε in the single chamber with index k is performed via $v(y,t) = v^\varepsilon(\varepsilon(k+y),t)$ and

$p(y,t) = p^\varepsilon(\varepsilon(k+y),t)$. Accordingly, spatial derivatives scale with factors of ε and we have to consider in Σ_Y, with independent variables y and t,

$$\varepsilon\bar{\rho}\,\partial_t v = -\nabla p + \frac{1}{\varepsilon}\mu_\varepsilon \Delta v\,, \tag{2.2}$$

$$\varepsilon\partial_t p + \bar{p}\,\nabla\cdot v = 0\,. \tag{2.3}$$

Given a pressure field $p \in H^1(0,T;H^{1/2}(\Gamma_Y))$ along the active interface, we can solve the equations (2.2)–(2.3) in the domain $\Sigma_Y \subset \mathbb{R}^d$ with the Dirichlet boundary condition imposed by p on Γ_Y. We always use the initial condition $v|_{t=0} \equiv 0$. Part of the solution is a field $v : \Sigma_Y \times (0,T) \to \mathbb{R}^d$ and we can extract, for $y \in \Gamma_Y$, the field $w(y,t) = e_d \cdot v(y,t)$. More precisely, we define w as the trace of the normal velocity along Γ_Y and obtain $w \in L^2(0,T;H^{-1/2}(\Gamma_Y))$. This defines $N_Y^\varepsilon(p) := w$.

Limit operator for Stokes chamber. Equation (2.3) allows to calculate with the theorem of Gauß, for $p_* : [0,T] \to \mathbb{R}$,

$$N_*^\varepsilon(p_*) = \fint_{\Gamma_Y} e_d \cdot v = -\frac{1}{|\Gamma_Y|}\int_{\Sigma_Y} \nabla\cdot v = \frac{\varepsilon}{\bar{p}\,|\Gamma_Y|}\int_{\Sigma_Y} \partial_t p\,. \tag{2.4}$$

When, for a sequence of solutions to (2.2)–(2.3), the integral over $\partial_t p$ remains bounded (for fixed boundary data p_*), then $N_*^\varepsilon(p_*)$ is vanishing for $\varepsilon \to 0$. We find $N_0(p_*) = 0$ for every argument p_*. The effective limit equation is given by the sound-hard boundary condition only, i.e., the behavior is as if there were no chambers. In particular, the effective law does not lead to sound absorption in order $\varepsilon^0 = 1$.

A different result is obtained at first order. With the above calculation, we find

$$N_1(p_*) = \lim_{\varepsilon\to 0} \frac{1}{\bar{p}\,|\Gamma_Y|}\int_{\Sigma_Y} \partial_t p\,. \tag{2.5}$$

Stokes chamber I, limit operator for $\varepsilon^{-1}\mu_\varepsilon \to 0$. We show in Appendix A that the physical parameters suggest $\varepsilon^{-1}\mu_\varepsilon \to 0$. When this is the case, the formal limit of the first equation is $\nabla p = 0$, and we expect that the pressure is constant in the chamber. In this case, the pressure p in the chamber coincides with the boundary data p_* and we find from (2.5)

$$N_1(p_*) = \frac{|\Sigma_Y|}{\bar{p}\,|\Gamma_Y|}\partial_t p_*\,. \tag{2.6}$$

This is a nontrivial limit map N_1.

Unfortunately, as a matter of fact, the effective law N_1 of (2.6) does not lead to sound absorption. We discuss this in Subsection 3.

Stokes chamber II, limit operator for $\varepsilon^{-1}\mu_\varepsilon \to \mu_* > 0$. Let us investigate the case $\varepsilon^{-1}\mu_\varepsilon \to \mu_* > 0$ – even though this case is not included in our theorems. The limit system for (2.2)–(2.3) in Σ_Y then reads

$$\nabla p = \mu_* \Delta v\,, \tag{2.7}$$

$$\nabla\cdot v = 0\,. \tag{2.8}$$

This is a stationary incompressible Stokes equation. It defines a Dirichlet-to-Neumann map, an approximation of N_Y^ε, defined as $p|_{\Gamma_Y} \mapsto e_d \cdot v|_{\Gamma_Y}$. By Gauß theorem, integrals over $e_d \cdot v|_{\Gamma_Y}$ vanish. This is consistent with $N_0 = 0$, which was already observed after (2.4). The solution of (2.7)–(2.8) can be given explicitly: For (spatially) constant boundary data p_*, the solution is $v \equiv 0$ and $p \equiv p_*$ (in the sense that $p(y,t) = p_*(t)$ for every $y \in \Sigma_Y$ and every $t \in [0,T]$). This allows to calculate N_1 from (2.5); with the same calculation as above, we obtain once more (2.6).

2.2. **Darcy law in chambers.** In applications, the chambers are (partially) filled with rock-wool or a similar material in order to obtain a damping mechanism.

We recall that the shape of the single chamber is described by Σ_Y. Rock-wool has a micro-structure that is so small that the Stokes system is overdamped. This means that a Darcy law (well-known from porous media modelling) is often appropriate to describe the physics of this material. In this setting, equation (2.2) is replaced by

$$\nabla p = -M_\varepsilon\, v\,. \tag{2.9}$$

We refer to Table 2 for typical values; in applications, M_ε is called the flow resistivity. Inserting $v = -M_\varepsilon^{-1}\nabla p$ into (2.3), we see that we have to solve in Σ_Y the diffusion equation

$$\frac{1}{\bar p} M_\varepsilon\, \varepsilon\; \partial_t p = \Delta p\,. \tag{2.10}$$

Given pressure boundary data $p \in H^1(0,T; H^{1/2}(\Gamma_Y))$, we can solve (2.10) in the domain Σ_Y (we recall that we always use trivial initial conditions, $p|_{t=0} \equiv 0$). This defines an operator N_Y^ε.

Darcy chamber I, limit operator for $M_\varepsilon\, \varepsilon \to 0$. Let us first discuss the case $M_\varepsilon\, \varepsilon \to 0$. In this case, the limit $\varepsilon \to 0$ in (2.10) yields $\Delta p = 0$ in Σ_Y, the Laplace-equation. For constant (in space) boundary data p_* at Γ_Y, the solution is a constant pressure $p(.,t) = p_*(t)$ in the chamber Σ_Y. The corresponding maps N_0 and N_1 of the active element can easily be calculated as in (2.5) and (2.6): There holds $N_0 = 0$ and $N_1(p_0) = |\Sigma_Y|\, (\bar p\, |\Gamma_Y|)^{-1}\, \partial_t p_0$. In particular, there is no sound absorption.

Darcy chamber II, limit operator for $M_\varepsilon\, \varepsilon \to M_* > 0$. We will now finally obtain a useful limit system. In the case $M_\varepsilon\, \varepsilon \to M_* > 0$, the limit $\varepsilon \to 0$ in (2.10) yields the diffusion equation

$$\frac{1}{\bar p} M_* \partial_t p = \Delta p \tag{2.11}$$

in Σ_Y. The solution provides the effective map N_1, given as

$$N_1(p_*) = \frac{1}{\bar p\, |\Gamma_Y|} \int_{\Sigma_Y} \partial_t p\,, \tag{2.12}$$

compare the derivation in (2.5). The system (2.11)–(2.12) is a useful limit system that can explain sound absorption.

In order to understand the last claim, let us investigate the time-harmonic situation with frequency ω. Replacing in (2.11) the time derivative with a multiplication by $i\omega$, we find the following equation in Σ_Y:

$$\frac{i\,\omega M_*}{\bar p}\, p = \Delta p\,. \tag{2.13}$$

For the typical data that we choose for our analysis in Appendix A, we find that the non-dimensionalized pre-factor on the left hand side has approximately the value 5. This shows that the two sides of (2.10) are balanced and the solution is not close to a constant.

In the time-harmonic setting, the limiting operator of (2.12) is given by

$$N_1(p_*) = \frac{i\,\omega}{\bar{p}\,|\Gamma_Y|} \int_{\Sigma_Y} p\,. \tag{2.14}$$

Using (2.14) in the effective law $\partial_\nu p_1 = -\bar\rho\alpha\,\partial_t N_1(p_0)$ of (1.20) and $\alpha = |\Gamma_Y|$, we find

$$\partial_\nu p_1 = -\frac{\bar\rho\,\omega^2}{\bar p} \int_{\Sigma_Y} p\,, \tag{2.15}$$

where p is the solution to (2.13) with boundary data p_0.

We see in Appendix A that the non-dimensionalized pre-factor on the right hand side has approximately the value 10. This implies that, when p_0 is corrected by the term $\varepsilon\, p_1$ with $\varepsilon = 1/20$ (as chosen in our data set), we obtain a relevant correction.

Remark 2.1 (The importance of a balanced Darcy velocity)**.** *For sound absorption, it is crucial to have the two sides in* (2.10) *balanced. Indeed, only a nontrivial limit equation as in* (2.11) *can provide a pressure p in Σ_Y that does* not *coincide with p_*.*

When the number M_ε is too large, the effective system is $\Delta p = 0$ and we have $p = p_*$ as in Stokes I or Darcy I.

When the number M_ε is too small, the effective system is $\partial_t p = 0$, in which case (2.12) *yields $N_1 = N_0 = 0$. Once more, the system does not induce sound absorption.*

Remark 2.2 (Impedance)**.** *One defines the impedance Z as the factor that relates pressure and normal velocity: $v\cdot\nu = Z^{-1}p$. Because of $\bar\rho\,\partial_t v = -\nabla p$, in the time-harmonic case with dependence $e^{i\omega t}$, there holds $\bar\rho\, i\,\omega\, v\cdot\nu = -\partial_\nu p$, hence the impedance boundary condition is $\partial_\nu p = -i\,(\bar\rho\,\omega/Z)\,p$.*

With the limit operator $N_1(p) = \beta\partial_t p$ of (2.6)*, the limiting boundary condition of* (1.22) *reads $\partial_\nu p = -\bar\rho\,\alpha\,\varepsilon\,\beta\,\partial_t^2 p$ along Γ. In the time-harmonic case, we find $\partial_\nu p = \bar\rho\,\alpha\,\varepsilon\,\beta\,\omega^2 p$. A comparison with the impedance boundary condition yields the purely imaginary impedance*

$$Z = \frac{i}{\alpha\,\varepsilon\,\beta\,\omega}\,. \tag{2.16}$$

We find it useful to work with the number $z = z(\omega) \in \mathbb{C}$ such that the boundary condition in the time-harmonic case reads $\partial_\nu p = z\,p$. For the operator N_1 under consideration, we find $z = (\bar\rho\, i\,\omega/Z) = \bar\rho\,\alpha\,\varepsilon\,\beta\,\omega^2$. This is a real factor and, therefore, the law $N_1(p) = \beta\partial_t p$ does not induce sound absorption, compare Section 3.

In the time-harmonic setting, the limit operator $N_1(p) = \beta\partial_t p$ is $N_1(p) = i\omega\beta p$. We observed that this phase shift of $90°$ does not imply sound absorption. In contrast, the operator (2.14) *for the Darcy chamber does not have a phase shift of $90°$ since there is a non-vanishing phase shift between p_* and the average of p.*

2.3. **Brinkman law and further extensions.** A more complete model is to include the inertia term in the Darcy law, which leads to the Brinkman model. The rescaled compressible Brinkman system reads, in Σ_Y:

$$\varepsilon\bar\rho\,\partial_t v = -\nabla p + M_\varepsilon\, v\,, \tag{2.17}$$

$$\varepsilon\partial_t p + \bar p\,\nabla\cdot v = 0\,. \tag{2.18}$$

Also these equations define a map of Dirichlet-to-Neumann type, $N_Y^\varepsilon(p) = e_d\cdot v|_{\Gamma_Y}$, a trivial limit map $N_0 = 0$, and, possibly, a nontrivial limit map N_1. Nevertheless, we do not expect that the Brinkman model in the above form gives a new limit operators N_1.

Combinations of the above. For applications, the most interesting cell-problem is a combination of the above: One part of the cell is filled with absorber, the other part is not. A relevant model would by that $\Sigma_Y = \Sigma_Y^1 \cup \Sigma_Y^2 \cup \gamma$ where Σ_Y^1 and Σ_Y^2 are disjoint open domains and $\gamma = \partial\Sigma_Y^1 \cap \partial\Sigma_Y^2$. One would impose, e.g. (2.2)–(2.3) in Σ_Y^1, and (2.10) in Σ_Y^2. This also defines a Dirichlet-to-Neumann map $N_Y^\varepsilon(p) = e_d \cdot v|_{\Gamma_Y}$.

The aim of the contribution at hand is to separate the two aspects of the problem. On the one hand, there is the task to average the operators N_Y^ε and to find a limit problem in Ω; this is the focus of the present analysis. On the other hand, there is the question of an (optimal) design and the analysis of appropriate (limit) laws in the chamber; this is a task for future work.

3. Conservation of energy

In the first analysis of the Stokes chambers, eventually, we neglected everywhere the viscous term. When we do that from the start, then the limit equation is given, in the whole domain Ω_ε, by the wave equation (A.1). Testing the wave equation in the complex geometry with $\partial_t p$, we find, when all boundaries prescribe either $p = 0$ or $\partial_\nu p = 0$,

$$\partial_t \left\{ \int_{\Omega_\varepsilon} |\partial_t p|^2 + c^2 |\nabla p|^2 \right\} = 0\,.$$

This is energy conservation. This calculation (which includes the microscopic elements) shows that we cannot expect that energy dissipation occurs when we work only with the Stokes system and neglect viscosity.

An ineffective limit operator. In the first Darcy setting, we found the law $N_1(p) = \beta\partial_t p$, see (2.6). In order to understand the effect of this boundary condition, let us investigate the corresponding limit model (1.21)–(1.22), i.e., the wave equation in Ω with the boundary condition $\partial_\nu p = -\beta\partial_t^2 p$ at the part Γ of the boundary. Testing the wave equation in Ω with $\partial_t p$, we find, for this boundary condition,

$$\partial_t \left\{ \int_\Omega |\partial_t p|^2 + c^2 |\nabla p|^2 \right\} = 2\int_\Gamma c^2 \partial_\nu p\, \partial_t p = -2\int_\Gamma c^2 \beta \partial_t^2 p\, \partial_t p = -c^2\beta\, \partial_t \int_\Gamma |\partial_t p|^2\,.$$

We see that the last integral defines an extra term in the energy. This boundary integral can be interpreted as the energy that is stored in the chambers.

Our result is that there is still energy conservation – even though the energy expression is modified. The limit operators N_0 and N_1 obtained from Darcy I, given as $N_0 = 0$ and $N_1(p_0) = \bar{p}^{-1}|\Sigma_Y||\Gamma_Y|^{-1}\partial_t p_0$, do not induce energy losses. Therefore, they cannot explain sound attenuation.

The effect of a complex prefactor z. For this analysis, we use an ansatz with an exponential time dependence. The aim is to find a solution with decay. We assume $p(.,t) = p(.)\, e^{\lambda t}$ for $\lambda \in \mathbb{C}$ with $\omega := \operatorname{Im}\lambda > 0$. The wave equation in Ω transforms into $\lambda^2 p = c^2 \Delta p$. Let us assume that this equation is complemented with the boundary condition $\partial_\nu p = z\, p$ along Γ. Testing the equation with the complex conjugate of p (here denoted as p^*), we obtain

$$\left\{ \int_\Omega \lambda^2 |p|^2 + c^2 |\nabla p|^2 \right\} = - \int_\Gamma c^2 \partial_\nu p \, p^* = -c^2 z \int_\Gamma |p|^2 \,.$$

In the case that z is real, the imaginary part of the right hand side vanishes. When p is not vanishing identically, then the imaginary part of λ^2 must vanish. We conclude that λ is purely imaginary, $\lambda = i\omega$. This reflects an oscillatory behavior without damping.

Instead, in the case $\operatorname{Im} z > 0$, there holds $\operatorname{Im} \lambda^2 < 0$, and hence $\operatorname{Re} \lambda < 0$. This implies that the system is damped. This calculation is the background of one of the comments in Remark 2.2.

4. Homogenization of active interfaces

The main part of this section regards the mapping properties of the operators N^ε and their limit behavior as $\varepsilon \to 0$.

4.1. **Estimates for norms.** We prepare our analysis with some results for $H^{1/2}$ and $H^{-1/2}$-norms.

To simplify notation and without loss of generality, we assume here that we can choose the height $h = 1$ in our assumptions on the geometry, i.e.: $\Omega_1 = (0,1)^{d-1} \times (-1, 0)$ is contained in Ω. For a function $\phi : \Gamma \to \mathbb{R}$ we can introduce a norm via extensions to the domain Ω_1 with upper boundary Γ:

$$\|\phi\|_{H^{1/2}(\Gamma)} := \inf \left\{ \|\tilde\phi\|_{H^1(\Omega_1,\mathbb{R})} \,\middle|\, \tilde\phi = \phi \text{ on } \Gamma \right\} . \tag{4.1}$$

Obviously, the functions $\tilde\phi$ must be elements of $H^1(\Omega_1, \mathbb{R})$ and boundary values of $\tilde\phi$ are understood in the sense of traces.

We also want to introduce a norm for a function on the domain $\Gamma_Y \subset (0,1)^{d-1} \subset \mathbb{R}^{d-1}$. We define similarly to (4.1), using the domain $Y_* = (0,1)^{d-1} \times (-1, 0)$ below Γ_Y: For a function $\psi : \Gamma_Y \to \mathbb{R}$ we use the norm

$$\|\psi\|_{H^{1/2}(\Gamma_Y)} := \inf \left\{ \|\tilde\psi\|_{H^1(Y_*,\mathbb{R})} \,\middle|\, \tilde\psi = \psi \text{ on } \Gamma_Y \right\} . \tag{4.2}$$

Let us note that this definition makes a choice regarding boundary conditions outside Γ_Y: We impose no condition (such as $\tilde\psi = 0$) on $(0,1)^{d-1} \setminus \Gamma_Y$.

An important preparation of the subsequent analysis regards the connections between the global norm (4.1) and the sum of local norms as in (4.2). We will investigate in several cases the situation that a function $\phi : \Gamma \to \mathbb{R}$ (e.g., of class $L^2(\Gamma)$ or $H^{1/2}(\Gamma)$) is locally given by functions $\psi_k : \Gamma_Y \to \mathbb{R}$ in the sense that

$$\phi(\varepsilon(k + y)) = \psi_k(y) \qquad \text{for all } k \in K_\varepsilon \,, \quad \text{a.e. } y \in \Gamma_Y \,. \tag{4.3}$$

We recall that the set of relevant indices K_ε was defined before (1.6), with our normalization to $\Gamma = (0,1)^{d-1} \times \{0\} \equiv (0,1)^{d-1}$, the number of boundary pieces Γ_k^ε is exactly $|K_\varepsilon| = 1/\varepsilon^{d-1}$.

Lemma 4.1 (Relation of two norms of $H^{1/2}$-type)**.** *Let ϕ be locally given by $(\psi_k)_k$ as in* (4.3)*. Then there holds*

$$\sum_{k \in K_\varepsilon} \varepsilon^{d-1} \|\psi_k\|^2_{H^{1/2}(\Gamma_Y)} \le \|\phi\|^2_{H^{1/2}(\Gamma)} \,. \tag{4.4}$$

Proof. Let $\eta > 0$ be an arbitrary error quantifier. Using the definition of the norm in (4.1), we choose a nearly optimal extension $\tilde\phi : \Omega_1 \to \mathbb{R}$ of ϕ. We demand that the infimum of (4.1) is attained up to an error η for the squared norms. Using the extension $\tilde\phi$ we set, for $\hat y \in (0,1)^{d-1}$ and $y_d \in (-1,0)$,

$$\tilde\psi_k(\hat y, y_d) := \tilde\phi(\varepsilon(k+\hat y), y_d)\,. \tag{4.5}$$

Note that we stretch $d-1$ variables and leave one variable unchanged. This defines extensions of $\psi_k : \Gamma_Y \to \mathbb{R}$ to functions $\tilde\psi_k : Y_* \to \mathbb{R}$. In the subsequent calculation, we use the elongated thin cylinders $S_k^\varepsilon := \varepsilon(k + (0,1)^{d-1}) \times (-1,0)$. We evaluate the left hand side of (4.4) and use, in this order, (i) the definition of the $H^{1/2}$-norm in (4.2), (ii) the definition of the H^1-norm, (iii) the transformation formula, (iv) $\varepsilon \le 1$:

$$\begin{aligned}
\sum_{k\in K_\varepsilon} \varepsilon^{d-1} \|\psi_k\|^2_{H^{1/2}(\Gamma_Y)} &\le \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \|\tilde\psi_k\|^2_{H^1(Y_*)} \\
&= \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \int_{Y_*} |\tilde\psi_k|^2 + \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \int_{Y_*} |\nabla_y \tilde\psi_k|^2 \\
&= \sum_{k\in K_\varepsilon} \int_{S_k^\varepsilon} |\tilde\phi|^2 + \sum_{k\in K_\varepsilon} \int_{S_k^\varepsilon} \Big(|\varepsilon(\partial_{x_1}, ..., \partial_{x_{d-1}})\tilde\phi|^2 + |\partial_{x_d}\tilde\phi|^2 \Big) \\
&\le \int_{\Omega_1} |\tilde\phi|^2 + \int_{\Omega_1} |\nabla_x \tilde\phi|^2 = \|\tilde\phi\|^2_{H^1(\Omega_1)} \le \|\phi\|^2_{H^{1/2}(\Gamma)} + \eta\,.
\end{aligned}$$

Since $\eta > 0$ was arbitrary, this proves (4.4). □

The next lemma provides an improvement: When averages of the functions are subtracted, we even gain the pre-factor ε in the estimate. We use, as before, the integral average $\fint_A f := |A|^{-1} \int_A f$.

Lemma 4.2 (Norms of $H^{1/2}$-type with subtracted averages)**.** *Let ϕ be locally described by $(\psi_k)_k$ as in* (4.3)*. We consider the local averages*

$$\psi_{k,0} := \fint_{\Gamma_Y} \psi_k \in \mathbb{R} \qquad \forall k \in K_\varepsilon\,. \tag{4.6}$$

Then there holds, for some constant $C > 0$ that depends only on Γ_Y:

$$\sum_{k\in K_\varepsilon} \varepsilon^{d-1} \|\psi_k - \psi_{k,0}\|^2_{H^{1/2}(\Gamma_Y)} \le C\,\varepsilon\, \|\phi\|^2_{H^{1/2}(\Gamma)}\,. \tag{4.7}$$

The subsequent proof uses other local extensions than the proof of Lemma 4.1.

Proof. Let $\eta > 0$ be an arbitrary error quantifier. We choose a nearly optimal extension $\tilde\phi : \Omega_1 \to \mathbb{R}$ of ϕ with a deviation of the squared norms less than η. With this extension we set

$$\tilde\psi_k(\hat y, y_d) := \tilde\phi(\varepsilon(k+\hat y), \varepsilon y_d)\,. \tag{4.8}$$

These are extensions of $\psi_k : \Gamma_Y \to \mathbb{R}$ to functions $\tilde\psi_k : Y_* \to \mathbb{R}$. We calculate with cubes $S_k^\varepsilon := \varepsilon(k + (0,1)^{d-1}) \times (-\varepsilon, 0)$ and the strip $S_\varepsilon := \Gamma \times (-\varepsilon, 0)$. The subsequent calculation uses the following: (i) The definition of the $H^{1/2}$-norm in (4.2), (ii) the Poincaré inequality for functions with vanishing average on a part of the boundary,

(iii) the chain rule for the evaluation of the gradient and a transformation formula with a factor ε^{-d} for the integral.

$$\begin{aligned}\sum_{k\in K_\varepsilon} \varepsilon^{d-1} \|\psi_k - \psi_{k,0}\|^2_{H^{1/2}(\Gamma_Y)} &\le \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \|\tilde\psi_k - \psi_{k,0}\|^2_{H^1(Y_*)} \\ \le C \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \int_{Y_*} |\nabla_y \tilde\psi_k|^2 &\le C \sum_{k\in K_\varepsilon} \varepsilon^{-1} \int_{S_k^\varepsilon} |\varepsilon \nabla_x \tilde\phi|^2 \\ \le C\,\varepsilon \int_{S_\varepsilon} |\nabla_x \tilde\phi|^2 \le C\,\varepsilon \|\tilde\phi\|^2_{H^1(\Omega_1)} &\le C\,\varepsilon \left(\|\phi\|^2_{H^{1/2}(\Gamma)} + \eta \right) .\end{aligned}$$

Since $\eta > 0$ was arbitrary, this yields (4.7). □

The spaces $H^{-1/2}$ are defined as the dual spaces of $H^{1/2}$-spaces. The norms are defined accordingly, e.g., by

$$\|W\|_{H^{-1/2}(\Gamma)} := \sup \left\{ \langle W, \phi\rangle \,\middle|\, \|\phi\|_{H^{1/2}(\Gamma)} \le 1 \right\} . \tag{4.9}$$

When $W \in H^{-1/2}(\Gamma)$ is represented by a function $W \in L^2(\Gamma)$, the application to a test-function $\phi \in H^{1/2}(\Gamma)$ is given by an integral,

$$\langle W, \phi \rangle = \int_\Gamma W\,\phi\,. \tag{4.10}$$

Below, we will use a sloppy notation and write always integrals for the application to test-functions. Similarly, we will write, e.g., $y \mapsto W(\varepsilon(k+y))$ for the transformed functional even when W is not given by a function.

Lemma 4.3 (Relation of two norms of $H^{-1/2}$-type)**.** *Let $W : \Gamma \to \mathbb{R}$ be of class $H^{-1/2}(\Gamma)$ and let $w_k : \Gamma_Y \to \mathbb{R}$ be of class $H^{-1/2}(\Gamma_Y)$ for every $k \in K_\varepsilon$. We assume that W satisfies $W(\varepsilon(k+y)) = w_k(y)$ for almost all $y \in \Gamma_Y$ and $W(\varepsilon(k+y)) = 0$ for almost all $y \in (0,1)^{d-q} \setminus \Gamma_Y$, for all $k \in K_\varepsilon$. A concise description of this relation is:*

$$\langle W, \phi\rangle = \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \langle w_k, \phi(\varepsilon(k+.))\rangle \qquad \forall \phi \in H^{1/2}(\Gamma)\,. \tag{4.11}$$

In this situation holds

$$\|W\|^2_{H^{-1/2}(\Gamma)} \le \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \|w_k\|^2_{H^{-1/2}(\Gamma_Y)}\,. \tag{4.12}$$

We note that inequality (4.12) can be regarded as being dual to (4.4).

Proof. The definition of the dual norm in (4.9) requires to consider an arbitrary function $\phi : \Gamma \to \mathbb{R}$ with $\|\phi\|_{H^{1/2}(\Gamma)} \le 1$. We define ψ_k as the rescaled versions of the restrictions, $\psi_k(y) = \phi(\varepsilon(k+y))$ for $y \in (0,1)^{d-1}$. The norm comparison of (4.4) implies $\sum_{k\in K_\varepsilon} \varepsilon^{d-1} \|\psi_k\|^2_{H^{1/2}(\Gamma_Y)} \le 1$. This allows to calculate

$$\begin{aligned}\langle W, \phi\rangle_\Gamma = \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \langle w_k, \psi_k\rangle_{\Gamma_Y} &\le \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \|w_k\|_{H^{-1/2}(\Gamma_Y)} \|\psi_k\|_{H^{1/2}(\Gamma_Y)} \\ &\le \left(\sum_{k\in K_\varepsilon} \varepsilon^{d-1} \|w_k\|^2_{H^{-1/2}(\Gamma_Y)} \right)^{1/2} \left(\sum_{k\in K_\varepsilon} \varepsilon^{d-1} \|\psi_k\|^2_{H^{1/2}(\Gamma_Y)} \right)^{1/2}\end{aligned}$$

$$\leq \left(\sum_{k\in K_\varepsilon} \varepsilon^{d-1} \|w_k\|^2_{H^{-1/2}(\Gamma_Y)} \right)^{1/2} .$$

This provides inequality (4.12). □

4.2. **The combined map N^ε.** After these preparations regarding properties of norms, we can now analyze operators N^ε that are given by local operators N^ε_Y.

Lemma 4.4 (Mapping properties of N^ε)**.** *Let N^ε_Y be a map for the domain Γ_Y as in* (1.10)*, we assume that it is Lipschitz continuous. Let N^ε be the corresponding map for the domain Γ as in* (1.8)*, defined locally on every set Γ^ε_k by the operator N^ε_Y and extended by zero, compare* (1.9)*. Then N^ε Lipschitz continuous.*

Proof. We consider an argument $\phi \in H^1(0,T;H^{1/2}(\Gamma))$ for the map N^ε. We define local functions $\psi_k \in H^1(0,T;H^{1/2}(\Gamma_Y))$ by restriction and rescaling as in (4.3). The local maps provide $w_k := N^\varepsilon_Y(\psi_k)$. In the second line of the subsequent calculation we use, for arbitrary $t \in [0,T]$, Lemma 4.3 for the function $W = N^\varepsilon(\phi)(t)$, which is given, locally, by w_k. The third line exploits the mapping property of N^ε_Y and the fourth line uses the estimate of Lemma 4.1 for every t. The two equalities use only the definition of the norm in the Bochner space.

$$\begin{aligned}
\|N^\varepsilon(\phi)\|^2_{L^2(0,T;H^{-1/2}(\Gamma))} &= \int_0^T \|N^\varepsilon(\phi)(t)\|^2_{H^{-1/2}(\Gamma))}\, dt \\
&\leq \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \int_0^T \|w_k(t)\|^2_{H^{-1/2}(\Gamma_Y)}\, dt \\
&\leq C \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \left(1 + \int_0^T \left\{ \|\psi_k(t)\|^2_{H^{1/2}(\Gamma_Y)} + \|\partial_t \psi_k(t)\|^2_{H^{1/2}(\Gamma_Y)} \right\} dt \right) \\
&\leq C + C \int_0^T \left\{ \|\phi(t)\|^2_{H^{1/2}(\Gamma)} + \|\partial_t \phi(t)\|^2_{H^{1/2}(\Gamma)} \right\} dt \\
&= C \left(1 + \|\phi\|^2_{H^1(0,T;H^{1/2}(\Gamma))} \right) .
\end{aligned}$$

This provides that N^ε is bounded on bounded sets.

The calculation for Lipschitz continuity is similar. We have to show that, for a constant $C > 0$ and arbitrary ϕ^j and ϕ,

$$\|N^\varepsilon(\phi^j) - N^\varepsilon(\phi)\|^2_{L^2(0,T;H^{-1/2}(\Gamma))} \leq C \|\phi^j - \phi\|^2_{H^1(0,T;H^{1/2}(\Gamma))} . \tag{4.13}$$

This follows with a calculation that is completely analogous to the above. Obviously, for linear maps N^ε_Y, Lipschitz continuity follows already from boundedness. □

4.3. **The limit $\varepsilon \to 0$ for the maps N^ε.** The following assumption makes precise how we understand the convergence property (1.13).

Assumption 4.5. *Let the maps N^ε_Y of* (1.10) *be uniformly Lipschitz continuous. We construct the averages N^ε_* as in* (1.12) *and assume, for some limit operator $N_0 : H^1(0,T;\mathbb{R}) \to L^2(0,T;\mathbb{R})$, that the following holds:*

1. The limit operator N_0 is Lipschitz continuous and has the following weak convergence property: For every weakly convergent sequence of arguments, $\psi^\varepsilon \rightharpoonup \psi$ in $H^1(0,T;\mathbb{R})$, there holds $N_0(\psi^\varepsilon) \rightharpoonup N_0(\psi)$ in $L^2(0,T;\mathbb{R})$.

2. The convergence $N_*^\varepsilon \to N_0$ holds in the following sense: For every temporal test-function $\varphi \in L^2(0,T;\mathbb{R})$ there is a sequence of numbers $\|N_*^\varepsilon - N_0\|_\varphi \to 0$ as $\varepsilon \to 0$ such that, for every argument $\psi \in H^1(0,T;\mathbb{R})$, there holds

$$\left| \int_0^T (N_*^\varepsilon(\psi) - N_0(\psi))(t)\, \varphi(t)\, dt \right| \le \|N_*^\varepsilon - N_0\|_\varphi\, \|\psi\|_{H^1(0,T;\mathbb{R})}\,. \tag{4.14}$$

Remark 4.6 (Criterion for operator convergence)**.** *Let N_Y^ε and N_*^ε be as in* (1.10) *and* (1.12)*. Assumption 4.5 is satisfied when the following criterion holds:*

The maps N_Y^ε are linear and the family of operators $N_Y^\varepsilon : H^1(0,T;H^{1/2}(\Gamma_Y)) \to L^2(0,T;H^{-1/2}(\Gamma_Y))$ is bounded. For some bounded and linear limit operator $N_0 : H^1(0,T;\mathbb{R}) \to L^2(0,T;\mathbb{R})$ holds: For every weakly convergent sequence of arguments, $\psi^\varepsilon \rightharpoonup \psi$ in $H^1(0,T;\mathbb{R})$, there holds

$$N_*^\varepsilon(\psi^\varepsilon) \rightharpoonup N_0(\psi) \quad \text{in } L^2(0,T;\mathbb{R})\,. \tag{4.15}$$

Proof. Every bounded linear operator is Lipschitz continuous and the Lipschitz constant is given by the norm. This shows that, in the situation of Remark 4.6, the first sentence of Assumption 4.5 holds.

Item 1. of Assumption 4.5 follows from linearity. Indeed, for an arbitrary linear functional $\lambda : L^2(0,T;\mathbb{R}) \to \mathbb{C}$, the concatenation $\lambda \circ N_0$ is a linear functional on $H^1(0,T;\mathbb{R})$. The weak convergence $\psi^\varepsilon \rightharpoonup \psi$ implies $\lambda \circ N_0(\psi^\varepsilon) \to \lambda \circ N_0(\psi)$ and thus the weak convergence $N_0(\psi^\varepsilon) \rightharpoonup N_0(\psi)$.

By linearity, it is sufficient to show (4.14) for arguments ψ with norm 1. It is therefore sufficient to show, for fixed φ, that

$$\sup_{\|\psi\|=1} \left| \int_0^T (N_*^\varepsilon(\psi) - N_0(\psi))(t)\, \varphi(t)\, dt \right| \to 0\,. \tag{4.16}$$

If (4.16) fails to hold, there exists a family of arguments ψ^ε, bounded in $H^1(0,T;\mathbb{R})$ by 1, such that the integral with arguments ψ^ε does not vanish in the limit. Since the space is reflexive, we find a limit and a weakly convergent subsequence $\psi^\varepsilon \rightharpoonup \psi$ in $H^1(0,T;\mathbb{R})$. In this situation, the weak convergence $N_0(\psi^\varepsilon) \rightharpoonup N_0(\psi)$ and (4.15) provide a contradiction. □

Proposition 4.7 (Zero order limit property of N^ε)**.** *Let Assumption 4.5 hold. Then N^ε converges to the pointwise defined operator αN_0. More precisely, for every sequence*

$$p^\varepsilon \to p \quad \text{weakly in } H^1(0,T;H^{1/2}(\Gamma))\,, \tag{4.17}$$

there holds

$$N^\varepsilon(p^\varepsilon) \to \alpha N_0(p) \quad \text{weakly in } L^2(0,T;H^{-1/2}(\Gamma))\,. \tag{4.18}$$

Proof. We study a sequence p^ε as in (4.17). Since it is bounded in $H^1(0,T;H^{1/2}(\Gamma))$, by Lemma 4.4, the corresponding flux functions $w^\varepsilon := N^\varepsilon(p^\varepsilon)$ are bounded in $L^2(0,T;H^{-1/2}(\Gamma))$. Since this latter space is reflexive, we can choose a weakly convergent subsequence and find a limit function $w \in L^2(0,T;H^{-1/2}(\Gamma))$ such that

$$N^\varepsilon(p^\varepsilon) = w^\varepsilon \rightharpoonup w \quad \text{in } L^2(0,T;H^{-1/2}(\Gamma))\,. \tag{4.19}$$

It remains to characterize the limit and to verify $w \stackrel{!}{=} W := \alpha N_0(p)$.

Given the pressure function p^ε, we extract the local pressure boundary data in the active interfaces; after rescaling, we find $p_k^\varepsilon : \Gamma_Y \times (0,T) \to \mathbb{R}$. To be precise: $\psi_k =$

$p_k^\varepsilon(t)$ is related to $\phi = p^\varepsilon(t)$ through (4.3), for every t. We define the corresponding averages as $p_{k,0}^\varepsilon : (0,T) \to \mathbb{R}$ as in (4.6). Lemma 4.2 provides with (4.7) the estimate

$$\sum_{k\in K_\varepsilon} \varepsilon^{d-1} \int_0^T \|p_k^\varepsilon(t) - p_{k,0}^\varepsilon(t)\|^2_{H^{1/2}(\Gamma_Y)}\, dt \le C\,\varepsilon \int_0^T \|p^\varepsilon(t)\|^2_{H^{1/2}(\Gamma)}\, dt \le C\,\varepsilon\,. \tag{4.20}$$

We can apply (4.7) also to the time derivative, since $\partial_t p_{k,0}^\varepsilon(t)$ is indeed the average of the time derivative $\partial_t p_k^\varepsilon(t)$. An integration over $[0,T]$ yields

$$\sum_{k\in K_\varepsilon} \varepsilon^{d-1} \int_0^T \|\partial_t p_k^\varepsilon(t) - \partial_t p_{k,0}^\varepsilon(t)\|^2_{H^{1/2}(\Gamma_Y)}\, dt \le C\,\varepsilon \int_0^T \|\partial_t p^\varepsilon(t)\|^2_{H^{1/2}(\Gamma)}\, dt \le C\,\varepsilon\,. \tag{4.21}$$

The local fluxes are $w_k^\varepsilon = N_Y^\varepsilon(p_k^\varepsilon)$, they constitute w^ε. We recall that, here and below, functions that are defined by some operator on all subsets Γ_k^ε, are extended by zero to all of Γ. We define different approximations, locally defined as follows:

$$\begin{aligned}
&w^\varepsilon \text{ locally given by } N_Y^\varepsilon(p_k^\varepsilon) && \text{(flux induced by local pressures)}\,,\\
&W_I^\varepsilon \text{ locally given by } N_Y^\varepsilon(p_{k,0}^\varepsilon) && \text{(evaluation for piecewise constant pressures)}\,,\\
&W_{II}^\varepsilon \text{ locally given by } N_*^\varepsilon(p_{k,0}^\varepsilon) && \text{(replace by piecewise constant flux)}\,,\\
&W_{III}^\varepsilon \text{ locally given by } N_0(p_{k,0}^\varepsilon) && \text{(use limit } N_0)\,,\\
&W_{IV}^\varepsilon \text{ pointwise given by } \alpha N_0(p^\varepsilon) && \text{(use limit } N_0 \text{ and the argument } p^\varepsilon)\,.
\end{aligned}$$

The proof now consists of the analysis of the five corresponding differences.

First difference: $w^\varepsilon - W_I^\varepsilon$. The subsequent calculation uses: (i) Squared norms are estimated in (4.12) by local representations. (ii) The uniform (in ε) Lipschitz continuity of N_Y^ε. (iii) Estimates (4.20) and (4.21). Suppressing the argument $t \in [0,T]$ of the functions everywhere, we find

$$\begin{aligned}
\int_0^T \|w^\varepsilon - W_I^\varepsilon\|^2_{H^{-1/2}(\Gamma)}\, dt &\le \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \int_0^T \|N_Y^\varepsilon(p_k^\varepsilon) - N_Y^\varepsilon(p_{k,0}^\varepsilon)\|^2_{H^{-1/2}(\Gamma_Y)}\, dt\\
&\le C \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \int_0^T \left\{ \|p_k^\varepsilon - p_{k,0}^\varepsilon\|^2_{H^{1/2}(\Gamma_Y)} + \|\partial_t p_k^\varepsilon - \partial_t p_{k,0}^\varepsilon\|^2_{H^{1/2}(\Gamma_Y)} \right\} dt \le C\,\varepsilon\,.
\end{aligned}$$

In particular, $w^\varepsilon - W_I^\varepsilon \to 0$ in $L^2(0,T;H^{-1/2}(\Gamma))$.

Second difference: $W_I^\varepsilon - W_{II}^\varepsilon$. For the next difference we use weak convergence methods. By choosing an appropriate subsequence, we can assume $W_I^\varepsilon \rightharpoonup W_I$ and $W_{II}^\varepsilon \rightharpoonup W_{II}$ in $L^2(0,T;H^{-1/2}(\Gamma))$. Let θ be a smooth test-function in Ω_1. We find a sequence θ^ε that is piecewise constant in each Γ_k^ε such that $\theta^\varepsilon \to \theta$ in $H^1(\Omega_1)$; the simple argument for the existence of such a function θ^ε is sketched in Appendix C. For an arbitrary test-function in the time domain, $\varphi \in C^0([0,T],\mathbb{R})$, we consider the space-time test functions $\theta^\varepsilon\varphi$ with the property $\theta^\varepsilon\varphi \to \theta\varphi$ in $L^2(0,T;H^1(\Omega_1))$. The subsequent calculation uses the notation $\Gamma_T := \Gamma \times (0,T)$ and: (i) Re-writing the integral. (ii) The definition of W_{II}^ε which implies that W_I^ε and W_{II}^ε act identically on functions that are constant in the interfaces. (iii) The strong convergence $\theta^\varepsilon \to \theta$

and the weak convergence of $W_I^\varepsilon - W_{II}^\varepsilon$.

$$\int_{\Gamma_T} (W_I^\varepsilon - W_{II}^\varepsilon)\, \theta\varphi = \int_{\Gamma_T} (W_I^\varepsilon - W_{II}^\varepsilon)\, \theta^\varepsilon\varphi - \int_{\Gamma_T} (W_I^\varepsilon - W_{II}^\varepsilon)\, (\theta^\varepsilon - \theta)\varphi$$
$$= - \int_{\Gamma_T} (W_I^\varepsilon - W_{II}^\varepsilon)\, (\theta^\varepsilon - \theta)\varphi \to 0\,.$$

Since θ and φ are arbitrary, the above shows that the weak limits coincide, $W_I^\varepsilon - W_{II}^\varepsilon \rightharpoonup W_I - W_{II} = 0$ in $L^2(0,T;H^{-1/2}(\Gamma))$.

Third difference: $W_{II}^\varepsilon - W_{III}^\varepsilon$. In this step we use test functions φ, θ and approximations θ^ε as above. The subsequent calculation of the third difference uses: (i) The definitions of W_{II}^ε and W_{III}^ε, we write θ_k^ε for the constant value of θ^ε in Γ_k^ε. (ii) Inequality (4.14), which quantifies $N_*^\varepsilon \approx N_0$, with $\psi = p_{k,0}^\varepsilon$. (iii) $\alpha \le 1$ and Young's inequality. (iv) Boundedness of $\|p^\varepsilon|_\Gamma\|_{H^1(0,T;H^{1/2}(\Gamma))}$ and of $\|\theta^\varepsilon|_\Gamma\|_{L^2(0,T;H^{1/2}(\Gamma))}$.

$$\begin{aligned}
&\left|\int_{\Gamma_T} (W_{II}^\varepsilon - W_{III}^\varepsilon)\, \theta^\varepsilon\varphi\right| \le \left|\sum_{k\in K_\varepsilon} \varepsilon^{d-1} \int_0^T \int_{\Gamma_Y} (N_*^\varepsilon(p_{k,0}^\varepsilon) - N_0(p_{k,0}^\varepsilon))(t)\, \theta_k^\varepsilon\, \varphi(t)\, dt\right| \\
&\quad\le \alpha \sum_{k\in K_\varepsilon} \varepsilon^{d-1}\, |\theta_k^\varepsilon|\, \|N_*^\varepsilon - N_0\|_\varphi\, \|p_{k,0}^\varepsilon\|_{H^1(0,T;\mathbb{R})} \\
&\quad\le \|N_*^\varepsilon - N_0\|_\varphi \sum_{k\in K_\varepsilon} \varepsilon^{d-1} \left[\|p_{k,0}^\varepsilon\|^2_{H^1(0,T;\mathbb{R}))} + |\theta_k^\varepsilon|^2\right] \\
&\quad\le C\|N_*^\varepsilon - N_0\|_\varphi \to 0\,.
\end{aligned}$$

The strong convergence $\theta^\varepsilon\varphi \to \theta\varphi$ in $L^2(0,T;H^1(\Omega_1))$ implies $W_{II}^\varepsilon - W_{III}^\varepsilon \rightharpoonup W_{II} - W_{III} = 0$ in $L^2(0,T;H^{-1/2}(\Gamma))$.

Fourth difference: $W_{III}^\varepsilon - W_{IV}^\varepsilon = W_{III}^\varepsilon - \alpha N_0(p^\varepsilon)$. Again, φ, θ and θ^ε are chosen as above. We proceed by using: (i) The definition of W_{III}^ε, it is $N_0(p_{k,0}^\varepsilon)$ in the boundary piece Γ_k^ε; after rescaling, the function is $N_0(p_{k,0}^\varepsilon)$ in Γ_Y and it vanishes in $(0,1)^{d-1}\setminus\Gamma_Y$. We decompose the integral in a sum of, with a parametrization, integrals over $(0,1)^{d-1}\times\{0\} \equiv (0,1)^{d-1}$. (ii) Fubini and triangle inequality. (iii) Boundedness of φ. (iv) Lipschitz property of N_0 in the function space. (v) Cauchy-Schwarz inequality.

$$\begin{aligned}
&\left|\int_0^T \int_\Gamma (W_{III}^\varepsilon - \alpha N_0(p^\varepsilon))(t)\, \theta^\varepsilon\, \varphi(t)\, dt\right| \\
&\quad\le \sum_{k\in K_\varepsilon} \varepsilon^{d-1}\, |\theta_k^\varepsilon| \left|\int_0^T \left(\alpha\, N_0(p_{k,0}^\varepsilon) - \alpha \int_{(0,1)^{d-1}} N_0(p^\varepsilon(\varepsilon(k+y),.))\, dy\right)(t)\, \varphi(t)\, dt\right| \\
&\quad\le \alpha \sum_{k\in K_\varepsilon} \varepsilon^{d-1}\, |\theta_k^\varepsilon| \int_{(0,1)^{d-1}} \left|\int_0^T \left[N_0(p_{k,0}^\varepsilon(.))(t) - N_0(p^\varepsilon(\varepsilon(k+y),.))(t)\right]\, \varphi(t)\, dt\right| dy \\
&\quad\le C \sum_{k\in K_\varepsilon} \varepsilon^{d-1}\, |\theta_k^\varepsilon| \int_{(0,1)^{d-1}} \left\|N_0(p_{k,0}^\varepsilon(.))(t) - N_0(p^\varepsilon(\varepsilon(k+y),.))(t)\right\|_{L^2(0,T;\mathbb{R})}\, dy \\
&\quad\le C \sum_{k\in K_\varepsilon} \varepsilon^{d-1}\, |\theta_k^\varepsilon| \int_{(0,1)^{d-1}} \left\|p_{k,0}^\varepsilon(.) - p^\varepsilon(\varepsilon(k+y),.)\right\|_{H^1(0,T;\mathbb{R})}\, dy
\end{aligned}$$

$$\leq C \left\| \sum_{k\in K_\varepsilon} \mathbf{1}_{\varepsilon(k+(0,1)^{d-1})}\ p^\varepsilon_{k,0} - p^\varepsilon \right\|_{H^1(0,T;L^2(\Gamma))} \to 0\,.$$

The last convergence follows from boundedness of $p^\varepsilon \in H^1(0,T;H^{1/2}(\Gamma))$. Boundedness in this space implies that functions are well approximated by their local spatial averages.

Fifth difference: $W^\varepsilon_{IV} - \alpha N_0(p) = \alpha N_0(p^\varepsilon) - \alpha N_0(p)$. We use once more weak convergence methods with continuous test functions φ and θ as above. We claim that

$$\begin{aligned}&\int_0^T \int_\Gamma \left(\alpha N_0(p^\varepsilon) - \alpha N_0(p)\right)(t)\ \theta\varphi(t)\,dt\\ &\quad= \alpha \int_\Gamma \theta(y) \left[\int_0^T \left(N_0(p^\varepsilon(y,.)) - N_0(p(y,.))\right)(t)\varphi(t)\,dt\right] dy \to 0\,.\end{aligned} \tag{4.22}$$

This convergence can be concluded from Assumption 4.5 as follows. We choose a countable set of continuous functions, dense in $H^{-1}(0,T;\mathbb{R})$. We fix one of these functions and denote it as φ. We can now consider the family of maps $\Gamma \ni y \mapsto \langle (p^\varepsilon - p)(y,.), \varphi\rangle_{H^1,H^{-1}}$. This family is bounded in $L^2(\Gamma)$ and has the weak limit 0. Moreover, because of the $H^{1/2}$-regularity in space, the family is compact in $L^2(\Gamma)$. We therefore find a set $B \subset \Gamma$ with $|B| = 0$ and a subsequence $\varepsilon \to 0$ such that $\langle (p^\varepsilon - p)(y,.), \varphi\rangle \to 0$ for every $y \in G := \Gamma\backslash B$. Since φ is chosen from a countable set, we find a single set B and a single subsequence $\varepsilon \to 0$ such that $\langle (p^\varepsilon - p)(y,.), \varphi\rangle \to 0$ for every φ and every $y \in G := \Gamma \setminus B$. This implies $(p^\varepsilon - p)(y,.) \rightharpoonup 0$ in $H^1(0,T;\mathbb{R})$ for every $y \in G$.

Item 1. of Assumption 4.5 implies a pointwise almost everywhere weak convergence: $N_0(p^\varepsilon(y,.)) \rightharpoonup N_0(p(y,.))$ in $L^2(0,T;\mathbb{R})$ for every $y \in G$. This yields the pointwise (a.e. y) convergence of the squared bracket in (4.22).

In order to apply Lebesgue dominated convergence, we additionally need an upper bound for the sequence. The norms $g_\varepsilon : \Gamma \ni y \mapsto \|p^\varepsilon(y,.)\|_{H^1(0,T;\mathbb{R})}$ are bounded in $L^2(\Gamma)$, but, because of the $H^{1/2}$-regularity, they are even compact in $L^2(\Gamma)$. We therefore find a strongly convergent subsequence $g_\varepsilon \to g$ in $L^2(\Gamma)$. Because of Lipschitz continuity of N_0 in the function space, the integrands of (4.22) are dominated by (a multiple of) the strongly convergent sequence g_ε. This allows to apply Lebesgue dominated convergence and to conclude the convergence in (4.22).

Conclusion. All the differences converge weakly, we therefore have the weak convergence $w^\varepsilon - W = w^\varepsilon - \alpha N_0(p) \rightharpoonup 0$ and thus (4.18). □

4.4. **First order convergence.**

Assumption 4.8. *Given N^ε_Y as in (1.10), we assume that $\varepsilon^{-1}N^\varepsilon_Y$ is bounded and uniformly Lipschitz continuous. We construct the averages N^ε_* as in (1.12) and demand a first order convergence: For some Lipschitz continuous limit operator $N_1 : H^1(0,T;\mathbb{R}) \to L^2(0,T;\mathbb{R})$ with weak continuity property as in Item 1. of Assumption 4.5, we assume convergence $\varepsilon^{-1}N^\varepsilon_* \to N_1$ as in (4.14).*

Proposition 4.7 has the following consequence.

Corollary 4.9 (First order limit property of N^ε)**.** *Let Assumption 4.8 hold. Then $\varepsilon^{-1}N^\varepsilon$ converges to the pointwise defined operator αN_1. More precisely, for every* *sequence*

$$p^\varepsilon \to p \quad \textit{weakly in } H^1(0,T;H^{1/2}(\Gamma))\,, \tag{4.23}$$

there holds

$$\varepsilon^{-1}N^\varepsilon(p^\varepsilon) \to \alpha N_1(p) \quad \textit{weakly in } L^2(0,T;H^{-1/2}(\Gamma))\,. \tag{4.24}$$

Proof. It suffices to apply Proposition 4.7 to the sequence $\varepsilon^{-1}N^\varepsilon$. □

4.5. **Homogenization of active interfaces.** After these preparations, the proofs for Theorem 1.1 and Theorem 1.2 are straightforward.

Proof of Theorem 1.1. We study a sequence of pressures p^ε as in Theorem 1.1. We want to derive the effective wave equation (1.15)–(1.17).

Let $\varphi : \bar\Omega \times (0,T) \to \mathbb{R}$ be a smooth function. Using $\partial_t\varphi$ as a test-function in the equations (1.1)–(1.2) for p^ε and v^ε provides

$$\begin{aligned}
\int_{\Omega_T} \partial_t p\, \partial_t\varphi \leftarrow \int_{\Omega_T} \partial_t p^\varepsilon\, \partial_t\varphi &= -\bar p \int_{\Omega_T} \nabla\cdot v^\varepsilon\, \partial_t\varphi \\
&= \bar p \int_{\Omega_T} v^\varepsilon \cdot \nabla\partial_t\varphi - \bar p \int_\Gamma e_d\cdot v^\varepsilon\, \partial_t\varphi - \bar p \int_{\partial\Omega\setminus\Gamma} g\, \partial_t\varphi \\
&= -\bar p \int_{\Omega_T} \partial_t v^\varepsilon \cdot \nabla\varphi - \bar p \int_\Gamma N^\varepsilon(p^\varepsilon)\, \partial_t\varphi - \bar p \int_{\partial\Omega\setminus\Gamma} g\, \partial_t\varphi \\
&\to -\bar p \int_{\Omega_T} \partial_t v \cdot \nabla\varphi - \bar p\, \alpha \int_\Gamma N_0(p)\, \partial_t\varphi - \bar p \int_{\partial\Omega\setminus\Gamma} g\, \partial_t\varphi\,.
\end{aligned}$$

In the last line, we used Proposition 4.7.

Because of $\mu_\varepsilon \to 0$, the weak limits satisfy, in the sense of distributions (and hence also strongly) the relation $\bar\rho\partial_t v = -\nabla p$ in Ω_T. Writing this relation as $\bar p\partial_t v = -c^2\nabla p$, we therefore obtain

$$-\int_{\Omega_T} \partial_t^2 p\,\varphi = c^2 \int_{\Omega_T} \nabla p\,\nabla\varphi - \bar p\,\alpha \int_\Gamma N_0(p)\,\partial_t\varphi - \bar p \int_{\partial\Omega\setminus\Gamma} g\,\partial_t\varphi\,. \tag{4.25}$$

This relation encodes all equations. Considering φ with compact support in $\Omega\times(0,T)$, we find $\partial_t^2 p = c^2\Delta p$. On the other hand, using $\bar p/\bar\rho = c^2$, we find along Γ the boundary condition

$$\partial_\nu p = -\bar\rho\alpha\, \partial_t N_0(p)\,, \tag{4.26}$$

and (1.17) along $\partial\Omega\setminus\Gamma$. □

The proof of Theorem 1.2 can be performed similarly with Corollary 4.9.

Proof of Theorem 1.2. In the situation of Theorem 1.2, Theorem 1.1 can be applied; we find that p_0 satisfies the effective wave equation (1.15)–(1.17) with $N_0 = 0$.

We define v_0 through the equation $\bar\rho\partial_t v_0 = -\nabla p_0$ and calculate for (v_0,p_0) the relation $\partial_t[\bar p\nabla\cdot v_0] = -c^2\Delta p_0 = -\partial_t^2 p_0 = \partial_t[-\partial_t p_0]$. This shows that (v_0,p_0) solves the compressible Stokes equations (1.1)–(1.2) with viscosity $\mu_\varepsilon = 0$.

We define the first order corrector functions as

$$(v_1^\varepsilon, p_1^\varepsilon) := \frac{1}{\varepsilon}\left[(v^\varepsilon,p^\varepsilon) - (v_0,p_0)\right] \rightharpoonup (v_1,p_1) \quad \text{in } H^1(0,T;H^1(\Omega,\mathbb{R}^{d+1}))\,. \tag{4.27}$$

By linearity of these equations, also $(v_1^\varepsilon, p_1^\varepsilon)$ satisfies (1.1)–(1.2), or, more precisely

$$\bar\rho\, \partial_t v_1^\varepsilon = -\nabla p_1^\varepsilon - \varepsilon^{-1}\mu_\varepsilon \Delta v_0\,, \tag{4.28}$$

$$\partial_t p_1^\varepsilon + \bar p\, \nabla\cdot v_1^\varepsilon = 0\,. \tag{4.29}$$

Because of $\nu\cdot v_0 = 0$ along Γ, the boundary condition reads

$$\nu\cdot v_1^\varepsilon|_\Gamma = \varepsilon^{-1} N^\varepsilon(p^\varepsilon|_\Gamma)\,. \tag{4.30}$$

On the other part of the boundary, we have the homogeneous condition $\nu\cdot v_1^\varepsilon = 0$, since the prescribed normal velocity g is realized by v_0.

The equations can be treated exactly as in the last proof, the proof of Theorem 1.1. We exploit the assumption $\varepsilon^{-1}\mu_\varepsilon \to 0$ and $\varepsilon^{-1}N^\varepsilon(p^\varepsilon) \rightharpoonup \alpha N_1(p_0)$ in (4.24) of Corollary 4.9 to conclude the equivalent of (4.25), namely

$$-\int_{\Omega_T} \partial_t^2 p_1\, \varphi = c^2 \int_{\Omega_T} \nabla p_1\, \nabla\varphi - \bar p\, \alpha \int_\Gamma N_1(p_0)\, \partial_t\varphi\,. \tag{4.31}$$

This is the weak formulation of the claimed system, namely the effective wave equation

$$\partial_t^2 p_1 = c^2 \Delta p_1 \qquad \text{in } \Omega\,, \tag{4.32}$$

$$\partial_\nu p_1 = -\bar\rho\alpha\, \partial_t N_1(p_0) \qquad \text{along } \Gamma\,, \tag{4.33}$$

together with $\partial_\nu p_1 = 0$ on $\partial\Omega\setminus\Gamma$. □

5. Verification of the assumptions for Darcy chambers

We study the operators N_Y^ε and N_*^ε that are defined through the Darcy chamber in the case $M_\varepsilon\varepsilon \to M_* > 0$. We claim that the operators satisfy Assumptions 4.5 and 4.8. We will use the criteria of Remark 4.6, which can be used also for the operator family $\varepsilon^{-1}N_Y^\varepsilon$.

We start with the description of $N_Y^\varepsilon : H^1(0,T; H^{1/2}(\Gamma_Y)) \to L^2(0,T; H^{-1/2}(\Gamma_Y))$. As in (2.10), for a pressure evolution $P^\varepsilon \in H^1(0,T; H^{1/2}(\Gamma_Y))$, we are interested in the solution $p^\varepsilon \in L^2(0,T; H^1(\Sigma_Y))$ of

$$\frac{1}{\bar p} M_\varepsilon \varepsilon\; \partial_t p^\varepsilon = \Delta p^\varepsilon \tag{5.1}$$

with homogeneous Neumann boundary condition $\partial_\nu p^\varepsilon = 0$ along $\partial\Sigma_Y \setminus \Gamma_Y$ and the Dirichlet boundary condition $p^\varepsilon = P^\varepsilon$ along Γ_Y, together with trivial initial values $p^\varepsilon|_{t=0} \equiv 0$.

A priori estimates for this equation can be obtained as follows. We use the harmonic extension of the Dirichlet boundary data to all of Σ_Y (with Neumann boundary data), and denote this extension again by $P^\varepsilon \in H^1(0,T; H^1(\Sigma_Y))$. Testing the evolution equation (5.1) with $\varphi = p^\varepsilon - P^\varepsilon$, we find uniform bounds for $p^\varepsilon \in L^2(0,T; H^1(\Sigma_Y))$. Testing equation (5.1) with $\varphi = \partial_t(p^\varepsilon - P^\varepsilon)$ yields bounds for $p^\varepsilon \in H^1(0,T; L^2(\Sigma_Y))$.

The map N_*^ε is obtained by using constant (in space) boundary data, i.e., instead of P^ε we use boundary data $p_* \in H^1(0,T;\mathbb{R})$. As in (2.4), the map N_*^ε is given by

$$N_*^\varepsilon(p_*)(t) = \frac{\varepsilon}{\bar p\, |\Gamma_Y|} \int_{\Sigma_Y} \partial_t p^\varepsilon(t)\,. \tag{5.2}$$

The a priori estimates imply the boundedness of $p^\varepsilon \in H^1(0,T;L^2(\Sigma_Y))$. In particular, both N_*^ε and $\varepsilon^{-1}N_*^\varepsilon$ are linear and bounded. Similarly, using general arguments P^ε, one obtains also that N_Y^ε and $\varepsilon^{-1}N_Y^\varepsilon$ are linear and bounded.

Let us now check the properties for convergent arguments. The boundedness of the solution sequence $p^\varepsilon \in L^2(0,T;H^1(\Sigma_Y)) \cap H^1(0,T;L^2(\Sigma_Y))$ implies that we find a subsequence and a weak limit p in the same space, solving

$$\frac{1}{\bar{p}} M_* \partial_t p = \Delta p \,, \tag{5.3}$$

with the same boundary and initial conditions. The solution of this equation is unique; in particular, the whole sequence converges to p.

By boundedness of the solution sequence, (5.2) yields $N_0 = 0$ and the convergence property. We turn to the corresponding calculation for N_1, which is defined by

$$N_1(p_*)(t) = \frac{1}{\bar{p}\,|\Gamma_Y|} \int_{\Sigma_Y} \partial_t p(t) \,. \tag{5.4}$$

We recall that this was already stated in (2.12).

We continue verifying the criteria of Remark 4.6. Regarding (4.15), it remains to show that, for $p_*^\varepsilon \rightharpoonup p_*$ in $H^1(0,T;\mathbb{R})$, there holds

$$\frac{1}{\varepsilon} N_*^\varepsilon(p_*^\varepsilon) \rightharpoonup N_1(p_*) \quad \text{ in } L^2(0,T;\mathbb{R}) \,.$$

This follows from the weak convergence $\partial_t p^\varepsilon \rightharpoonup \partial_t p$ in $L^2(0,T;L^2(\Sigma_Y))$.

Appendix A. Elementary physics

Our aim in this section is to analyze which terms of the fundamental equations (1.1)–(1.2) are relevant and what are the limit laws that we can expect for the pressure. We collect typical orders of magnitude of the relevant quantities in Table 1. The quantities $\bar{p}$, $\bar{\rho}$, μ and c are physical constants for air around sea level, the variation p is taken as the acoustic pressure of a car in 10 m distance, ρ is calculated from the linearized gas-law. For the frequency we choose the typical test frequency of 1000 Hertz, the wave-length is calculated from $\lambda = \mathrm{c}/\omega$. Finally, the typical speed of particles in the sound wave, v, must be calculated from (1.1)–(1.2), we perform the calculation below.

symbol	units	typical value	meaning
$\bar{p}$	Pa	10^5	pressure
$\bar{\rho}$	kg/m^3	1	density
μ	kg/(m s)	$2 \cdot 10^{-5}$	dynamic viscosity
c	m/s	$3 \cdot 10^2$	speed of sound
p	Pa	10^{-1}	pressure variation
ρ	kg/m^3	10^{-6}	density variation
ω	1/s	10^3	frequency
λ	m	$3 \cdot 10^{-1}$	wave-length
v	m/s	$3 \cdot 10^{-4}$	speed of air in sound wave

Table 1. Units and orders of magnitude of physical quantities.

In this section, our convention is that all variables carry the corresponding unit, e.g., c $= 300\,$m/s. The unit of Pascal ist given as a force per area, $1\,\text{Pa} = 1\,\text{N/m}^2 = 1\,\text{kg/(m\,s}^2)$. In particular, an equivalent unit of the dynamic viscosity μ is Pa s.

A.1. **A limit equation away from boundaries.** It is interesting to compare the orders of magnitude of the three terms of (1.1) in air (by which we intend: far from any walls or obstacles). The units of all quantities is $\text{kg/(m}^2\,\text{s}^2)$. We estimate the effect of a time derivative by a multiplication with the frequency ω. With this rule, the order of the first term is obtained by calculating $\bar\rho\, v\, \omega$. With the data of our table, we find the order to be $1 \cdot 3 \cdot 10^{-4} \cdot 10^3 = 0.3$. To estimate the order of the pressure gradient, instead of taking the spatial derivative, we divide by λ. Evaluating p/λ, we estimate the order of the second term to be $10^{-1}/(3 \cdot 10^{-1}) = 1/3$. In fact, this calculation justifies the order of magnitude of the typical speed v. Regarding the latter, we note that one can also use the specific acoustic impedance from a table, it is about $400\,\text{Pa/(m/s)}$, which means that a typical speed can be obtained by dividing a typical pressure by this number, which yields $10^{-1}/400 = 0.25 \cdot 10^{-3}$ with units m/s and justifies our values. We recall that we provide the orders of physical quantities in order to justify approximations, we are not interested in exact numbers (which anyway depend on the specific problem).

Regarding approximations, let us now check the order of the third term in the momentum equation, the effect of the viscosity: Replacing second spatial derivative by $1/\lambda^2$, this term is of the order $2 \cdot 10^{-5} \cdot (1/3)^2 \cdot 10^2 \cdot 3 \cdot 10^{-4} = (2/3) \cdot 10^{-7}$. We conclude that this term is negligable in comparison with the other two terms of (1.1), which are of order 1.

Accordingly, we will neglect the term containing μ in the following calculation. We take the time derivative of (1.2) and find

$$\partial_t^2 p = -\bar p \nabla \cdot (\partial_t v) = \frac{\bar p}{\bar\rho} \Delta p = c^2 \Delta p\,. \tag{A.1}$$

This provides the wave equation. The speed of sound is given by $c^2 = \bar p/\bar\rho$, which is of the order $1 \cdot 10^5$ with units $\text{m}^2/\text{s}^2 = (\text{Pa}\,\text{m}^3)/\text{kg}$.

A.2. **Description of chambers.** We next analyze the flow in chambers. Our interest is to see which terms in the equations are of relevance. We use the typical quantities of Table 2, which are taken from a typical sound absorbing wall, e.g., used in a lecture room. It also reflects the typical values of the commercial element that is shown in Figure 1. The table contains ε, which is a parameter that can be chosen freely. We use here the periodicity of the hole pattern and divide by one meter, which is a typical length scale of Ω.

Viscosity in chambers. In order to have a feeling for typical velocities in the chamber, let us perform the following calculation that should provide an upper bound j: Assuming that the pressure in some part of the chamber is constant in time, pressure differences in the chamber are in the order of p, i.e., in the order 10^{-1}, the unit is Pa. The pressure gradient is then of the order of $p/D \approx 10^{-1}/(5 \cdot 10^{-2}) = 2$, the unit is Pa/m. We consider equation (1.1), replace the time derivative by a multiplication with ω and conclude, on this basis, that typical velocities in the chamber are of the order $p/(D\omega\bar\rho) \approx 2 \cdot 10^{-3}$, the unit is $(\text{Pa}\,\text{s}\,\text{m}^2)/\text{kg} = \text{m/s}$. This rough estimate shows that the upper bound j might be about a factor 10 larger than the outside velocity v.

symbol	units	typical value	meaning
D	m	$5 \cdot 10^{-2}$	periodicity of holes (and typical diameter of chambers)
α	1	10^{-1}	area fraction of holes
ε	1	$5 \cdot 10^{-2}$	Our choice of a small parameter
j	m/s	$2 \cdot 10^{-3}$	bound for the velocity in chambers
M_ε	Pa s/m^2	10^4	flow resistivity = Darcy coefficient

Table 2. Geometry and physical quantities related to holes and chambers.

We next assume that the order of second derivatives are dictated by the typical dimensions; to be on the save side, we do not use D, but the reduced unit of $\delta = 1 \cdot 10^{-2}\,\mathrm{m}$, which might reflect diameters of channels. We find the following upper bound for the order of the viscous term: $\mu\Delta v \sim 2\cdot 10^{-5}\cdot 10^4\cdot 2\cdot 10^{-3} = 4\cdot 10^{-4}$, the unit is $\mathrm{kg}/(\mathrm{m}^2\,\mathrm{s}^2) = \mathrm{Pa/m}$. Above, we have calculated for the other two terms (one is the pressure gradient) in the Stokes equation the order 1. We can therefore conclude that viscous terms are at least three or four orders of magnitude smaller than the other terms. We can certainly neglect their effect for the considered data.

Darcy law in chambers. Regarding the Darcy law in the chambers, we find, for our data, $M_\varepsilon \varepsilon \omega/\bar{p} \approx 10^4 \cdot 5 \cdot 10^{-2} \cdot 10^3/10^5 = 5$, the unit is $1/\mathrm{m}^2$. We see that the two sides of (2.10) are balanced.

The order of the pre-factor in (2.15) is $\bar{\rho}\omega^2/\bar{p} \approx 1 \cdot 10^6/10^5 = 10$, the unit is $\mathrm{kg}/(\mathrm{m}^3\mathrm{s}^2\mathrm{Pa}) = 1/\mathrm{m}^2$. A pre-factor of the order 10 can imply that p_1 is larger than p_0, and the correction with $\varepsilon\, p_1$ can be relevant for $\varepsilon = 0.05$.

On the order of nonlinear terms. For completeness, we additionally estimate the order of the inertia term $\bar{\rho}(v\cdot\nabla)v$ that was neglected in (1.1). Using once more that j is of the order $2\cdot 10^{-3}$ and derivatives are of the order that is given by dividing by $\delta = 10^{-2}\,\mathrm{m}$, we find that the order of $\bar{\rho}(v\cdot\nabla)v$ is $4\cdot 10^{-6}/10^{-2} = 4\cdot 10^{-4}$. The unit is, once more, $\mathrm{kg}/(\mathrm{m}^2\,\mathrm{s}^2)$. This, again, has to be compared with values of order 1. We conclude that it is justified to neglect the inertia term. In particular, a turbulent effect as indicated in Figure 1 is certainly not relevant for the considered data.

Helmholtz resonator geometry. The chambers can also be constructed such that they produce a resonance. Essentially, one has to connect the chambers with very thin channels to the domain Ω. When the pressure in the chamber is q and the pressure outside is p, then $p - q$ is a driving force for the average upward velocity j and we can expect a law such as $\bar{\rho}\partial_t j = L^{-1}(p-q)$, where L stands for the length of the channel. Mass conservation and gas law in the chamber require a law $V\partial_t q = -\bar{p}\int \nabla\cdot v = \bar{p}\,A\,j$, where A is the opening area of the channel and V is the volume of the chamber. Taking the time derivative and inserting the first law, we find $V\partial_t^2 q = \bar{p}\,A\,\partial_t j = c^2 A\,L^{-1}(p-q)$. This elementary calculation already shows that, when $c^2 A/(VL)$ is comparable to the squared frequency ω^2, a resonance effect is possible.

We make an estimate for the appropriate size of the channel cross-section using $V = D^2\cdot 10^{-2}\,\mathrm{m}$, $L = 10^{-2}\,\mathrm{m}$, and other data as collected in the table. We find for the critical surface area A_*, defined as $A_* = \omega^2\,V\,L/c^2$, the value $10^6\cdot 25\cdot 10^{-6}\cdot 10^{-2}/10^5 = 2.5\cdot 10^{-6}$, the unit is m^2. The square root of this number is about $10^{-3}\,\mathrm{m}$ and we conclude that the diameter of the channels must be in the order of a millimeter in order to create the resonance: Very thin holes, so called "microperforations" are necessary to create the effect in the considered situation. Indeed, among others, holes with a diameter of 3 mm are considered in [12] in this context. In order to perform a limit analysis of this effect, one usually has to use an ε-dependent scaling of the holes. In [5, 13, 14] they are chosen as follows: Order ε^3 in two space dimensions and order ε^2 in three space dimensions.

We note that a resonance effect needs even smaller opening cross sections A_* when the frequency is smaller: Considering the frequency 100/Hz, we reduce the frequency by a factor of 10 which lowers A_* by a factor of 100 and hence the typical channel diameter $\sqrt{A_*}$ by a factor of 10. For this frequency, one has to use hole sizes well below a millimeter.

Appendix B. Comments on the formal effective system (1.21)–(1.22)

In order to justify the system of Remark 1.3, one would proceed as follows. The aim is to compare the solution p^ε of the original system with the solution $\tilde{p}^\varepsilon$ of the effective system (1.21)–(1.22). In Theorem 1.2 appear two solutions p_0 and p_1 from which we can define $\hat{p}^\varepsilon := p_0 + \varepsilon p_1$. The difference that we have to analyze can be written as $p^\varepsilon - \tilde{p}^\varepsilon = (p^\varepsilon - \hat{p}^\varepsilon) + (\hat{p}^\varepsilon - \tilde{p}^\varepsilon)$. Theorem 1.2 provides with (1.18) for the first contribution

$$\frac{1}{\varepsilon}\left(p^\varepsilon - \hat{p}^\varepsilon\right) \rightharpoonup 0\,. \tag{B.1}$$

Our wish is therefore to show

$$q^\varepsilon := \frac{1}{\varepsilon}\left(\hat{p}^\varepsilon - \tilde{p}^\varepsilon\right) \rightharpoonup 0\,. \tag{B.2}$$

The quantity q^ε solves a simple system of equations: It satisfies the homogeneous wave equation on Ω with trivial initial data and homogeneous Neumann conditions on $\partial\Omega \setminus \Gamma$. Along Γ holds

$$\partial_\nu q^\varepsilon = -\bar{\rho}\alpha\left[\partial_t N_1(p_0) - \partial_t N_1(\tilde{p}^\varepsilon)\right]\,. \tag{B.3}$$

The two arguments on the right hand side are p_0 and $\tilde{p}^\varepsilon$, the latter can be written as

$$\tilde{p}^\varepsilon = \hat{p}^\varepsilon - \varepsilon q^\varepsilon = p_0 + \varepsilon p_1 - \varepsilon q^\varepsilon\,. \tag{B.4}$$

Hence, formally, the right hand side of (B.3) can be estimated by one term of order ε and another term of order $\varepsilon q^\varepsilon$. On that basis, one would expect that (B.3) (which is the driving force in the wave equation for q^ε) leads to an estimate that q^ε is of the order ε.

Unfortunately, we do not see how to make this argument rigorous. One of the obstacles is the appearance of two time derivatives in (B.3), one of them being the explicit time derivative, the other is hidden in the operator properties of N_1.

Appendix C. An approximate sequence of test-functions

We want to sketch the construction of a sequence of functions that was used repeatedly in Proposition 4.7. The aim is to find, for a given smooth test-function θ in Ω_1, a sequence θ^ε that is piecewise constant in each Γ_k^ε such that $\theta^\varepsilon \to \theta$ in $H^1(\Omega_1)$.

To verify the existence of θ^ε, we use a decomposition and a rescaling as in Lemma 4.2: We work with cubes $S_k^\varepsilon := \varepsilon(k+(0,1)^{d-1})\times(-\varepsilon,0)$. The function $\theta|_{S_k^\varepsilon} : S_k^\varepsilon \to \mathbb{R}$ is modified to a function $\theta^\varepsilon|_{S_k^\varepsilon} : S_k^\varepsilon \to \mathbb{R}$, which has the boundary values of θ on $\partial S_k^\varepsilon \setminus \varepsilon(k+(0,1)^{d-1})\times\{0\}$, but a constant value on Γ_k^ε. This is possible with $\|\theta^\varepsilon-\theta\|_\infty \le C$ and $\|\nabla\theta^\varepsilon-\nabla\theta\|_\infty \le C$, where C depends on the C^1-norm of θ.

This defines a function θ^ε on the strip $S_\varepsilon := \Gamma\times(-\varepsilon,0)$, we set $\theta^\varepsilon=\theta$ on $\Omega_1\setminus S_\varepsilon$. The resulting function is continuous and there holds $\|\theta^\varepsilon-\theta\|^2_{H^1(\Omega_1)} \le C\,|S_\varepsilon| \le C\,\varepsilon$.

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