Representation of solutions to wave equations with profile functions

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Abstract: Solutions to the wave equation with constant coefficients in $\mathbb{R}^d$ can be represented explicitly in Fourier space. We investigate a reconstruction formula, which provides an approximation of solutions $u(.,t)$ to initial data $u_0(.)$ for large times. The reconstruction consists of three steps: 1) Given u_0, initial data for a profile equation are extracted. 2) A profile evolution equation determines the shape of the profile at time $\tau = \varepsilon^2 t$. 3) A shell reconstruction operator transforms the profile to a function on $\mathbb{R}^d$. The sketched construction simplifies the wave equation, since only a one-dimensional problem in an $O(1)$ time span has to be solved. We prove that the construction provides a good approximation to the wave evolution operator for times t of order ε^{-2}.

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1 Introduction

In many applications, one observes solutions of a wave equation that have the shape of a ring. This can be understood as an effect of large times: the initial data of the problem are concentrated in a bounded domain and send waves in every direction. Each of the different waves travels at the same speed c and after a large time t, the perturbance of the medium is visible mainly at the distance ct. We observe a ring-like structure (shell-like in three dimensions).

In more mathematical terms, we are interested (in the simplest setting) in the long time behavior of solutions u to the linear wave equation

$$\partial_t^2 u(x,t) - c^2 \Delta u(x,t) = 0\,. \tag{1.1}$$

In this equation, $x \in \mathbb{R}^d$ is the spatial variable and $t \in [0,\infty)$ is the time variable, the operator $\Delta = \sum_{j=1}^d \partial_{x_j}^2$ acts only on the spatial variables, and $c > 0$ is

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a prescribed velocity parameter. The equation is complemented with the initial conditions $u(x,0) = u_0(x)$ and $\partial_t u(x,0) = u_1(x)$.

Our aim is to characterize the shape of solutions in the limit of large times. We write $\tau = \varepsilon^2 t$ for a rescaled time variable. Our result gives an approximate formula for the function $x \mapsto u(x, \varepsilon^{-2}\tau)$. The approximate formula is given by a sequential execution of three operators: An operator $\mathcal{R}$ extracts from the initial data u_0 initial data for a profile evolution equation. An evolution operator J_b describes the evolution of the profile. Finally, a shell operator $\mathcal{S}$ reconstructs, from a profile V, a shell like solution u; the operator $\mathcal{S}$ maps the profile V to a function u which looks like V along every ray through 0, whereby the profile is centered in the point ct.

We present a mathematical proof that the described reconstruction operator provides, in the large time limit $\varepsilon \to 0$, an approximation of the solution u. The result is based on a stationary phase method.

The equation (1.1) is a partial differential equation with constant coefficients on the full space $\mathbb{R}^d$. This allows to write the solution explicitely in terms of its Fourier transform. One solution of the wave equation is given by

$$\hat{u}(k,t) = e^{-ic|k|t}\hat{u}_0(k)\,, \tag{1.2}$$

another by the same formula upon replacing $-ic|k|t$ by $+ic|k|t$. In this work, we always assume that the initial data u_1 are such that the solution u is given by (1.2). This is not a restriction. General initial data can be treated by an appropriate decomposition, see [9] for details.

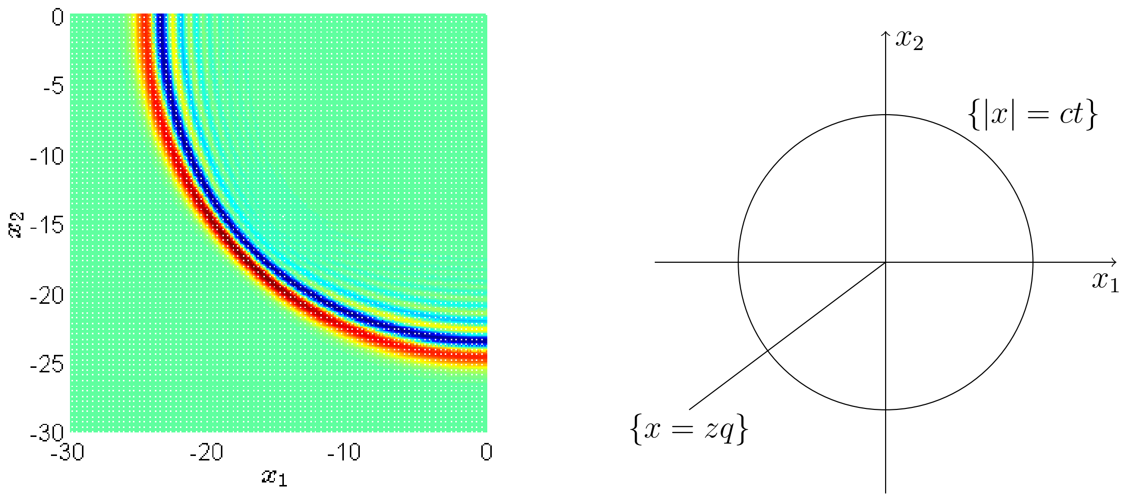

Figure 1: Left: The solution to a wave equation. The initial data are essentially supported in a unit ball around $x = 0$. The wave speed of the equation is $c = 1$. After time $t = 25$, the disturbance of the medium is concentrated in a neighborhood of the ring $\{x \in \mathbb{R}^2 : |x| = ct\}$. The figure shows one quadrant and was calculated by T. Dohnal. Right: Sketch for the construction of the shell operator $\mathcal{S}$. A profile $z \mapsto V(z;q)$ with a direction parameter $q \in S^{d-1}$ is used along the ray $x = zq$; the profile is centered in order to have the main pulse near $\{x \in \mathbb{R}^2 : |x| = ct\}$.

A simplified version of our main result can be stated as follows. We define a reconstruction operator in Definition 2.6. Essentially, the operator extracts profile information from $\hat{u}_0$ and maps the profiles to a shell-like solution. The shell solution is obtained by using the profiles, in each direction, and centering them at the distance $c\tau/\varepsilon^2$ from the origin.

Theorem 1.1 (Simplified version of the main theorem)**.** *Let $\hat{u}_0 : \mathbb{R}^d \to \mathbb{C}$ be smooth initial data with compact support with $d \in \{1, 2, 3\}$. For arbitrary $\rho > 0$, let $\hat{\mathcal{Q}}_0^\rho$ be the reconstruction operator of Definition 2.6. Then, for every $\tau > 0$ and every $k \in \mathbb{R}^d$ in Fourier space with $|k| > \rho$, the reconstruction is similar to the solution of the wave equation: As $\varepsilon \to 0$,*

$$\hat{\mathcal{Q}}_0^\rho \hat{u}_0(k, \tau/\varepsilon^2) - e^{-ic|k|\tau/\varepsilon^2} \hat{u}_0(k) \to 0 \,. \tag{1.3}$$

Our main result is stated in Theorem 2.7 below, and it treats a much more general situation. It allows to treat weakly dispersive wave equations such as, e.g., $\partial_t^2 u(x,t) - c^2 \Delta u(x,t) + \varepsilon^2 B \Delta^2 u = 0$. We only assume that the solution can be represented in Fourier-space as $\hat{u}(k,t) = e^{-ic|k|t} e^{-ib(|k|)\varepsilon^2 t} \hat{u}_0(k)$ for some dispersion function $b = b(|k|)$. Our theorem yields that the solution u can be obtained as described above with a shell-like reconstruction from profiles. In the case of weakly dispersive wave equations, the profile equation becomes nontrivial: In the leading order case $b(|k|) = b_3|k|^3$, we obtain a linearized KdV equation $\partial_\tau V(z,q,\tau) = b_3 \partial_z^3 V(z,q,\tau)$ for the evolution of the profile $V(.,q,\tau)$ in the direction $q \in S^{d-1}$. We note that the factor in the equation is ε^2, since the equation appears as an effective equation for a problem with micro-structure with length scale ε, see [4].

Let us discuss briefly the complexity of the two problems under consideration. In order to solve a dispersive wave equation on a time interval of order ε^{-2}, one has to use a computational spatial domain of order $(\varepsilon^{-2})^d$, the complexity is of the order $(\varepsilon^{-2})^{d+1}$. To calculate the approximation by the shell reconstruction operator, one has to extract, for every direction $q \in S^{d-1}$, a profile function (which is concentrated on a domain of order 1. One has to solve (again, for every q) a profile evolution equation on a time interval of order 1. In the third step, the profiles are combined to a shell like solution. In particular, the complexity of the reconstruction process is independent of ε.

Literature

It is a classical problem to investigate the long time behavior of solutions to a wave equation. In fact, most research treats more difficult problem classes than we treat here. We recall that only linear wave equations with constant coefficients are investigated here; we assume that the solution can be described in Fourier space by a multiplication operator that uses the dispersion relation of the equation.

One of the more difficult problem classes regards homogenization. In this context, one is interested in a medium that has a periodic microstructure and asks for the behavior of solutions after long times. An important contribution in this area is [8]; essentially, the second order wave equation in a heterogeneous medium can be replaced by a weakly dispersive wave equation in a homogeneous medium. Rigorous results have been obtained in [4] and [5], numerical approaches are discussed in [1]. The same question in a stochastic medium was addressed in [3].

Our analysis can be understood as a continuation and improvement of [9], where the authors studied the long time behavior for a lattice wave equation. They derived, on the one hand, that a weakly dispersive wave equation in a homogeneous medium is a valid replacement for the lattice wave equation. On the other hand, [9] introduced the shell reconstruction operator; one result regards the approximate reconstruction of the solution from profiles that are obtained as solutions of a linearized KdV equation.

The work at hand studies the shell reconstruction operator on a more abstract level. We do not apply the results to the discrete wave equation (even though this is possible); we merely investigate an arbitrary evolution of initial data in Fourier space, where the evolution is given by harmonic functions through some dispersion relation. For very general equations, we show that the shell reconstruction operator provides an approximation of the solution.

We improve the results of [9] in two ways. On the one hand, we can now treat the dimension $d = 3$. On the other hand, we can decouple the effect of dispersion from the analysis of the shell operator. This makes the analysis more flexible.

An important tool for our method is a stationary phase method. We show the necessary result in Section 4. It regards the convergence of an oscillatory integral on the sphere. For other stationary phase results we refer to the book [10].

In [2], dispersive limit equations are derived for a linear wave equation in the context of homogenization. For the long time behavior of waves in a nonlinear system we mention [6]. The monograph [7] contains many representation formulas for solutions of equations related to the wave equation.

2 The reconstruction operator

We now introduce the three operators that were announced in the introduction. The concatenation of these operators provides the reconstruction operator $\mathcal{Q}$. In the construction, we have to switch several times between the physical space and the Fourier space.

On the space $X := L^2(\mathbb{R}^d; \mathbb{C})$ we use the standard d-dimensional Fourier transform $\mathcal{F}_d : X \to X$,

$$(\mathcal{F}_d u_0)(k) := \hat{u}_0(k) := \int_{\mathbb{R}^d} u_0(x) e^{-ik\cdot x}\, dx\,. \tag{2.1}$$

The inverse Fourier transform is $\mathcal{F}_d^{-1} : X \to X$,

$$(\mathcal{F}_d^{-1}\hat{u}_0)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot k}\hat{u}_0(k)\, dk\,.$$

By Parseval's identity, $\frac{1}{(2\pi)^{d/2}} \|\mathcal{F}_d u_0\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}$.

The first operator of our construction has the character of a restriction: functions on $\mathbb{R}^d$ are mapped to a family of functions on $\mathbb{R}$ (parametrized by a directional variable q). We use the space

$$X_S := L^2(\mathbb{R} \times S^{d-1}; \mathbb{C}), \tag{2.2}$$

where $S^{d-1} \subset \mathbb{R}^d$ denotes the $(d-1)$-dimensional sphere.

Definition 2.1 (The operator $\mathcal{R}$)**.** *The linear operator $\mathcal{R}$ maps functions on $\mathbb{R}^d$ to one-dimensional profiles. We define $\mathcal{R} : X \to X_S$ through*

$$\mathcal{R}\hat{u}_0(\xi, q) := \left(\frac{|\xi|}{2\pi i}\right)^{(d-1)/2} \mathbf{1}_{\{\xi>0\}}\ \hat{u}_0(|\xi|q). \tag{2.3}$$

It is straightforward to see that

$$\|\mathcal{R}\hat{u}_0\|_{X_S} = (2\pi)^{-(d-1)/2} \|\hat{u}_0\|_X . \tag{2.4}$$

Indeed,

$$\begin{aligned}
\|\mathcal{R}\hat{u}_0(\xi, q)\|_{X_S}^2 &= \int_{\mathbb{R}} \int_{S^{d-1}} |\mathcal{R}\hat{u}_0(\xi, q)|^2 \, dS(q) d\xi \\
&= \frac{1}{(2\pi)^{(d-1)}} \int_0^\infty \int_{S^{d-1}} |\xi|^{d-1} \left|\hat{u}_0(|\xi|q)\right|^2 \, dS(q) d\xi \\
&= \frac{1}{(2\pi)^{(d-1)}} \int_{\mathbb{R}^d} |\hat{u}_0(x)|^2 \, dx = \frac{1}{(2\pi)^{(d-1)}} \|\hat{u}_0\|_X^2 .
\end{aligned}$$

In order to obtain our results we have to regularize the function

$$W(\xi) := |\xi|^{(d-1)/2} \mathbf{1}_{\{\xi>0\}} .$$

For a small parameter $\rho > 0$ and $d \in \{1, 2, 3\}$ we consider functions W_ρ with the following properties: $W_\rho \in C^{d-1}(\mathbb{R}; \mathbb{R})$ and

$$W_\rho(\xi) = 0 \ \forall \xi \le 0, \quad W_\rho(\xi) = |\xi|^{(d-1)/2} \ \forall \xi \ge \rho, \quad 0 \le W_\rho(\xi) \le |\xi|^{(d-1)/2} \ \forall \xi \ge 0 . \tag{2.5}$$

We use the smooth functions W_ρ to define regularized versions of the operator $\mathcal{R}$.

Definition 2.2 (The operator $\mathcal{R}_\rho$)**.** *Let $\rho > 0$ and let W_ρ be as in* (2.5)*. The linear operator $\mathcal{R}_\rho : X \to X_S$ is defined through*

$$\mathcal{R}_\rho \hat{u}_0(\xi, q) := \left(\frac{1}{2\pi i}\right)^{(d-1)/2} W_\rho(\xi) \hat{u}_0(|\xi|q). \tag{2.6}$$

As in (2.4), by $0 \le W_\rho(\xi) \le |\xi|^{(d-1)/2}$, the regularized operators satisfy the estimate $\|\mathcal{R}_\rho \hat{u}_0\|_{X_S} \le (2\pi)^{-(d-1)/2} \|\hat{u}_0\|_X$.

The next operator associates to an initial profile (in Fourier space) an evolution of profiles (in Fourier space).

Definition 2.3 (The operator J_b)**.** *Let $b : \mathbb{R} \to \mathbb{R}$ be a function. The linear operator J_b maps the (Fourier transform of) a profile to an evolution of profiles. We define the linear operator $J_b : X_S \to L^\infty(0, \infty; X_S)$ through*

$$(J_b \hat{V}_0)(\xi, q, \tau) := e^{-ib(\xi)\tau} \hat{V}_0(\xi, q) . \tag{2.7}$$

We emphasize that the time variable is τ and not t, which means that the evolution of profiles is studied in a new time scale. We will use $\tau = \varepsilon^2 t$, where $\varepsilon > 0$ is a small scaling variable. In the following, two choices of b will be relevant.

1) For $b(\xi) = 0$ one has $J_b \hat{V}_0(\xi, q, \tau) = J_0 \hat{V}_0(\xi, q, \tau) = \hat{V}_0(\xi, q)$. In this case, the time evolution of the profile is trivial, the profile remains unchanged. In physical space, this operator describes the trivial evolution equation $\partial_\tau V(z, q, \tau) = 0$ with initial datum $V(z, q, 0) = (\mathcal{F}_1^{-1} \hat{V}_0(\cdot, q))(z)$.

2) For $b(\xi) = b_3 \xi^3$ with $b_3 \in \mathbb{R}$ one has $(J_b \hat{V}_0)(\xi, q, \tau) = e^{-ib_3 \xi^3 \tau} \hat{V}_0(\xi, q)$. With this choice, the profile evolution in physical space is given by the linearized KdV-equation $\partial_\tau V(z, q, \tau) = b_3 \partial_z^3 V(z, q, \tau)$ with initial data $V(z, q, 0) = (\mathcal{F}_1^{-1} \hat{V}_0(\cdot, q))(z)$.

We note that, independently of the choice of the function b, for every time instance τ, the operator $J_b|_\tau$ is an isometry because of $|e^{-ib(\xi)\tau}| = 1$.

We finally introduce the shell reconstruction operator. The reconstruction was also used in [9]; it maps a family of profiles to a ring-like ($d = 2$) or shell like ($d = 3$) function on $\mathbb{R}^d$. An important ingredient is the rescaling factor $(ct)^{-(d-1)/2}$, which has the effect that L^2-norms of reconstructed functions are bounded.

Definition 2.4 (The operator $\mathcal{S}$)**.** *We introduce an operator $\mathcal{S}$ that maps profiles to functions on $\mathbb{R}^d$. For a small parameter $\varepsilon > 0$ we define the linear operator $\mathcal{S} : L^\infty(0, \infty; X_S) \to L^\infty(0, \infty; X)$ through*

$$(\mathcal{S}V)(x, t) := \frac{1}{(ct)^{(d-1)/2}} \mathbf{1}_{\{|x| < 2ct\}} \, V\left(|x| - ct, \frac{x}{|x|}, \varepsilon^2 t\right) . \tag{2.8}$$

The operator $\mathcal{S}$ constructs, starting from a slowly varying function V, a shell-like solution. The main pulse of the shell-like solution is near $|x| = ct$ and moves with constant speed c; its profile is given by V. The construction depends on the small parameter ε, which we suppress in most calculations for the sake of readability.

Lemma 2.5. *The operator $\mathcal{S} : L^\infty(0, \infty; X_S) \to L^\infty(0, \infty; X)$ is bounded. It satisfies, for every $V \in L^\infty(0, \infty; X_S)$*

$$\|\mathcal{S}V\|_{L^\infty(0,\infty;X)} \le 2^{(d-1)/2} \|V\|_{L^\infty(0,\infty;X_S)} .$$

Proof. For every $t \in (0, \infty)$ one has

$$\begin{aligned} \|\mathcal{S}V(\cdot, t)\|_X^2 &= \int_{\mathbb{R}^d} \frac{1}{(ct)^{d-1}} \left| V\left(|x| - ct, \frac{x}{|x|}, \varepsilon^2 t\right)\right|^2 \mathbf{1}_{\{|x|<2ct\}} \, dx \\ &= \int_0^{2ct} \int_{S^{d-1}} \frac{r^{d-1}}{(ct)^{d-1}} \left| V\left(r - ct, q, \varepsilon^2 t\right)\right|^2 \, dS(q) \, dr \\ &\le 2^{d-1} \int_0^{2ct} \int_{S^{d-1}} \left| V\left(r - ct, q, \varepsilon^2 t\right)\right|^2 \, dS(q) \, dr \\ &\le 2^{d-1} \|V(\cdot, \cdot, \varepsilon^2 t)\|_{X_S}^2 \le 2^{d-1} \|V\|_{L^\infty(0,\infty;X_S)}^2 , \end{aligned}$$

which provides the claim. □

With the above operators at hand we are now in the position to introduce our main object of interest, the reconstruction operator $\mathcal{Q}_b$. It can be described in words as the following concatenation: From a Fourier transform $\hat{u}_0$ of initial values, profile initial data are extracted with the operator $\mathcal{R}$, then the profile evolution J_b is applied and the profile is interpreted in physical space with the inverse Fourier transform $\mathcal{F}_1^{-1}$. Finally, the shell operator $\mathcal{S}$ is applied in order to reconstruct an evolution of functions on $\mathbb{R}^d$.

Definition 2.6 (The reconstruction operator Q_b)**.** *We define the linear reconstruction operator* $\mathcal{Q}_b : X \to L^\infty(0,\infty;X)$ *through*

$$\mathcal{Q}_b = \mathcal{S} \circ \mathcal{F}_1^{-1} \circ J_b \circ \mathcal{R}\,. \tag{2.9}$$

The operator in Fourier space is denoted as $\hat{\mathcal{Q}}_b := \mathcal{F}_d \circ \mathcal{Q}_b$. *For* $\rho > 0$ *we define the regularized operators* $\mathcal{Q}_b^\rho$ *and* $\hat{\mathcal{Q}}_b^\rho$ *by replacing* $\mathcal{R}$ *with* $\mathcal{R}_\rho$.

We can now state our main result, which compares two objects. On the one hand, the solution of a (dispersive) wave equation, which is given by a multiplication with $e^{-i\left(c|k|/\varepsilon^2 + b(|k|)\right)\tau}$ in Fourier space. On the other hand, the reconstruction $\hat{\mathcal{Q}}_b\hat{u}_0$. The result is that the two operators coincide in the limit $\varepsilon \to 0$.

Theorem 2.7 (Approximation result for reconstructions)**.** *Let* $\hat{u}_0 \in X$ *be continuous initial data with compact support, let the dimension be* $d \in \{1,2,3\}$, *and let* $b : \mathbb{R} \to \mathbb{R}$ *be a dispersion function. Let* ρ *and* W_ρ *be as in* (2.5). *We assume that the regularized profile evolution* $V^\rho := (\mathcal{F}_1^{-1} \circ J_b \circ \mathcal{R}_\rho)\hat{u}_0$ *satisfies the smoothness and decay properties of Assumption 3.1. Then, for every* $\tau > 0$ *and every* $k \in \mathbb{R}^d$ *with* $|k| > \rho$, *there holds*

$$(\hat{\mathcal{Q}}_b^\rho \hat{u}_0)(k, \tau/\varepsilon^2) e^{ic|k|\tau/\varepsilon^2} \to e^{-ib(|k|)\tau}\hat{u}_0(k)\,. \tag{2.10}$$

Moreover, for every $\tau > 0$, *there holds weak convergence for the non-regularized reconstruction operators,*

$$(\hat{\mathcal{Q}}_b \hat{u}_0)(k, \tau/\varepsilon^2) e^{ic|k|\tau/\varepsilon^2} \rightharpoonup e^{-ib(|k|)\tau}\hat{u}_0(k) \tag{2.11}$$

weakly in $L^2(\mathbb{R}^d)$ *as functions in* $k \in \mathbb{R}^d$ *as* $\varepsilon \to 0$.

As outlined in Remark 3.2 below, the assumptions of Theorem 2.7 are satisfied as soon as $\hat{u}_0$ and b are sufficiently smooth.

The fundamental approximation result of this article is presented in the next section as Theorem 3.3. Our main theorem, Theorem 2.7 above, can be regarded as a corollary thereof. We present its proof here, using Theorem 3.3.

Proof of Theorem 2.7. We have to show a pointwise and a weak convergence.

Step 1: Pointwise convergence of regularized profiles. We set $\hat{V}^\rho := (J_b \circ \mathcal{R}_\rho)(\hat{u}_0)$ and apply Theorem 3.3 to these regularized profile evolutions $\hat{V}^\rho = \hat{V}^\rho(\xi, q, \tau)$. The function $V^\rho := \mathcal{F}_1^{-1}\hat{V}^\rho$ satisfies Assumption 3.1 by the assumptions of Theorem 2.7. Theorem 3.3 provides, for fixed $k \neq 0$,

$$(\mathcal{F}_d \circ \mathcal{S})(V^\rho)(k, \tau/\varepsilon^2) e^{ic|k|\tau/\varepsilon^2} \to \left(\frac{|k|}{2\pi i}\right)^{-(d-1)/2} \hat{V}^\rho\left(|k|, \frac{k}{|k|}, \tau\right) \tag{2.12}$$

as $\varepsilon \to 0$. It remains to calculate the two sides of this relation.

The term $(\mathcal{F}_d \circ \mathcal{S})(V^\rho)$ on the left hand side of (2.12) is $(\mathcal{F}_d \circ \mathcal{S})(V^\rho) = (\mathcal{F}_d \circ \mathcal{S} \circ \mathcal{F}_1^{-1} \circ J_b \circ \mathcal{R}_\rho)(\hat{u}_0) = \hat{\mathcal{Q}}_b^\rho \hat{u}_0$. We see that the left hand side in (2.12) coincides with the left hand side in (2.10).

For $k \neq 0$, we calculate for the right hand side of (2.12), using $\hat{V}_0^\rho := \mathcal{R}_\rho \hat{u}_0$,

$$
\begin{aligned}
&\left(\frac{|k|}{2\pi i}\right)^{-(d-1)/2} \hat{V}^\rho \left(\xi = |k|, q = \frac{k}{|k|}, \tau\right) \\
&\quad = \left(\frac{|k|}{2\pi i}\right)^{-(d-1)/2} e^{-ib(|k|)\tau} \hat{V}_0^\rho \left(\xi = |k|, q = \frac{k}{|k|}\right) \\
&\quad = \begin{cases} e^{-ib(|k|)\tau} |k|^{-(d-1)/2} W_\rho(|k|) \hat{u}_0(k) & \text{for } |k| < \rho \\ e^{-ib(|k|)\tau} \hat{u}_0(k) & \text{for } |k| \geq \rho\,. \end{cases}
\end{aligned}
$$

We have used that $W_\rho(|k|) = |k|^{(d-1)/2}$ for $|k| > \rho$. For $|k| \geq \rho$ the right hand side of (2.12) coincides with the right hand side of (2.10). This provides the pointwise convergence.

Step 2: Weak convergence. The operators $\hat{\mathcal{Q}}_b$ are bounded, uniformly in $\varepsilon > 0$. Therefore, the left hand side of (2.11) is bounded in $L^2(\mathbb{R}^d)$, for every $\tau > 0$. Upon choosing a subsequence $\varepsilon \to 0$, for some limit function $L_\tau : \mathbb{R}^d \to \mathbb{C}$, $L_\tau = L_\tau(k)$, we can assume

$$
(\hat{\mathcal{Q}}_b \hat{u}_0)(\cdot, \tau/\varepsilon^2) e^{ic|\cdot|\tau/\varepsilon^2} \rightharpoonup L_\tau \tag{2.13}
$$

weakly in $L^2(\mathbb{R}^d)$. It remains to identify the limit L_τ as $e^{-ib(|\cdot|)\tau}\hat{u}_0$. The pointwise convergence of Step 1 implies that, for every $\rho > 0$ and every $\tau > 0$,

$$
\left[\hat{\mathcal{Q}}_b^\rho \hat{u}_0(k, \tau/\varepsilon^2) e^{ic|k|\tau/\varepsilon^2} - e^{-ib(|k|)\tau}\hat{u}_0(k)\right] \mathbf{1}_{\{|k|\geq\rho\}} \rightharpoonup 0\,. \tag{2.14}
$$

Identification of L_τ. Let $f \in C_c^\infty(\mathbb{R}^d)$ be a smooth test function. We calculate

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \left[(\hat{\mathcal{Q}}_b \hat{u}_0)(k, \tau/\varepsilon^2) e^{ic|k|\tau/\varepsilon^2} - e^{-ib(|k|)\tau}\hat{u}_0(k)\right] f(k)\, dk \\
&\quad = \int_{\mathbb{R}^d} \left(\left(\hat{\mathcal{Q}}_b - \hat{\mathcal{Q}}_b^\rho\right) \hat{u}_0\right)(k, \tau/\varepsilon^2) e^{ic|k|\tau/\varepsilon^2} f(k)\, dk \\
&\qquad + \int_{\mathbb{R}^d} (\hat{\mathcal{Q}}_b^\rho \hat{u}_0)(k, \tau/\varepsilon^2)(1 - \mathbf{1}_{\{|k|\geq\rho\}}) e^{ic|k|\tau/\varepsilon^2} f(k)\, dk \\
&\qquad + \int_{\mathbb{R}^d} \left[\left(\hat{\mathcal{Q}}_b^\rho \hat{u}_0\right)(k, \tau/\varepsilon^2) e^{ic|k|\tau/\varepsilon^2} - e^{-ib(|k|)\tau}\hat{u}_0(k)\right] \mathbf{1}_{\{|k|\geq\rho\}} f(k)\, dk \\
&\qquad + \int_{\mathbb{R}^d} e^{-ib(|k|)\tau}\hat{u}_0(k)(\mathbf{1}_{\{|k|\geq\rho\}} - 1) f(k)\, dk \\
&\quad =: I_{\varepsilon,\rho} + II_{\varepsilon,\rho} + III_{\varepsilon,\rho} + IV_{\varepsilon,\rho}\,.
\end{aligned}
$$

Regarding the error term $I_{\varepsilon,\rho}$, we use the fact that the operator $\mathcal{S} \circ \mathcal{F}_1^{-1} \circ J_b$ is bounded:

$$
\begin{aligned}
&\left\| \left(\left(\hat{\mathcal{Q}}_b - \hat{\mathcal{Q}}_b^\rho\right) \hat{u}_0\right)(\cdot, \tau/\varepsilon^2) e^{ic|\cdot|\tau/\varepsilon^2} \right\|_{L^2(\mathbb{R}^d)} \\
&\quad = \| \left(\left(\mathcal{S} \circ \mathcal{F}_1^{-1} \circ J_b \circ (\mathcal{R} - \mathcal{R}_\rho)\right) \hat{u}_0\right)(\cdot, \tau/\varepsilon^2) \|_{L^2(\mathbb{R}^d)} \\
&\quad \leq C \|(\mathcal{R} - \mathcal{R}_\rho)\hat{u}_0\|_{X_S} \leq \tilde{C} \|W - W_\rho\|_{L^2(\mathbb{R})} \to 0
\end{aligned}
$$

as $\rho \to 0$. In the last step we have used that $\hat{u}_0$ is bounded. This allows to calculate $I_{\varepsilon,\rho}$ in the limit $\rho \to 0$,

$$\begin{aligned} |I_{\varepsilon,\rho}| &\leq \left\| \left(\left(\hat{\mathcal{Q}}_b - \hat{\mathcal{Q}}_b^\rho \right) \hat{u}_0 \right) (\cdot, \tau/\varepsilon^2) e^{ic|\cdot|\tau/\varepsilon^2} \right\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} \\ &\leq \tilde{C} \|W - W_\rho\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R}^d)} \to 0 \,. \end{aligned}$$

For the second error term we calculate

$$\begin{aligned} |II_{\varepsilon,\rho}| &\leq \int_{\mathbb{R}^d} \left| (\hat{\mathcal{Q}}_b^\rho \hat{u}_0)(k, \tau/\varepsilon^2) \right| \mathbf{1}_{\{|k|<\rho\}} |f(k)| \, dk \\ &\leq \|f\|_\infty \|(\hat{\mathcal{Q}}_b^\rho \hat{u}_0)(\cdot, \tau/\varepsilon^2)\|_{L^2(\mathbb{R}^d)} \, |\{|k| < \rho\}|^{1/2} \\ &\leq C \|f\|_\infty \|\hat{u}_0\|_{L^2(\mathbb{R}^d)} \rho^{d/2} \,, \end{aligned}$$

where we have used that the linear operators $\hat{\mathcal{Q}}_b^\rho$ are bounded, independent of ρ.

For the third error term $III_{\varepsilon,\rho}$ we exploit, for $\rho > 0$ fixed, the weak convergence (2.14). Finally, $IV_{\varepsilon,\rho}$ is estimated by

$$|IV_{\varepsilon,\rho}| \leq \int_{\mathbb{R}^d} |\hat{u}_0(k)| \mathbf{1}_{\{|k|<\rho\}} |f(k)| \, dk \leq C \|\hat{u}_0\|_{L^2(\mathbb{R}^d)} \, \rho^{d/2} \|f\|_\infty \,.$$

In order to conclude the identification of the weak limit L_τ, we first choose $\rho > 0$ small such that $I_{\varepsilon,\rho}, II_{\varepsilon,\rho}$ and $IV_{\varepsilon,\rho}$ are small. Afterwards, we choose $\varepsilon > 0$ to achieve smallness in $III_{\varepsilon,\rho}$. We find

$$\int_{\mathbb{R}^d} (\hat{\mathcal{Q}}_b \hat{u}_0)(k, \tau/\varepsilon^2) e^{ic|k|\tau/\varepsilon^2} f(k) \, dk \to \int_{\mathbb{R}^d} e^{-ib(|k|)\tau} \hat{u}_0(k) f(k) \, dk$$

as $\varepsilon \to 0$. Since $f \in C_c^\infty(\mathbb{R}^d)$ was arbitrary, we conclude

$$L_\tau(k) = e^{-ib(|k|)\tau} \hat{u}_0(k) \,.$$

This shows (2.11) and concludes the proof. □

Interpretation. Two choices of the function b are of particular interest.

1) $b(\xi) = 0$ for all $\xi \in \mathbb{R}$. We recall that, by our assumption on the initial data, the solution of the linear wave equation is given in Fourier space by

$$\hat{u}(k, \tau/\varepsilon^2) = e^{-ic|k|\tau/\varepsilon^2} \hat{u}_0(k) \,. \tag{2.15}$$

Theorem 2.7 implies that, in the limit $\varepsilon \to 0$, the solution $\hat{u}$ is close to the function $\hat{\mathcal{Q}}_b \hat{u}_0$. This means that the ring solution with profile function $V = \mathcal{F}_1^{-1} \circ J_b \circ \mathcal{R} \hat{u}_0$ is a good approximation of u. The pointwise convergence (2.10) implies Theorem 1.1.

2) $b(\xi) = b_3 \xi^3$ for all $\xi \in \mathbb{R}$. The weakly dispersive equation

$$\partial_t^2 u(x,t) - c^2 \Delta_x u(x,t) + \varepsilon^2 d_0 \Delta_x^2 u(x,t) = 0 \tag{2.16}$$

with $d_0 > 0$ is an effective model to describe waves in heterogeneous media or in discrete media, see [4] and [9]. The Fourier transform of u satisfies

$$\partial_t^2 \hat{u}(k,t) + c^2 |k|^2 \hat{u}(k,t) + \varepsilon^2 d_0 |k|^4 \hat{u}(k,t) = 0 \,. \tag{2.17}$$

With appropriate initial data, the solution to (2.16) is given in Fourier space by

$$\hat{u}(k,t) = e^{-i\sqrt{c^2|k|^2+\varepsilon^2 d_0|k|^4}\,t}\hat{u}_0(k)\,. \tag{2.18}$$

Expanding the square route in a Taylor series and considering large times $t = \tau/\varepsilon^2$ we find that

$$\begin{aligned}\sqrt{c^2|k|^2+\varepsilon^2 d_0|k|^4}\tau/\varepsilon^2 &= \left(\sqrt{c^2|k|^2} + \frac{\varepsilon^2 d_0|k|^4}{2\sqrt{c^2|k|^2}}\right)\tau/\varepsilon^2 + O(\varepsilon^2)\\ &= c|k|\tau/\varepsilon^2 + \frac{d_0|k|^3}{2c}\tau + O(\varepsilon^2)\,.\end{aligned}$$

We set $b_3 := \frac{d_0}{2c}$ and use Theorem 2.7. We conclude that, in the limit $\varepsilon \to 0$, the solution u is well approximated by $\mathcal{Q}_b\hat{u}_0$: The profile function $V = \mathcal{F}_1^{-1} \circ J_b \circ \mathcal{R}\hat{u}_0$ provides a good approximation of the solution of the weakly dispersive equation (2.16). The profile V satisfies the linearized KdV-equation

$$\partial_\tau V(z,q,\tau) = b_3\partial_z^3 V(z,q,\tau)\,.$$

With this result we recover the profile analysis of [9] in dimension $d = 1$ and $d = 2$, and extend it to dimension $d = 3$.

3 Analysis of the reconstruction operator

Our main result Theorem 2.7 states that solutions to a (dispersive) wave equation can be recovered approximately by the reconstruction operator $\mathcal{Q}_b$. This requires a study of the expression $\hat{\mathcal{Q}}_b\hat{u}_0 = \mathcal{F}_d \circ \mathcal{S} \circ \mathcal{F}_1^{-1} \circ J_b \circ \mathcal{R}(\hat{u}_0)$. As we have already seen, the core result regards the outer part, the expression $(\mathcal{F}_d \circ \mathcal{S})V$. This part is analyzed in Theorem 3.3 below.

Assumption 3.1. *Let the dimension be $d \in \{1,2,3\}$. On $V \in L^\infty(0,\infty;X_S)$ we assume the following.*

(i) There exist $C,\alpha > 0$ such that for every $\tau \in (0,\infty)$ and $q \in S^{d-1}$

$$|V(z,q,\tau)| \le C(1+|z|)^{-d-\alpha}\,. \tag{3.1}$$

(ii) The Fourier transform $\hat{V} := \mathcal{F}_1 V$ has the property that, for every $\tau \in (0,\infty)$, the function

$$\mathbb{R} \times S^{d-1} \ni (\xi,q) \mapsto \hat{V}(\xi,q,\tau) \in \mathbb{C}$$

is of class $C^{d-1}(\mathbb{R}\times S^{d-1};\mathbb{C})$.

In Theorems 2.7 and 3.3, we demand that $V^\rho := (\mathcal{F}_1^{-1} \circ J_b \circ \mathcal{R}_\rho)\hat{u}_0$ satisfies Assumption 3.1. Actually, this is not too restrictive.

Remark 3.2. *Let $\hat{u}_0$ be a smooth function with compact support. Then $\hat{V}^\rho := (J_b \circ \mathcal{R}_\rho)\hat{u}_0$ has also compact support. Moreover, since $\mathcal{R}_\rho$ uses the regularization of $|\xi|^{(d-1)/2}\mathbf{1}_{\{\xi>0\}}$, the smoothness of $\hat{u}_0$ is inherited by $\hat{V}^\rho$. Smoothness of $\hat{V}^\rho$ implies the decay property (3.1) of $V^\rho = \mathcal{F}_1^{-1}\hat{V}^\rho$ in z. We conclude that Assumption 3.1 is satisfied.*

We are now in the position to prove our core result.

Theorem 3.3 (The shell operator in Fourier space)**.** *In dimension $d \in \{1,2,3\}$ let $\hat{V} \in L^\infty(0,\infty;X_s)$ satisfy $\hat{V}(\xi,q,\tau) = 0$ for every $\xi < 0$ and let $V := \mathcal{F}_1^{-1}\hat{V}^\rho \in L^\infty(0,\infty;X_s)$ satisfy Assumption 3.1. Consider the ring-like solution $\mathcal{S}V$ and its Fourier transform $\mathcal{F}_d \circ \mathcal{S}(V)$. For every $k \in \mathbb{R}^d \setminus \{0\}$ and every $\tau > 0$ holds*

$$(\mathcal{F}_d \circ \mathcal{S})(V)(k,\tau/\varepsilon^2) e^{ic|k|\tau/\varepsilon^2} \to \left(\frac{|k|}{2\pi i}\right)^{-(d-1)/2} \hat{V}\left(|k|, q = \frac{k}{|k|}, \tau\right) \tag{3.2}$$

as $\varepsilon \to 0$. Moreover, the convergence holds as weak convergence in $L^2(\mathbb{R}^d)$.

Proof of Theorem 3.3. It suffices to prove, for $k \neq 0$,

$$Q^\varepsilon(k,\tau) := e^{ic|k|\tau/\varepsilon^2}\, (\mathcal{F}_d \circ \mathcal{S})(V)(k,\tau/\varepsilon^2) \to \left(\frac{|k|}{2\pi i}\right)^{-(d-1)/2} \hat{V}\left(|k|, q = \frac{k}{|k|}, \tau\right). \tag{3.3}$$

Indeed, since the operator $\mathcal{F}_d \circ \mathcal{S}$ is bounded and since $|e^{ic|k|\tau/\varepsilon^2}| = 1$, for every $\tau > 0$, the sequence $Q^\varepsilon(\cdot,\tau)$ is uniformly bounded in $L^2(\mathbb{R}^d)$. Therefore there exists, up to a subsequence, a weak limit in $L^2(\mathbb{R}^d)$. Since weak and pointwise limits always coincide, we conclude the weak convergence of $Q^\varepsilon(\cdot,\tau)$ to the right hand side of (3.3).

We show the pointwise convergence in five steps.

Step 1: Calculation of the quantity of interest. We calculate the left hand side of (3.3). Definition 2.4 of the shell operator $\mathcal{S}$ provides

$$\mathcal{S}V(x,t) = \frac{1}{(ct)^{(d-1)/2}} V\left(|x| - ct, \frac{x}{|x|}, \varepsilon^2 t\right) \mathbf{1}_{\{|x|<2ct\}}\,.$$

We calculate the Fourier transform in polar coordinates, $x = rq$ with $r > 0$ and $q \in S^{d-1}$,

$$\begin{aligned}(\mathcal{F}_d \circ \mathcal{S})(V)(k,t) &= \int_{\mathbb{R}^d} e^{-ix\cdot k} (\mathcal{S}V)(x,t)\, dx \\ &= \int_0^\infty \int_{S^{d-1}} r^{d-1} e^{-irq\cdot k} (\mathcal{S}V)(rq,t)\, dS(q)\, dr\,.\end{aligned}$$

We insert $\mathcal{S}V$ from above. Evaluating in $t = \tau/\varepsilon^2$ we find

$$Q^\varepsilon(k,\tau) = e^{ic|k|\tau/\varepsilon^2} \int_0^{2c\tau/\varepsilon^2} \int_{S^{d-1}} \frac{r^{d-1} e^{-irq\cdot k}}{(c\tau/\varepsilon^2)^{(d-1)/2}} V\left(r - c\frac{\tau}{\varepsilon^2}, q, \tau\right) dS(q)\, dr\,. \tag{3.4}$$

To simplify, we write $r = c\tau/\varepsilon^2 + z$ with a new variable $z \in \mathbb{R}$; the integration over r is replaced by an integration over z. We find

$$Q^\varepsilon(k,\tau) = e^{ic|k|\tau/\varepsilon^2} \int_{-c\tau/\varepsilon^2}^{c\tau/\varepsilon^2} \int_{S^{d-1}} \frac{(c\tau/\varepsilon^2 + z)^{d-1}}{(c\tau/\varepsilon^2)^{(d-1)/2}} e^{-iq\cdot k\, c\tau/\varepsilon^2} e^{-izq\cdot k}\, V(z,q,\tau)\, dS(q)\, dz\,.$$

Step 2: Approximation. We treat the cases $d \in \{1,2\}$ and $d = 3$ differently.

Case $d \in \{1,2\}$. We use the approximations $\int_{-c\tau/\varepsilon^2}^{c\tau/\varepsilon^2} \approx \int_{\mathbb{R}}$ and $\frac{(c\tau/\varepsilon+z)^{d-1}}{(c^\tau/\varepsilon^2)^{(d-1)/2}} \approx (c\tau/\varepsilon^2)^{(d-1)/2}$ and write

$$Q^\varepsilon(k,\tau) = A_0^\varepsilon(k,\tau) + G_0^\varepsilon(k,\tau) \tag{3.5}$$

with

$$\begin{aligned} A_0^\varepsilon(k,\tau) &= e^{ic|k|\tau/\varepsilon^2} \int_{\mathbb{R}} \int_{S^{d-1}} (c\tau/\varepsilon^2)^{(d-1)/2} e^{-iq\cdot k\, c\tau/\varepsilon^2} e^{-izq\cdot k}\, V(z,q,\tau)\, dS(q)\, dz\,, \\ G_0^\varepsilon(k,\tau) &= e^{ic|k|\tau/\varepsilon^2} \int_{\mathbb{R}} \int_{S^{d-1}} e^{-iq\cdot k(c\tau/\varepsilon^2+z)} V(z,q,\tau) \times \\ &\qquad \times \left[\frac{(c\tau/\varepsilon^2+z)^{d-1}}{(c\tau/\varepsilon^2)^{(d-1)/2}} \mathbf{1}_{\{|z|<c\tau/\varepsilon^2\}} - (c\tau/\varepsilon^2)^{(d-1)/2} \right] dS(q)\, dz\,. \end{aligned}$$

Case d=3: In three dimensions, we use higher order approximations: $\int_{-c\tau/\varepsilon^2}^{c\tau/\varepsilon^2} \approx \int_{\mathbb{R}}$ and $\frac{(c\tau/\varepsilon^2+z)^2}{(c\tau/\varepsilon^2)} \approx c\tau/\varepsilon^2 + 2z$. This allows to write

$$Q^\varepsilon(k,\tau) = A_1^\varepsilon(k,\tau) + G_1^\varepsilon(k,\tau) \tag{3.6}$$

with

$$\begin{aligned} A_1^\varepsilon(k,\tau) &= e^{ic|k|\tau/\varepsilon^2} \int_{\mathbb{R}} \int_{S^{d-1}} (c\tau/\varepsilon^2 + 2z) e^{-iq\cdot k\, c\tau/\varepsilon^2} e^{-izq\cdot k}\, V(z,q,\tau)\, dS(q)\, dz\,, \\ G_1^\varepsilon(k,\tau) &= e^{ic|k|\tau/\varepsilon^2} \int_{\mathbb{R}} \int_{S^{d-1}} e^{-iq\cdot k(c\tau/\varepsilon^2+z)} V(z,q,\tau) \times \\ &\qquad \times \left[\frac{(c\tau/\varepsilon^2+z)^2}{(c\tau/\varepsilon^2)} \mathbf{1}_{\{|z|<c\tau/\varepsilon^2\}} - (c\tau/\varepsilon^2 + 2z) \right] dS(q)\, dz\,. \end{aligned}$$

Step 3: Simplifying the expression for A_0^ε, A_1^ε. One of the integrals in the formulas for A_i^ε can be evaluated. Indeed, in A_0^ε and in one of the two terms of A_1^ε, we recognize

$$\int_{\mathbb{R}} V(z,q,\tau) e^{-izq\cdot k}\, dz = \hat{V}(q\cdot k, q, \tau)\,.$$

In dimension $d = 3$, we find

$$\int_{\mathbb{R}} z\, V(z,q,\tau) e^{-izq\cdot k}\, dz = i\partial_\xi \hat{V}(\xi = q\cdot k, q, \tau)\,,$$

where integrability of all terms is assured by Assumption 3.1.

The formula for A_0^ε simplifies to

$$\begin{aligned} A_0^\varepsilon(k,\tau) &= \int_{S^{d-1}} (c\tau/\varepsilon^2)^{(d-1)/2}\, e^{i(|k|-q\cdot k)\, c\tau/\varepsilon^2}\, \hat{V}(q\cdot k, q, \tau)\, dS(q) \\ &= \int_{S^{d-1}} \left(\frac{|k|}{2\pi i} c\tau/\varepsilon^2 \right)^{(d-1)/2} e^{i(1-q\cdot k/|k|)\, |k| c\tau/\varepsilon^2} \left[\left(\frac{|k|}{2\pi i} \right)^{-(d-1)/2} \hat{V}(q\cdot k, q, \tau) \right] dS(q)\,. \end{aligned} \tag{3.7}$$

The formula for A_1^ε simplifies to

$$
\begin{aligned}
A_1^\varepsilon(k,\tau) &= \int_{S^2} e^{i(|k|-q\cdot k)\,c\tau/\varepsilon^2}\,\left(c\tau/\varepsilon^2\hat V(q\cdot k,q,\tau)+2i\partial_\xi \hat V(q\cdot k,q,\tau)\right)\,dS(q)\\
&= \int_{S^2}\left(\frac{|k|}{2\pi i}c\tau/\varepsilon^2\right)\,e^{i(1-q\cdot k/|k|)\,|k|c\tau/\varepsilon^2}\,\left[\left(\frac{|k|}{2\pi i}\right)^{-1}\hat V(q\cdot k,q,\tau)\right]\,dS(q)\\
&\quad+\frac{2\varepsilon^2}{c\tau}i\int_{S^2}\left(\frac{|k|}{2\pi i}c\tau/\varepsilon^2\right)\,e^{i(1-q\cdot k/|k|)\,|k|c\tau/\varepsilon^2}\,\left[\left(\frac{|k|}{2\pi i}\right)^{-1}\partial_\xi\hat V(q\cdot k,q,\tau)\right]\,dS(q)\,.
\end{aligned}
\tag{3.8}
$$

Step 4: Application of a stationary phase limit. We consider the terms in squared brackets in (3.7) and (3.8) as test-functions. Denoting them as $\phi=\phi(q)$, we exploit Lemma 4.1 to calculate the limit $\varepsilon\to 0$. The lemma provides

$$
\int_{S^{d-1}}\left(\frac{|k|}{2\pi i}c\tau/\varepsilon^2\right)^{(d-1)/2} e^{i(1-q\cdot k/|k|)\,|k|c\tau/\varepsilon^2}\,\phi(q)\,dS(q)\to\phi(k/|k|)\,. \tag{3.9}
$$

Indeed, since $k\in\mathbb{R}^d\setminus\{0\}$ is held fixed, we can use Lemma 4.1 with $\kappa:=k/|k|$ and the sequence of numbers $N:=|k|c\tau/\varepsilon^2$, which tends to $+\infty$.

Let us check if the assumptions of Lemma 4.1 are satisfied. The lemma requires that $\phi: S^{d-1}\to\mathbb{C}$ is supported on the half sphere defined by κ. This requirement is satisfied since we demanded $\hat V(\xi,q,\tau)=0$ for every $\xi<0$. Moreover, Lemma 4.1 requires that $\phi: S^{d-1}\to\mathbb{C}$ is of class C^1. In dimension $d=1$, this is no further requirement. In dimension $d=2$, we need that $q\mapsto\hat V(q\cdot k,q,\tau)$ is of class C^1; this follows from Assumption 3.1, (ii). In dimension $d=3$, we need that both $q\mapsto\hat V(q\cdot k,q,\tau)$ and $q\mapsto\partial_\xi\hat V(q\cdot k,q,\tau)$ are of class C^1; also this follows from Assumption 3.1, (ii).

The second term in (3.8) vanishes in the limit as $\varepsilon\to 0$ due to (3.9) and the factor ε^2 in front of the integral. The limits of the remaining terms are determined by evaluating $\hat V(q\cdot k,q,\tau)$ in the point $q=\kappa=k/|k|$. We find $\hat V\left(\frac{k}{|k|}\cdot k,\frac{k}{|k|},\tau\right)=\hat V\left(|k|,\frac{k}{|k|},\tau\right)$. This yields, for $k\neq 0$,

$$
\lim_{\varepsilon\to0}A_0^\varepsilon(k,\tau)=\lim_{\varepsilon\to0}A_1^\varepsilon(k,\tau)=\left(\frac{|k|}{2\pi i}\right)^{-(d-1)/2}\hat V\left(|k|,\frac{k}{|k|},\tau\right)\,.
$$

This is the desired limit in (3.3). Once we show that the error terms G_0^ε and G_1^ε are small, we have shown (3.3) and hence the Theorem.

Step 5: Calculation of the error terms G_0^ε and G_1^ε. We show the result for the three dimensions separately.

Dimension $d=1$. In the case $d=1$ we have

$$
G_0^\varepsilon(k,\tau)=e^{ic|k|\tau/\varepsilon^2}\sum_{q=\pm1}\int_{\mathbb{R}} e^{-iq\cdot k(c\tau/\varepsilon^2+z)}V(z,q,\tau)\mathbf{1}_{\{|z|\geq c\tau/\varepsilon^2\}}\,dz\,.
$$

Exploiting $\left|e^{ic|k|\tau/\varepsilon^2}e^{-iq\cdot k(c\tau/\varepsilon^2+z)}\right| = 1$ we find

$$|G_0^\varepsilon(k,\tau)| \le \sum_{q=\pm1} \int_{\mathbb{R}} |V(z,q,\tau)| \, \mathbf{1}_{\{|z|\ge c\tau/\varepsilon^2\}} \, dz \to 0$$

as $\varepsilon \to 0$; here we exploit that Assumption 3.1 provides a decay rate that assures $V(\cdot,q,\tau) \in L^1(\mathbb{R})$ uniformly in q and τ.

Dimension $d=2$. In the case $d=2$ we find

$$|G_0^\varepsilon(k,\tau)| \le \int_{\mathbb{R}} \int_{S^1} |V(z,q,\tau)| \left| \frac{c\tau/\varepsilon^2 + z}{(c\tau/\varepsilon^2)^{1/2}} \mathbf{1}_{\{|z|<c\tau/\varepsilon^2\}} - (c\tau/\varepsilon^2)^{1/2} \right| dS(q)\, dz \,.$$

Since S^1 has the finite measure 2π, it suffices to show the convergence

$$\int_{\mathbb{R}} |V(z,q,\tau)| \left| \frac{c\tau/\varepsilon^2 + z}{(c\tau/\varepsilon^2)^{1/2}} \mathbf{1}_{\{|z|<c\tau/\varepsilon^2\}} - (c\tau/\varepsilon^2)^{1/2} \right| dz \to 0$$

as $\varepsilon \to 0$, uniformly in $q \in S^1$. We decompose the integral into two parts, $|z| \le \delta/\varepsilon$ and $|z| > \delta/\varepsilon$ with $\delta > 0$ to be chosen below. We only consider ε-values with $c\tau/\varepsilon > \delta$, such that

$$\begin{aligned} &\int_{|z|\le\delta/\varepsilon} |V(z,q,\tau)| \left| \frac{c\tau/\varepsilon^2 + z}{(c\tau/\varepsilon^2)^{1/2}} \mathbf{1}_{\{|z|<c\tau/\varepsilon^2\}} - (c\tau/\varepsilon^2)^{1/2} \right| dz \\ &\qquad = \int_{|z|\le\delta/\varepsilon} |V(z,q,\tau)| \left| \frac{c\tau/\varepsilon^2 + z}{(c\tau/\varepsilon^2)^{1/2}} - (c\tau/\varepsilon^2)^{1/2} \right| dz \,. \end{aligned}$$

Using

$$\left| \frac{c\tau/\varepsilon^2 + z}{(c\tau/\varepsilon^2)^{1/2}} - (c\tau/\varepsilon^2)^{1/2} \right| = \left| \frac{z}{(c\tau/\varepsilon^2)^{1/2}} \right| = \varepsilon \frac{z}{(c\tau)^{1/2}} \le \frac{\delta}{(c\tau)^{1/2}} \tag{3.10}$$

for $|z| \le \delta/\varepsilon$, we obtain

$$\begin{aligned} \int_{|z|\le\delta/\varepsilon} |V(z,q,\tau)| \left| \frac{c\tau/\varepsilon^2 + z}{(c\tau/\varepsilon^2)^{1/2}} - (c\tau/\varepsilon^2)^{1/2} \right| dz &\le \frac{\delta}{(c\tau)^{1/2}} \int_{|z|\le\delta/\varepsilon} |V(z,q,\tau)| \, dz \\ \le \frac{\delta}{(c\tau)^{1/2}} \int_{\mathbb{R}} |V(z,q,\tau)| \, dz \le C\delta \end{aligned}$$

with $C = C(\tau)$, where we have used that $V(\cdot,q,\tau) \in L^1(\mathbb{R})$ uniformly in q and τ. The integral over $|z| > \delta/\varepsilon$ is estimated exploiting

$$\begin{aligned} \left| \frac{c\tau/\varepsilon^2 + z}{(c\tau/\varepsilon^2)^{1/2}} \mathbf{1}_{\{|z|<c\tau/\varepsilon^2\}} - (c\tau/\varepsilon^2)^{1/2} \right| &\le \left| \frac{c\tau/\varepsilon^2 + z}{(c\tau/\varepsilon^2)^{1/2}} \right| + (c\tau/\varepsilon^2)^{1/2} \\ \le 2(c\tau/\varepsilon^2)^{1/2} + \left| \frac{z}{(c\tau/\varepsilon^2)^{1/2}} \right| &= \frac{2}{\varepsilon}(c\tau)^{1/2} + \varepsilon \frac{|z|}{(c\tau)^{1/2}} \,. \end{aligned}$$

We find

$$\int_{|z|>\delta/\varepsilon} |V(z,q,\tau)| \left| \frac{c\tau/\varepsilon^2 + z}{(c\tau/\varepsilon^2)^{1/2}} \mathbf{1}_{\{|z|<c\tau/\varepsilon^2\}} - (c\tau/\varepsilon^2)^{1/2} \right| dz$$

$$
\begin{aligned}
&\le \int_{|z|>\delta/\varepsilon} |V(z,q,\tau)| \left(\frac{2}{\varepsilon}(c\tau)^{1/2} + \varepsilon \frac{|z|}{(c\tau)^{1/2}} \right) dz \\
&\le C \int_{|z|>\delta/\varepsilon} |z|^{-2-\alpha} \left(\frac{2}{\varepsilon}(c\tau)^{1/2} + \varepsilon \frac{|z|}{(c\tau)^{1/2}} \right) dz \\
&\le C \left(\varepsilon^{-1}(\varepsilon/\delta)^{1+\alpha} + \varepsilon(\varepsilon/\delta)^{\alpha} \right) .
\end{aligned}
$$

In the last step we have exploited the assumption on V, namely $|V(z,q,\tau)| \le C(1+|z|)^{-2-\alpha}$. Choosing first $\delta > 0$ to have smallness in the first integral and then $\varepsilon > 0$ small to make the second integral small, we conclude

$$
|G_0^\varepsilon(k,\tau)| \to 0 \quad \text{for } \varepsilon \to 0 .
$$

Dimension $d = 3$. The case $d = 3$ is analogous to the case $d = 2$. For the integral over $|z| \le \delta/\varepsilon$ we use

$$
\begin{aligned}
&\frac{(c\tau/\varepsilon^2 + z)^2}{(c\tau/\varepsilon^2)} - (c\tau/\varepsilon^2 + 2z) = \frac{1}{\varepsilon^2}\left(\frac{(c\tau + \varepsilon^2 z)^2}{c\tau} - c\tau \right) - 2z \\
&\quad = \frac{1}{\varepsilon^2 c\tau}\left((c\tau + \varepsilon^2 z)^2 - (c\tau)^2 \right) - 2z = \frac{1}{\varepsilon^2 c\tau}\left(2c\tau\varepsilon^2 z + \varepsilon^4 z^2 \right) - 2z = \frac{\varepsilon^2 z^2}{c\tau} \le \frac{\delta^2}{c\tau}
\end{aligned}
$$

and the fact that $V(\cdot,q,\tau) \in L^1(\mathbb{R})$ uniformly in q. Concerning the integral over $|z| > \delta/\varepsilon$ we calculate

$$
\left| \frac{(c\tau/\varepsilon^2 + z)^2}{(c\tau/\varepsilon^2)} \mathbf{1}_{\{|z|<c\tau/\varepsilon^2\}} - (c\tau/\varepsilon^2 + 2z) \right| \le 3c\tau/\varepsilon^2 + \frac{2|z|^2}{c\tau/\varepsilon^2} + 2|z|
$$

and thus

$$
\begin{aligned}
&\int_{|z|>\delta/\varepsilon} |V(z,q,\tau)| \left| \frac{c\tau/\varepsilon^2 + z}{(c\tau/\varepsilon^2)^{1/2}} \mathbf{1}_{\{|z|<c\tau/\varepsilon^2\}} - (c\tau/\varepsilon^2)^{1/2} \right| dz \\
&\quad \le \int_{|z|>\delta/\varepsilon} |V(z,q,\tau)| \left(3c\tau/\varepsilon^2 + \frac{2|z|^2}{c\tau/\varepsilon^2} + 2|z| \right) dz \\
&\quad \le C \int_{|z|>\delta/\varepsilon} |z|^{-3-\alpha} \left(3c\tau/\varepsilon^2 + \frac{2|z|^2}{c\tau/\varepsilon^2} + 2|z| \right) dz \\
&\quad \le C \left(\varepsilon^{-2}(\varepsilon/\delta)^{2+\alpha} + \varepsilon^2(\varepsilon/\delta)^{\alpha} + (\varepsilon/\delta)^{1+\alpha} \right) .
\end{aligned}
$$

In the last step we have exploited Assumption 3.1 on V, namely $|V(z,q,\tau)| \le C(1+|z|)^{-3-\alpha}$ uniformly in q and τ.

Once more, we choose first $\delta > 0$ small to have the integral over $|z| \le \delta/\varepsilon$ small. We then choose $\varepsilon > 0$ small to have the other integral small. We obtain that the error terms G_0^ε and G_1^ε vanish in the limit $\varepsilon \to 0$. Up to the claim in (3.9), where we used the subsequent Lemma 4.1, the theorem is shown. □

4 A stationary phase convergence result

In the last section, the relevant small parameter was $\varepsilon > 0$; in this section, we work with the large parameter $N := |k|c\tau/\varepsilon^2$. We applied in Section 3 the subsequent Lemma 4.1 with the vector $\kappa := k/|k|$.

In the following, for arbitrary dimension $d \in \{1,2,3\}$, we will demand that the test-function $\phi : S^{d-1} \to \mathbb{R}$ is of class $C^1(S^{d-1})$ and that it is supported on the half-sphere $\{q \in S^{d-1} \,|\, q \cdot \kappa \geq 0\}$.

Regarding the case $d=1$ we note that $S^{d-1} = \{+1,-1\}$ and that, for $\kappa = e_1 \equiv 1$, a function $\phi \in C^1(S^{d-1})$ with support in the half-sphere $\{q \in S^{d-1} \,|\, q \cdot \kappa \geq 0\} = \{1\}$ is a function $\phi : \{+1,-1\} \to \mathbb{R}$ with $\phi(-1) = 0$.

Lemma 4.1. *Let the dimension be $d \in \{1,2,3\}$. Let $\kappa \in S^{d-1}$ be a point on the sphere and let $\phi \in C^1(S^{d-1};\mathbb{R})$ be supported in $\{q \in S^{d-1} \,|\, q \cdot \kappa \geq 0\}$. Then there holds*

$$A_\phi^N := (2\pi i)^{-(d-1)/2} \int_{S^{d-1}} N^{(d-1)/2}\, e^{i(1-q\cdot\kappa)\,N}\, \phi(q)\, dS(q) \to \phi(\kappa) \tag{4.1}$$

as $N \to \infty$.

Proof. By radial symmetry it is sufficient to consider the case $\kappa := e_1$. We show the result for the three dimensions separately.

Step 1: Dimension $d=1$. In the case $d=1$, the integral in (4.1) is a sum of two terms,

$$A_\phi^N = \sum_{q\in\{\pm 1\}} e^{i(1-q\cdot 1)\,N}\, \phi(q) = \phi(1) + e^{2i\,N}\, \phi(-1) = \phi(1)\,. \tag{4.2}$$

This shows (4.1).

Step 2: Dimension $d=3$. We use spherical coordinates

$$q(\theta,\vartheta) := \begin{pmatrix} \cos(\theta) \\ \sin(\theta)\cos(\vartheta) \\ \sin(\theta)\sin(\vartheta) \end{pmatrix}$$

with angles $\theta \in (0,\pi)$ and $\vartheta \in (0,2\pi)$ and surface element $J := \sqrt{\det(Dq^T Dq)} = \sin(\theta)$. We calculate the expression of (4.1) for $d=3$ with spherical coordinates as

$$\begin{aligned}
A_\phi^N &= (2\pi i)^{-1} \int_{S^2} N\, e^{i(1-q\cdot e_1)\,N}\, \phi(q)\, dS(q) \\
&= (2\pi i)^{-1} \int_0^\pi \int_0^{2\pi} N\, e^{i(1-\cos(\theta))\,N}\, \phi(q(\theta,\vartheta))\, d\vartheta\, \sin(\theta)\, d\theta \\
&= -\int_0^\pi iN\, e^{i(1-\cos(\theta))\,N} \sin(\theta) \underbrace{\left(\frac{1}{2\pi}\int_0^{2\pi} \phi(q(\theta,\vartheta))\, d\vartheta\right)}_{=:\tilde\phi(\theta)} d\theta \\
&= -\int_0^\pi \frac{d}{d\theta}\left[e^{i(1-\cos(\theta))\,N}\right] \tilde\phi(\theta)\, d\theta \\
&= -\left[e^{i(1-\cos(\theta))\,N}\tilde\phi(\theta)\right]_{\theta=0}^{\pi} + \int_0^\pi e^{i(1-\cos(\theta))\,N} \frac{d}{d\theta}\tilde\phi(\theta)\, d\theta \\
&= \tilde\phi(0) + \int_0^{\pi/2} e^{i(1-\cos(\theta))\,N} \frac{d}{d\theta}\tilde\phi(\theta)\, d\theta\,,
\end{aligned}$$

where integration by parts is allowed because of $\tilde{\phi} \in C^1([0,\pi])$. In the last line we also exploited $\tilde{\phi}(\theta) = 0$ for $\theta \in (\pi/2, \pi)$. For $\tilde{\phi}(0)$ we obtain

$$\tilde{\phi}(0) = \frac{1}{2\pi}\int_0^{2\pi} \phi(q(0,\vartheta))\, d\vartheta = \frac{1}{2\pi}\int_0^{2\pi} \phi(e_1)\, d\vartheta = \phi(e_1)\,.$$

We turn now to the treatment of the integral. We use the substitution $z = 1 - \cos(\theta)$ with $\frac{dz}{d\theta} = \sin(\theta) = \sqrt{1-\cos^2(\theta)} = \sqrt{1-(1-z)^2} = \sqrt{z}\sqrt{2-z}$ to obtain

$$\int_0^{\pi/2} e^{i(1-\cos(\theta))\,N} \frac{d}{d\theta}\tilde{\phi}(\theta)\, d\theta = \int_0^1 e^{izN} \frac{d}{d\theta}\tilde{\phi}(\arccos(1-z))\frac{1}{\sqrt{z}\sqrt{2-z}}\, dz\,.$$

The factor $z \mapsto e^{izN}$ is a sequence of highly oscillatory functions; it converges to the mean value $\frac{1}{2\pi}\int_0^{2\pi} e^{iy}\, dy = 0$ weakly in $L^p(0,1)$ for every $p \in (1,\infty)$. Since $\frac{d}{d\theta}\tilde{\phi}$ is bounded and $\frac{1}{\sqrt{2-z}} \le 1$ for $z \in (0,1)$, we find that

$$z \mapsto \frac{d}{d\theta}\tilde{\phi}(\arccos(1-z))\frac{1}{\sqrt{z}\sqrt{2-z}}$$

is in $L^q(0,1)$ for $q \in (1,2)$; it is thus an admissible test function for the weak convergence property. We obtain

$$\int_0^{\pi/2} e^{i(1-\cos(\theta))\,N} \frac{d}{d\theta}\tilde{\phi}(\theta)\, d\theta \to 0$$

as $N \to \infty$, which provides the claim (4.1) for $d = 3$.

Step 3: Dimension $d = 2$. We use the coordinates $q(\theta) := (\cos(\theta), \sin(\theta))$ with $\theta \in (-\pi,\pi)$, the line element is $J = 1$. The expression of (4.1) is

$$\begin{aligned} A_\phi^N &= (2\pi i)^{-1/2}\int_{S^1} N^{1/2}\, e^{i(1-q\cdot e_1)\,N}\, \phi(q)\, dS(q) \\ &= (2\pi i)^{-1/2}\int_{-\pi}^{\pi} N^{1/2}\, e^{i(1-\cos(\theta))\,N}\, \phi(q(\theta))\, d\theta \\ &= (2\pi i)^{-1/2}\int_0^{\pi/2} N^{1/2}\, e^{i(1-\cos(\theta))\,N}\, \tilde{\phi}(\theta)\, d\theta\,, \end{aligned}$$

where $\tilde{\phi}(\theta) := \phi(q(\theta)) + \phi(q(-\theta))$ denotes a symmetrized version of ϕ. We split the integral into two parts, $\theta \in (0,\delta)$ and $\theta \in (\delta,\pi)$, where the small parameter δ is chosen N-dependent, $\delta := N^{-\beta}$ with $\beta = 3/10$. We calculate

$$\begin{aligned} &\int_{N^{-3/10}}^{\pi/2} N^{1/2}\, e^{i(1-\cos(\theta))\,N}\, \tilde{\phi}(\theta)\, d\theta \\ &= \frac{1}{\sqrt{N}}\int_{N^{-3/10}}^{\pi/2} \sin(\theta) i\, N\, e^{i(1-\cos(\theta))\,N}\, \frac{\tilde{\phi}(\theta)}{i\sin(\theta)}\, d\theta \\ &= \frac{1}{\sqrt{N}}\left[e^{i(1-\cos(\theta))\,N}\frac{\tilde{\phi}(\theta)}{i\sin(\theta)}\right]_{\theta=N^{-3/10}}^{\pi/2} - \frac{1}{\sqrt{N}}\int_{N^{-3/10}}^{\pi/2} e^{i(1-\cos(\theta))\,N}\, \frac{\frac{d}{d\theta}\tilde{\phi}(\theta)}{i\sin(\theta)}\, d\theta \end{aligned}$$

$$+\frac{1}{\sqrt{N}}\int_{N^{-3/10}}^{\pi/2} e^{i(1-\cos(\theta))\,N}\,\frac{\tilde\phi(\theta)\cos(\theta)}{i\sin^2(\theta)}\,d\theta$$
$$=: I_N + II_N + III_N\,.$$

The terms I_N, II_N vanish in the limit as $N\to\infty$. Indeed, for N sufficiently large

$$|I_N| = \frac{1}{\sin(N^{-3/10})\sqrt{N}}|\tilde\phi(N^{-3/10})| \le C\frac{N^{3/10}}{\sqrt{N}} = CN^{3/10-1/2} \overset{N\to\infty}{\to} 0$$

and, since sine is monotonically increasing in $(0,\pi/2)$,

$$|II_N| \le \frac{1}{\sqrt{N}}\int_{N^{-3/10}}^{\pi/2}\left|\frac{\frac{d}{d\theta}\tilde\phi(\theta)}{\sin(\theta)}\right| d\theta \le C\frac{N^{3/10}}{\sqrt{N}}\int_{N^{-3/10}}^{\pi/2}\left|\frac{d}{d\theta}\tilde\phi(\theta)\right| d\theta$$
$$\le \tilde C N^{3/10-1/2} \overset{N\to\infty}{\to} 0\,,$$

where we have used $\tilde\phi\in C^1([0,\pi])$.

To treat III_N, we integrate by parts once more:

$$III_N = -N^{-3/2}\int_{N^{-3/10}}^{\pi/2} iNe^{i(1-\cos(\theta))\,N}\sin(\theta)\frac{\tilde\phi(\theta)\cos(\theta)}{\sin^3(\theta)}\,d\theta$$
$$= -N^{-3/2}\left[e^{i(1-\cos(\theta))\,N}\frac{\tilde\phi(\theta)\cos(\theta)}{\sin^3(\theta)}\right]_{\theta=N^{-3/10}}^{\pi/2}$$
$$+ N^{-3/2}\int_{N^{-3/10}}^{\pi/2} e^{i(1-\cos(\theta))\,N}\,\frac{\frac{d}{d\theta}\tilde\phi(\theta)\cos(\theta)-\tilde\phi(\theta)\sin(\theta)}{\sin^3(\theta)}\,d\theta$$
$$- N^{-3/2}\int_{N^{-3/10}}^{\pi/2} e^{i(1-\cos(\theta))\,N}\,\frac{3\cos^2(\theta)\tilde\phi(\theta)}{\sin^4(\theta)}\,d\theta\,.$$

Since $1/\sin^3(N^{-3/10})\le CN^{9/10}$, the term in square brackets is of order $N^{-3/2}N^{9/10} = N^{-3/5}\to 0$ as $N\to\infty$. For the integral expressions we note that $1/\sin^4(N^{-3/10})\le CN^{6/5}$ and by assumption $\tilde\phi\in C^1([0,\pi])$. We conclude that the last integral scales as $N^{-3/2}N^{6/5} = N^{-3/10}\to 0$ as $N\to\infty$. The second integral is of lower order. This proves that $III_N\to 0$.

We now treat the other part of A^N_ϕ, the integral over the interval $(0, N^{-3/10})$. We first note that, since $\tilde\phi$ is Lipschitz-continuous, for $\theta\in(0,N^{-3/10})$ one has $|\tilde\phi(\theta)-\tilde\phi(0)|\le C\theta\le CN^{-3/10}$ and thus

$$\left|\int_0^{N^{-3/10}} N^{1/2}\,e^{i(1-\cos(\theta))\,N}\,\tilde\phi(\theta)\,d\theta - \int_0^{N^{-3/10}} N^{1/2}\,e^{i(1-\cos(\theta))\,N}\,\tilde\phi(0)\,d\theta\right| \le CN^{1/2-3/5}\,,$$

which vanishes in the limit as $N\to\infty$. In view of this smallness, it remains to investigate the integral

$$\tilde\phi(0)\,(2\pi i)^{-1/2}\int_0^{N^{-3/10}} N^{1/2}\,e^{i(1-\cos(\theta))\,N}\,d\theta\,. \tag{4.3}$$

As a result we find, using Lemma A.1 in the appendix and recalling that $\tilde\phi(0) = 2\phi(e_1)$, as $N \to \infty$,

$$\tilde\phi(0)(2\pi i)^{-1/2} \int_0^{N^{-3/10}} N^{1/2}\, e^{i(1-\cos(\theta))\,N}\, d\theta \\ \to \tilde\phi(0)(2\pi i)^{-1/2}\frac{1}{2}(2\pi i)^{1/2} = \phi(e_1)\,.$$

This shows the claim (4.1) in dimension $d = 2$. □

A An oscillatory integral

We want to evaluate the limit of the integral (4.3).

Lemma A.1. *Let $\beta \in (1/6, 1/2)$. Then, as $N \to \infty$,*

$$I_N := \int_0^{N^{-\beta}} N^{1/2}\, e^{i(1-\cos(\theta))\,N}\, d\theta \to \frac{1}{2}\sqrt{\pi}(1+i) = \frac{1}{2}(2\pi i)^{1/2}\,. \tag{A.1}$$

Proof. The integral in (A.1) can be written with the substitution $z = (1-\cos(\theta))N$, leading to $d\theta = dz/(N\sin(\theta))$. We find

$$I_N = \int_0^{(1-\cos(N^{-\beta}))N} e^{iz}\frac{1}{N^{1/2}\sin(\theta)}\, dz\,. \tag{A.2}$$

Next we use the approximation $\frac{1}{N^{1/2}\sin(\theta)} \approx \frac{1}{\sqrt{2z}}$ and $(1-\cos(N^{-\beta}))N \approx \infty$. Indeed,

$$CN^{1-2\beta} \le (1-\cos(N^{-\beta}))N \le \tilde{C}N^{1-2\beta}\,.$$

Since $\beta < 1/2$ one finds $(1-\cos(N^{-\beta}))N \to \infty$ for $N \to \infty$. Regarding the approximation of $\frac{1}{N^{1/2}\sin(\theta)}$ we obtain

$$\sin(\theta) = \sqrt{1-\cos^2(\theta)} = \sqrt{1-(1-z/N)^2} \\ = \sqrt{\frac{2z}{N}}\sqrt{1-\frac{z}{2N}} = \sqrt{\frac{2z}{N}} + O\left((z/N)^{3/2}\right)$$

and thus, expanding the fraction,

$$\frac{1}{N^{1/2}\sin(\theta)} = \frac{1}{\sqrt{2z}+O\left(\sqrt{N}(z/N)^{3/2}\right)} = \frac{1}{\sqrt{2z}} + O\left(z^{1/2}N^{-1}\right)\,.$$

Since in the domain of integration $z \le \tilde{C}N^{1-2\beta}$, we finally find

$$\left|\int_0^{(1-\cos(N^{-\beta}))N} e^{iz}\left(\frac{1}{N^{1/2}\sin(\theta)} - \frac{1}{\sqrt{2z}}\right) dz\right| \\ \le C\left(N^{1-2\beta}\right)^{1/2} N^{-1}\, N^{1-2\beta} = CN^{1/2-3\beta} \to 0$$

as $N \to \infty$, since $\beta > 1/6$. To sum up, we obtain

$$\begin{aligned} I_N &= \int_0^{(1-\cos(N^{-\beta}))N} e^{iz} \frac{1}{N^{1/2}\sin(\theta)}\, dz \\ &\to \int_0^\infty e^{iz} \frac{1}{\sqrt{2z}}\, dz = \sqrt{2} \int_0^\infty e^{ip^2}\, dp = \frac{1}{2}\sqrt{\pi}(1+i) = \frac{1}{2}(2\pi i)^{1/2}\,. \end{aligned}$$

In the last line we used the substitution $z = p^2$ and Fresnel integrals: For real and imaginary part there holds $\int_0^\infty \sin(x^2)\, dx = \int_0^\infty \cos(x^2)\, dx = \sqrt{\pi}/(2\sqrt{2})$. This provides the claim of (A.1). □

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