Sound absorption by perforated walls along boundaries

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Abstract: We analyze the Helmholtz equation in a complex domain. A sound absorbing structure at a part of the boundary is modelled by a periodic geometry with periodicity $\varepsilon > 0$. A resonator volume of thickness ε is connected with thin channels (opening ε^3) with the main part of the macroscopic domain. For this problem with three different scales we analyze solutions in the limit $\varepsilon \to 0$ and find that the effective system can describe sound absorption.

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1 Introduction

We are interested in the mathematical analysis of a sound absorbing structure, e.g., along the wall of a room. The sound absorber consists of a combination of small-scale structures. For the simplest setting one should think of a wooden plate that is attached to the wall. The plate is attached in such a way that a thin gap remains between plate and wall. To create the sound absorption effect, little holes are drilled in the wood to connect the room with the thin volume behind the plate.

In order to analyze the effects of such a structure, we define a geometry with different small scales: The wood is modelled by a layer of thickness $\varepsilon > 0$, the (resonator) volume behind the wood has also a thickness of order ε, the holes are distributed periodically with periodicity ε. The width of the holes is assumed to be of order ε^3; this is the scaling in which a nontrivial limit behavior is observed. We study the Helmholtz equation in the domain that is filled with air, using homogeneous Neumann conditions along all boundaries. Denoting solutions by u^ε, we are

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interested in the behavior of u^ε in the limit $\varepsilon \to 0$. We find two effective systems; they describe sound waves in the volume with the sound absorbing structure.

We derive *two* limit systems, since the lowest order approximation is trivial. At order ε^0, the limit problem coincides with the original Helmholtz problem: The small structures along the boundary have no effect. In this sense, the complex geometry can only lead to an effect of order ε. We derive the effective system for this $O(\varepsilon)$ deviation in Theorem 1.4 below. Due to L^2-unboundedness of relevant functions, the proof is performed with L^1-based function spaces and limit measures for pressure and flux quantities.

The interesting question from the modelling perspective is: Why can the $O(\varepsilon)$ deviation be relevant for sound absorption? We see the answer in the effective equation of Theorem 1.4: The effective system contains the quantity $(\alpha/(LV)) - \omega^2$, where α, L, V are geometric quantities, and ω is the frequency. When the frequency is near to $\sqrt{\alpha/(LV)}$, resonance occurs and the solutions of the $O(\varepsilon)$-system can be very large. When they are of the same order as the inverse periodicity (i.e.: ε^{-1}), then the sound absorber can have a relevant effect. This is discussed towards the end of this introduction.

Geometry. We next describe the domain Ω_ε. It consists of a volume Ω_0 and some small scale structures that are attached to one part of the boundary of Ω_0. To keep the setting simple, we assume that Ω_0 is a rectangle in $\mathbb{R}^2$. With the two positive parameters $a, b > 0$ we denote by the interval $I := (0, a)$ the range of the horizontal coordinate x_1. The limit domain is

$$\Omega_0 := (0, a) \times (-b, 0) = I \times (-b, 0)\,,$$

the upper boundary of Ω_0 is the set $\Gamma_0 := I \times \{0\}$. By slight abuse of notation we will identify functions on I with functions on Γ_0.

Attached to Γ_0 is the resonator volume, which is connected with many thin channels to the volume Ω_0. The channels are distributed periodically with a spacing $\varepsilon > 0$; our analysis is concerned with the limit $\varepsilon \to 0$. We denote by $L > 0$ and $V > 0$ the relative length of the channels and the relative thickness of the resonator volume, respectively. The parameter $\alpha > 0$ denotes a relative width of the channels.

The domain Ω_ε is constructed as the union of three sets as described below (see Figure 1). For simplicity we always assume $a/\varepsilon \in \mathbb{N}$. The resonator strip and the channels are

$$S_\varepsilon := I \times (L\varepsilon, (L+V)\varepsilon)\,, \tag{1.1}$$

$$C_\varepsilon := \bigcup_{k=0}^{(a/\varepsilon)-1} (k\varepsilon, k\varepsilon + \alpha\varepsilon^3) \times [0, L\varepsilon]\,, \tag{1.2}$$

and the domain is defined as

$$\Omega_\varepsilon := \Omega_0 \cup S_\varepsilon \cup C_\varepsilon\,. \tag{1.3}$$

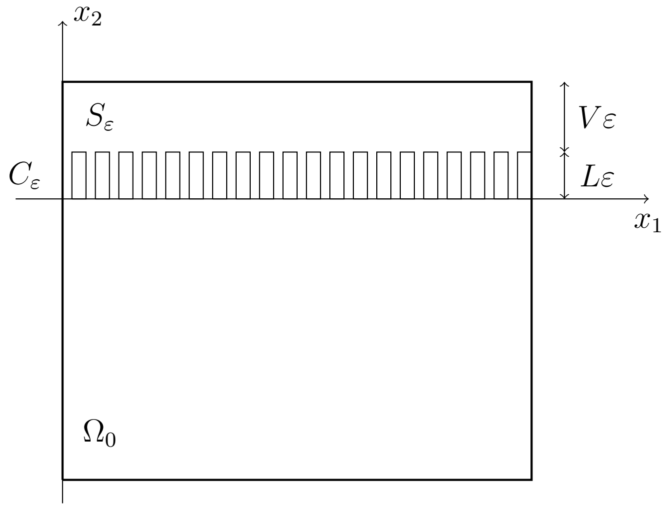

Figure 1: The geometry. The complex domain Ω_ε is given as the union of a limit domain Ω_0 (the domain below the x_1-axis), the set of channels C_ε, and the strip S_ε above the channels. The length of the channels is $L\varepsilon$, the width of the strip S_ε is $V\varepsilon$. The channels are distributed with periodicity ε, the width of the channels is $\alpha\varepsilon^3$.

The upper boundary of Ω_ε is $\Gamma_\varepsilon := I \times \{(L+V)\varepsilon\}$. We emphasize the fact that three scales are involved, since the channels have the width $\alpha\varepsilon^3$. The total volume of the channels is of the order (length $\times$ width $\times$ number) $|C_\varepsilon| \sim \varepsilon\, \varepsilon^3 \cdot \varepsilon^{-1} = \varepsilon^3$.

Main results. We are interested in the limit behavior of a sequence u^ε satisfying the Helmholtz equation

$$\begin{aligned} -\Delta u^\varepsilon - \omega^2 u^\varepsilon &= f \qquad \text{in } \Omega_\varepsilon\,,\\ \partial_n u^\varepsilon &= 0 \qquad \text{on } \partial\Omega_\varepsilon\,. \end{aligned} \tag{1.4}$$

Throughout, we assume that the frequency $\omega > 0$ and the right hand side $f \in L^2(\mathbb{R}^2)$ are given. To simplify calculations, we assume that f has support in Ω_0.

We first provide the following theorem in order to stress that the limit system for (1.4) is trivial.

Theorem 1.1 (Trivial limit equation)**.** *Let $(u^\varepsilon)_{\varepsilon>0}$ be a sequence of solutions to* (1.4) *for some sequence $\varepsilon \to 0$. We assume that $u^\varepsilon \in H^1(\Omega_\varepsilon)$ is bounded and that a weak limit $u \in H^1(\Omega_0)$ exists,*

$$u^\varepsilon|_{\Omega_0} \rightharpoonup u \quad \text{in } H^1(\Omega_0)\,. \tag{1.5}$$

Then u solves the trivial limit problem

$$\begin{aligned} -\Delta u - \omega^2 u &= f \qquad \text{in } \Omega_0\,,\\ \partial_n u &= 0 \qquad \text{on } \partial\Omega_0\,. \end{aligned} \tag{1.6}$$

Proof. Let $\varphi \in C^1(\mathbb{R}^2)$ be an arbitrary test function. In the following calculation, we use first the volume estimate $|\Omega_\varepsilon \setminus \Omega_0| = O(\varepsilon)$, then the weak form of (1.4), and finally decompose the integral and exploit the boundedness of the sequence u^ε:

$$\begin{aligned}\int_{\Omega_0} f\varphi \leftarrow \int_{\Omega_\varepsilon} f\varphi &= \int_{\Omega_\varepsilon} \nabla u^\varepsilon \cdot \nabla\varphi - \omega^2 \int_{\Omega_\varepsilon} u^\varepsilon \varphi \\ &= \int_{\Omega_0} \nabla u^\varepsilon \cdot \nabla\varphi + \int_{\Omega_\varepsilon\setminus\Omega_0} \nabla u^\varepsilon \cdot \nabla\varphi - \omega^2 \int_{\Omega_0} u^\varepsilon\varphi - \omega^2 \int_{\Omega_\varepsilon\setminus\Omega_0} u^\varepsilon\varphi \\ &\to \int_{\Omega_0} \nabla u \cdot \nabla\varphi - \omega^2 \int_{\Omega_0} u\,\varphi\end{aligned}$$

as $\varepsilon \to 0$. We thus obtained the weak form of (1.6). □

The interesting effect in the behavior of solutions becomes visible in the next order in ε. We define two new functions. The first encodes the averages of u^ε with respect to the variable x_2 in the resonator strip S_ε,

$$v^\varepsilon : I \to \mathbb{R}, \quad x_1 \mapsto \frac{1}{\varepsilon V} \int_{\varepsilon L}^{\varepsilon(L+V)} u^\varepsilon(x_1, x_2)\, dx_2\,, \tag{1.7}$$

and the second denotes the corrector from the trivial limit,

$$w^\varepsilon : \Omega_0 \to \mathbb{R}, \quad w^\varepsilon := \frac{u^\varepsilon - u}{\varepsilon}\,. \tag{1.8}$$

We work with the following assumption.

Assumption 1.2. *For some $v \in H^1(I)$ there holds $v^\varepsilon \rightharpoonup v$ in $L^2(I)$. Moreover, the sequence w^ε is bounded in $W^{1,1}(\Omega_0)$ and, for some $w \in W^{1,1}(\Omega_0)$, $w^\varepsilon \rightharpoonup w$ weak-$*$ in $BV(\bar\Omega_0)$. The sequence $u^\varepsilon \in H^1(\Omega_\varepsilon)$ is bounded and the vertical derivative of u^ε satisfies the following boundedness in the channels: For some constant $C > 0$, that does not depend on ε, holds*

$$\frac{1}{\varepsilon^2} \int_{C_\varepsilon} |\partial_2 u^\varepsilon| \le C\,. \tag{1.9}$$

The weak-$*$ convergence of $w^\varepsilon \to w$ in $BV(\bar\Omega_0)$ is equivalent to: $w^\varepsilon \to w$ in $L^1(\Omega_0)$ and $\int_{\Omega_0} \nabla w^\varepsilon \cdot \phi \to \int_{\Omega_0} \nabla w \cdot \phi$ for all $\phi \in C(\bar\Omega_0; \mathbb{R}^2)$.

For the heuristics of Assumption 1.2 we refer to Section 2.1 below.

Remark 1.3. *In what follows it would be sufficient to assume that ∇w is a measure, which is the natural assumption in the context of weak BV-convergence. For the sake of simplicity of notation we stick to the stronger assumption $w \in W^{1,1}(\Omega_0)$.*

We are now in a position to formulate the main result of this article. It determines the limit equation for the function w. By definition of w, the solution u^ε has the expansion $u^\varepsilon \approx u + \varepsilon w$.

Theorem 1.4 (Equations for the corrector)**.** *Let u^ε and u be as in Theorem 1.1. Let v^ε and w^ε be as in* (1.7) *and* (1.8)*. Let Assumption 1.2 hold with limits v and w. Then the equation for w is*

$$\begin{aligned} -\Delta w - \omega^2 w &= 0 && \text{in } \Omega_0\,, \\ \partial_n w &= V\,(\partial_1^2 + \omega^2) v && \text{on } \Gamma_0\,, \\ \partial_n w &= 0 && \text{on } \partial\Omega_0 \setminus \Gamma_0\,, \end{aligned} \tag{1.10}$$

and the equation for v is

$$\left(-\partial_1^2 + \left(\frac{\alpha}{LV} - \omega^2\right)\right) v = \frac{\alpha}{LV} u|_{\Gamma_0}\,. \tag{1.11}$$

The function v has the regularity $v \in W^{2,1}(I)$. System (1.10)–(1.11) *has to be understood in the weak sense: For every $\varphi \in C^1(\bar\Omega_0)$ holds*

$$\int_{\Omega_0} \nabla w \cdot \nabla\varphi - \omega^2 \int_{\Omega_0} w\,\varphi = -V \int_{\Gamma_0} (\partial_1 v\,\partial_1\varphi - \omega^2 v\,\varphi)\,, \tag{1.12}$$

and for every $\psi \in C^1(\bar\Gamma_0)$ holds

$$\int_{\Gamma_0} \partial_1 v\,\partial_1\psi + \int_{\Gamma_0} \left(\frac{\alpha}{LV} - \omega^2\right) v\,\psi = \int_{\Gamma_0} \frac{\alpha}{LV} u\,\psi\,. \tag{1.13}$$

We note that (1.13) *encodes not only* (1.11)*, but additionally the homogeneous Neumann boundary condition $\partial_1 v = 0$ at ∂I.*

We have formulated the limiting system in a form that shows the existence and uniqueness of solutions for almost all frequencies ω. The limit problem for u has a unique solution if ω is not an eigenvalue of the Neumann Laplace operator on Ω_0. Given $u \in H^1(\Omega_0)$ and its trace $u|_{\Gamma_0} \in H^{1/2}(\Gamma_0)$, equation (1.11) with Neumann boundary conditions $\partial_1 v = 0$ at ∂I can be solved for $v \in H^2(I)$. Finally, assuming again that ω is not an eigenvalue of the Neumann Laplace operator on Ω_0, we can solve system (1.10) for $w \in H^1(\Omega_0)$. This line of argument yields not only existence, but also uniqueness of solutions with w of class H^1.

We note that we required less regularity on w in Assumption 1.2. Our results imply that, if the limit has the additional regularity $w \in H^1(\Omega_0)$, then it necessarily coincides with the unique H^1-solution of the limit system (for ω not an eigenvalue of the Neumann Laplace operator of Ω_0).

The limit equation (1.11) can be re-written as

$$(\partial_1^2 + \omega^2) v = \frac{\alpha}{LV}(v - u) \quad \text{on } \Gamma_0\,. \tag{1.14}$$

The boundary condition for w along Γ_0 can therefore be expressed as $\partial_n w = \frac{\alpha}{L}(v-u)$ and equation (1.12) can be written as

$$\int_{\Omega_0} \nabla w \cdot \nabla\varphi - \omega^2 \int_{\Omega_0} w\varphi = \frac{\alpha}{L} \int_{\Gamma_0} (v-u)\varphi\,.$$

The derivation of (1.10) is actually not difficult, we present the proof in Proposition 2.1. The connections between u and v are more involved, we derive two relations in Propositions 3.1 and 3.3. Theorem 1.4 is proved after Proposition 3.3.

Interpretation of the main result. As stressed before, the limit solution u is not affected by the small scale structures along the boundary.

Let us study the limit equation (1.11). The function v depends only on the horizontal coordinate x_1. Let us consider solutions of the form $v(x_1) = v_0 \sin(kx_1)$ and $u(x_1, 0) = u_0 \sin(kx_1)$ for some real parameters $v_0, u_0 \in \mathbb{R}$. Equation (1.11) then reads

$$\left(k^2 + \frac{\alpha}{LV} - \omega^2\right) v_0 = \frac{\alpha}{LV} u_0 \,.$$

This relation implies that, for resonant frequencies ω, the factor v_0 can be much larger than the factor u_0. For small horizontal wave numbers k, this occurs when ω is close to the Helmholtz resonator frequency $\omega_H := \sqrt{\alpha/(LV)}$.

When all the functions w, u, and v have the dependence $\sin(kx_1)$ on x_1, then the problem for w is a homogeneous Helmholtz problem with the upper boundary condition

$$\begin{aligned} \partial_n w &= \frac{\alpha}{L}(v - u) = \frac{\alpha}{L}(v_0 - u_0) \sin(kx_1) \\ &= \frac{\alpha}{L} \left[\frac{\alpha}{LV} \left(k^2 + \frac{\alpha}{LV} - \omega^2\right)^{-1} - 1 \right] u_0 \sin(kx_1) \,. \end{aligned}$$

The factor in squared brackets can be large due to resonance (small denominator). This results in large values of the function w. In the reconstruction of u^ε we obtain $u^\varepsilon \approx u + \varepsilon w$, and the correction has the order $\varepsilon \|w\| = O(\varepsilon (k^2 - \omega^2 + \frac{\alpha}{LV})^{-1})$. Due to the resonance, this can constitute a visible (or, better: audable) contribution even for small periodicity length $\varepsilon > 0$.

Literature. Some of the first mathematical results in the field of homogenization regarded the derivation of limit equations for domains that are periodically perforated, see, e.g., [5]. Quickly, the interest shifted also to geometries where the perforations are along lower dimensional manifolds, we refer to [12, 14] for two early contributions. The periodic unfolding method was adapted to this kind of problems, see [4]. For the problem in the context of fluid mechanics, see [6].

As a “natural scaling” we regard the setting where the periodicity is $\varepsilon > 0$, and the typical size of the obstacles is also ε (in every direction). This scaling was also considered in the papers [7, 8, 9, 10, 17]. The aim of these papers is to provide a thorough analysis of the Neumann problem, for which no effects of order ε^0 are induced by the geometry. In order to derive limit equations one has to analyze higher order effects. Progress was possible in [17] with the consequent use of $W^{1,1}$-spaces: the expansion of the solution has natural bounds in the corresponding norms.

We emphasize that, in the natural scaling, where periodicity, width, and the length of the channels are all of order ε, no resonances can occur. In such a setting, one can only expect that deviations from the trivial limit solution u are of order ε.

We note that another scaling is used, e.g., in [3, 13, 18]: Here, a structure of finite width is analyzed. For a periodicity $\varepsilon > 0$ and a diameter of the channels of order ε, the length of the channels does not tend to 0 as $\varepsilon \to 0$. This scaling allows for resonances in the longitudinal direction of the channels. Yet another setting of the geometry was used, e.g., in [2]: One considers "perforations" in the boundary or in an interface of lower dimension. The resulting system has the character of an oscillatory boundary condition, we mention [1] as a contribution in this vast field.

The combination of two different small length scales in the obstacles can create resonant structures. This is well-known for the Helmholtz resonator and it was used for an analysis of spectral properties in [15]. Using the small Helmholtz resonator as a building block, one can create resonant bulk materials, see [11]. In that work, the resonators are distributed in the whole volume and not only along the boundary. For an overview regarding resonances and homogenization in this spirit, we mention [16].

2 Preliminaries and proof of (1.12)

2.1 Expected orders of different quantities

It might be surprising that we work with L^1-based spaces. The choice of the function space is important. In fact, we claim that working only in L^2-based function spaces is not adequate in the problem at hand. We note that a similar observation was made in [17].

Let us discuss heuristically the behavior of solutions. We expect that u^ε has values of order $\varepsilon^0 = 1$ everywhere, in the domain Ω_0 and in the resonator strip S_ε.

Since the channels are thin, there is is only a weak connection between the volume Ω_0 and the strip S_ε. There is no reason why the values of u^ε at both ends of the channel should be close. We can therefore expect that also the difference $v - u$ is of order 1. As a result, since the length of each channel is of order ε, the derivative $\partial_2 u^\varepsilon$ should be of order ε^{-1} in the channels.

With respect to (1.9) we recall that the total volume of the channels is of the order $|C_\varepsilon| \sim \varepsilon^3$. We can therefore expect that the quantity in (1.9), $\varepsilon^{-2} \int_{C_\varepsilon} |\partial_2 u^\varepsilon|$, is bounded.

We note that the boundedness of $\nabla u^\varepsilon \in L^2(\Omega_\varepsilon)$ implies the following property of horizontal derivatives:

$$\frac{1}{\varepsilon} \int_{C_\varepsilon} |\partial_1 u^\varepsilon| \leq \frac{1}{\varepsilon} \|\nabla u^\varepsilon\|_{L^2(\Omega_\varepsilon)} |C_\varepsilon|^{1/2} \leq C \varepsilon^{3/2-1} \to 0\,. \tag{2.1}$$

We include the warning that L^2-spaces are not adequate for this problem. We expect $\varepsilon^{-2} \int_{C_\varepsilon} |\partial_2 u^\varepsilon|^2 \sim \varepsilon^{-2} |C_\varepsilon| \varepsilon^{-2} \sim \varepsilon^{-1} \to \infty$. In particular, we do not expect that L^2-norms of ∇w^ε are bounded. This is why we work in the L^1-family of norms.

2.2 Proof of (1.12)

In this subsection we derive that the corrector w satisfies (1.10). More precisely, we derive the weak form (1.12).

Proposition 2.1. *Let the sequence u^ε be as in Theorem 1.4. Then the limit function w satisfies the effective equation* (1.12).

Proof. Let $\varphi \in C^1(\mathbb{R}^2)$ by an arbitrary test function. Decomposing the integral over Ω_ε into intgrals over Ω_0 and $S_\varepsilon \cup C_\varepsilon$, the equation for u^ε reads

$$\int_{\Omega_0} \nabla u^\varepsilon \cdot \nabla\varphi - \omega^2 \int_{\Omega_0} u^\varepsilon \varphi + \int_{S_\varepsilon \cup C_\varepsilon} \nabla u^\varepsilon \cdot \nabla\varphi - \omega^2 \int_{S_\varepsilon \cup C_\varepsilon} u^\varepsilon \varphi = \int_{\Omega_\varepsilon} f\varphi = \int_{\Omega_0} f\varphi \,.$$

On the other hand, the equation for u provides

$$\int_{\Omega_0} \nabla u \cdot \nabla\varphi - \omega^2 \int_{\Omega_0} u\varphi = \int_{\Omega_0} f\varphi \,.$$

We subtract the two equations, divide by ε, and insert the definition $w^\varepsilon = (u^\varepsilon - u)/\varepsilon$ to obtain

$$\int_{\Omega_0} \nabla w^\varepsilon \cdot \nabla\varphi - \omega^2 \int_{\Omega_0} w^\varepsilon \varphi = -\frac{1}{\varepsilon}\left\{ \int_{S_\varepsilon \cup C_\varepsilon} \nabla u^\varepsilon \cdot \nabla\varphi - \omega^2 \int_{S_\varepsilon \cup C_\varepsilon} u^\varepsilon \varphi \right\} . \tag{2.2}$$

We consider test functions φ that have the regularity $\varphi|_{\bar\Omega_0} \in C^1(\bar\Omega_0)$, assume that they are independent of x_2 for $x_2 \geq 0$ and that they satisfy $\partial_1\varphi = 0$ in the set $S_\varepsilon \cap \{x_1 < \delta \text{ or } x_1 > a - \delta\}$ for some $\delta > 0$.

Assumption 1.2 on w^ε implies that the left hand side of (2.2) converges, as $\varepsilon \to 0$,

$$\int_{\Omega_0} \nabla w^\varepsilon \cdot \nabla\varphi - \omega^2 \int_{\Omega_0} w^\varepsilon \varphi \to \int_{\Omega_0} \nabla w \cdot \nabla\varphi - \omega^2 \int_{\Omega_0} w\,\varphi \,.$$

We can use Assumption 1.2 also to calculate the right hand side of (2.2). All integrals over C_ε vanish in the limit $\varepsilon \to 0$ because of boundedness of u^ε and ∇u^ε in $L^2(\Omega_\varepsilon)$. For one of the remaining two integrals, we use $\partial_2\varphi = 0$ in S_ε and an integration by parts to find

$$-\frac{1}{\varepsilon}\int_{S_\varepsilon} \nabla u^\varepsilon \,\nabla\varphi = -\frac{1}{\varepsilon}\int_{S_\varepsilon} \partial_1 u^\varepsilon \,\partial_1\varphi = \frac{1}{\varepsilon}\int_{S_\varepsilon} u^\varepsilon \,\partial_1^2\varphi \to V\int_I v\,\partial_1^2\varphi = -V\int_I \partial_1 v\,\partial_1\varphi \,.$$

The last integral satisfies

$$\frac{1}{\varepsilon}\int_{S_\varepsilon} u^\varepsilon\varphi \to V\int_I v\varphi \,,$$

as $\varepsilon \to 0$. Combining these limits, we arrive at

$$\int_{\Omega_0} \nabla w \cdot \nabla\varphi - \omega^2 \int_{\Omega_0} w\,\varphi = V\int_I (-\partial_1 v(x_1)\,\partial_1\varphi(x_1,0) + \omega^2 v(x_1)\varphi(x_1,0))\,dx_1 \,.$$

By density of functions φ as above, this relation holds for all $\varphi \in C^1(\bar\Omega_0)$. We have obtained (1.12). □

2.3 A flux quantity

Relation (1.11) between u and v requires much more involved arguments. We introduce an additional quantity, the vertical flux j^ε and its limit j_*. We set

$$j^\varepsilon(x) := \frac{1}{L\varepsilon^2}\, \partial_2 u^\varepsilon(x)\, \mathbf{1}_{C_\varepsilon}(x)\,, \tag{2.3}$$

where $\mathbf{1}_{C_\varepsilon}$ is the characteristic function of the channels, $\mathbf{1}_{C_\varepsilon}(x) = 1$ for $x \in C_\varepsilon$ and $\mathbf{1}_{C_\varepsilon}(x) = 0$ for $x \notin C_\varepsilon$. The quantity j^ε measures, in a rescaled fashion, the x_2-derivative of u^ε in the channels.

Lemma 2.2. *Let u^ε be a sequence as in Theorem 1.4 and let j^ε be as in* (2.3). *Then there exists a subsequence $\varepsilon \to 0$ and a Radon measure $j_* \in \mathcal{M}(\mathbb{R}^2)$ with* $\mathrm{supp}(j_*) \subset \bar\Gamma_0$ *such that*

$$j^\varepsilon \overset{*}{\rightharpoonup} j_* \tag{2.4}$$

in the sense of Radon measures.

Proof. By Assumption (1.9) on $\partial_2 u^\varepsilon$, the current j^ε is uniformly bounded in $L^1(\mathbb{R}^2)$:

$$\|j^\varepsilon\|_{L^1(\mathbb{R}^2)} = \frac{1}{L\varepsilon^2} \int_{C_\varepsilon} |\partial_2 u^\varepsilon| \le \frac{C}{L}\,.$$

With the two-dimensional Lebesgue measure $\mathcal{L}^2$, we can consider $j^\varepsilon\,\mathcal{L}^2$ as a bounded family of measures. The weak star compactness of Radon measures implies the existence of a subsequence and of a limit measure $j_* \in \mathcal{M}(\mathbb{R}^2)$ with (2.4). The measure j_* is concentrated on $\bar\Gamma_0$ since the measures $j^\varepsilon\,\mathcal{L}^2$ are supported in the channels C_ε, hence in an ε-neighborhood on Γ_0. □

We will use Lemma 2.2 as follows: For every function $\varphi \in C(\mathbb{R}^2)$ there holds, as $\varepsilon \to 0$,

$$\frac{1}{L\varepsilon^2} \int_{C_\varepsilon} \partial_2 u^\varepsilon(x)\varphi(x)\, dx \to \int_{\bar\Gamma_0} \varphi(x)\, dj_*(x)\,. \tag{2.5}$$

As a preparation of one of the subsequent proofs, we note that the arguments of Lemma 2.2 can be repeated for the absolute values of j^ε: We consider $J^\varepsilon := |j^\varepsilon|$. The measures $J^\varepsilon\mathcal{L}^2$ are a bounded family of Radon measures. Along a subsequence $\varepsilon \to 0$ we can therefore assume, for some limit Radon measure $J_* \in \mathcal{M}(\mathbb{R}^2)$ with support in $\bar\Gamma_0$, that $J^\varepsilon \overset{*}{\rightharpoonup} J_*$ in the sense of Radon measures.

3 Relations between u and v

In this section we obtain equation (1.11) for u and v. It is obtained from two other relations that involve u, v, and the flux quantity j_*: The geometric flow rule (3.2) and the mass conservation (3.15). Upon eliminating the flux j_*, we obtain (1.11).

The first of these two new relations is the geometric flow rule and is shown in Proposition 3.1. This geometric rule can be perceived as follows: When u^ε has the typical value v at the upper end of the channel and the typical value u at the lower end of the channel, then the derivative has the typical value $\partial_2 u^\varepsilon \sim (v-u)/(L\varepsilon)$. For the integral of j^ε over a single channel (with length $L\varepsilon$ and width $\alpha\varepsilon^3$) we therefore expect to obtain $L\varepsilon\,\alpha\varepsilon^3/(L\varepsilon^2)\cdot(v-u)/(L\varepsilon) = (\alpha\varepsilon/L)(v-u)$. The factor ε denotes the periodicity. We therefore expect a relation of the form $j_* = (\alpha/L)(v-u)$. The argument is made precise in the following Proposition.

Proposition 3.1 (Geometric flow rule)**.** *Let j_* be as in Lemma 2.2, v and u as in Assumption 1.2 and Theorem 1.1. Then there exists a density function $j \in L^1(I)$ such that*

$$j_*(x) = j(x_1)\,\mathcal{H}^1|_{\Gamma_0}\,. \tag{3.1}$$

The density satisfies

$$j(x_1) = \frac{\alpha}{L}(v(x_1) - u(x_1,0))\,. \tag{3.2}$$

Proof. Once (3.1)–(3.2) are shown, the L^1-regularity of j follows directly from the fact that $u(\cdot,0)$ and v are of class $L^1(I)$. We only have to prove (3.1)–(3.2).

Let $[c,d] \subset [0,a] = \bar{I}$ be an interval. Since j_* is supported on $\bar{\Gamma}_0$, Proposition 3.1 is proved as soon as we can show that the limit measure j_* satisfies

$$\int_{[c,d]\times\{0\}} dj_* = \frac{\alpha}{L}\int_c^d (v(x_1) - u(x_1,0))\,dx_1\,. \tag{3.3}$$

We will use the following function θ^ε with large gradients:

$$\theta^\varepsilon(x_2) := \begin{cases} 0 & \text{for } x_2 \le 0\,,\\ x_2/(\varepsilon L) & \text{for } 0 < x_2 < \varepsilon L\,,\\ 1 & \text{for } x_2 \ge \varepsilon L\,. \end{cases} \tag{3.4}$$

We want to use a localization function $\psi_\varepsilon : [0,a] \to \mathbb{R}$. As a test function we then consider $\varphi^\varepsilon(x_1,x_2) := \psi_\varepsilon(x_1)\theta^\varepsilon(x_2)$. The proof of (3.3) consists in calculating the quantity

$$B_\varepsilon := \frac{1}{\varepsilon}\int_{C_\varepsilon} \nabla u^\varepsilon \cdot \nabla\varphi^\varepsilon \tag{3.5}$$

in two different ways.

As localization function ψ_ε we cannot use the characteristic function $\chi_{[c,d]} : [0,a] \to \{0,1\}$ of the interval $[c,d]$, since the jumps of this function can occur within a channel. We choose to consider all cells that touch the interval $[c,d]$: We define a set $\mathcal{K}_\varepsilon$ of indices as

$$\mathcal{K}_\varepsilon := \{k_1 \in \mathbb{Z}\,|\,\varepsilon k_1 \in [0, a-\varepsilon] \text{ and } (k_1\varepsilon, k_1\varepsilon+\varepsilon)\cap[c,d] \ne \emptyset\}\,. \tag{3.6}$$

The number of elements of $\mathcal{K}_\varepsilon$ is of order $|\mathcal{K}_\varepsilon| = O(\varepsilon^{-1})$. We furthermore introduce $I^\varepsilon_{c,d} := \bigcup_{k_1\in\mathcal{K}_\varepsilon}(k_1\varepsilon, k_1\varepsilon+\varepsilon)$ and set

$$\psi_\varepsilon(x_1) := \begin{cases} 1 & \text{for } x_1 \in I^\varepsilon_{c,d}\,,\\ 0 & \text{else}\,. \end{cases} \tag{3.7}$$

First calculation of B_ε. We write B_ε with the flux variable j^ε as

$$B_\varepsilon = \frac{1}{\varepsilon}\int_{C_\varepsilon} \nabla u^\varepsilon \cdot \nabla \varphi^\varepsilon = \frac{1}{\varepsilon^2 L}\int_{C_\varepsilon \cap \{x_1 \in I^\varepsilon_{c,d}\}} \partial_2 u^\varepsilon = \int_{\{x_1\in I^\varepsilon_{c,d}\}} j^\varepsilon \, d\mathcal{L}^2 \,.$$

We claim that this implies, as $\varepsilon \to 0$,

$$B_\varepsilon \to \int_{[c,d]\times\{0\}} dj_* \,. \tag{3.8}$$

Indeed, for every $\delta > 0$ and for every $\varepsilon < \delta$, there holds, as $\varepsilon \to 0$,

$$\begin{aligned}\left| B_\varepsilon - \int_{\{x_1\in[c,d]\}} dj_* \right| &\le \left| \int_{\{x_1\in(0,a)\cap(c-\delta,d+\delta)\}} j^\varepsilon \, d\mathcal{L}^2 - \int_{\{x_1\in(c-\delta,d+\delta)\}} dj_* \right| \\ &\quad + \int_{\{x_1\in(c-\delta,c)\cup(d,d+\delta)\}} J^\varepsilon \, d\mathcal{L}^2 + \left| \int_{\{x_1\in(c-\delta,c)\cup(d,d+\delta)\}} dj_* \right| \\ &\to \int_{\{x_1\in(c,c+\delta)\cup(d-\delta,d)\}} dJ_* + \left| \int_{\{x_1\in(c,c+\delta)\cup(d-\delta,d)\}} dj_* \right| \,.\end{aligned}$$

By outer regularity of the Radon measures J_* and j_*, the right hand side is arbitrarily small for small $\delta > 0$. This verifies (3.8).

Second calculation of B_ε. The second calculation of B_ε is based on a quite elementary integration by parts: The integral of the derivative is given by the difference of values at top and bottom of the channels.

To perform the calculation, we need some additional notation. Recall that the microscopic channels are defined as $C_\varepsilon := \bigcup_{k_1=0}^{a/\varepsilon-1}(k_1\varepsilon, k_1\varepsilon + \alpha\varepsilon^3)\times[0, L\varepsilon]$. We define the union of the lower and upper channel boundaries in the interval (c,d) as

$$\Gamma^U_\varepsilon := \bigcup_{k_1\in\mathcal{K}_\varepsilon} (k_1\varepsilon, k_1\varepsilon+\alpha\varepsilon^3)\times\{0\} \quad \text{and} \quad \Gamma^V_\varepsilon := \bigcup_{k_1\in\mathcal{K}_\varepsilon} (k_1\varepsilon, k_1\varepsilon+\alpha\varepsilon^3)\times\{\varepsilon L\} \,.$$

With this notation, an integration by parts provides

$$B_\varepsilon = \frac{1}{\varepsilon^2 L}\int_{C_\varepsilon\cap\{x_1\in I^\varepsilon_{c,d}\}} \partial_2 u^\varepsilon = \frac{1}{\varepsilon^2 L}\left(\int_{\Gamma^V_\varepsilon} u^\varepsilon - \int_{\Gamma^U_\varepsilon} u^\varepsilon \right) \,. \tag{3.9}$$

It remains to determine the limit on the right hand side of (3.9). We will prove that

$$\lim_{\varepsilon\to0} \frac{1}{\varepsilon^2}\int_{\Gamma^V_\varepsilon} u^\varepsilon = \alpha \int_c^d v(x_1)\, dx_1 \,, \tag{3.10}$$

$$\lim_{\varepsilon\to0} \frac{1}{\varepsilon^2}\int_{\Gamma^U_\varepsilon} u^\varepsilon = \alpha \int_c^d u(x_1,0)\, dx_1 \,. \tag{3.11}$$

Once (3.10)–(3.11) is shown, the proof of the proposition is complete: together with (3.8), we obtain

$$\int_{[c,d]\times\{0\}} dj_* = \lim_{\varepsilon\to0} B_\varepsilon = \frac{\alpha}{L}\int_c^d (v(x_1) - u(x_1,0))\, dx_1 \,.$$

Since $[c,d]$ was arbitrary, relations (3.1)–(3.2) are verified.

Verification of (3.10)–(3.11). We consider the unit cell $Y := (0,1)\times(-1,L+V)$ and the index set $\mathcal{K}_\varepsilon$ of (3.6) and study the following averaged functions on Y:

$$U^\varepsilon(y_1,y_2) := \frac{1}{|\mathcal{K}_\varepsilon|}\sum_{k_1\in\mathcal{K}_\varepsilon} u^\varepsilon(\varepsilon(k_1+y_1),\varepsilon y_2) \quad \text{for } (y_1,y_2)\in Y\,.$$

The (rescaled) channel in the periodicity cell Y is $C_Y^\varepsilon := (0,\alpha\varepsilon^2)\times[0,L]\subset Y$. Its lower and upper boundary are the sets $\Gamma_Y^{\varepsilon,U} := [0,\alpha\varepsilon^2]\times\{0\}$ and $\Gamma_Y^{\varepsilon,V} := [0,\alpha\varepsilon^2]\times\{L\}$. The domain below the channel is $Y_U := (0,1)\times(-1,0)$, the domain above the channel is $Y_V := (0,1)\times(L,L+V)$.

Our first aim is to prove that the restrictions $U^\varepsilon|_{Y_U}$ and $U^\varepsilon|_{Y_V}$ converge (weakly in L^2) to constant functions.

Using Jensen's inequality and the fact that $|\mathcal{K}_\varepsilon|^{-1} = \varepsilon/(d-c)+O(\varepsilon^2)$, we find

$$\begin{aligned}
\int_{Y_U}|U^\varepsilon(y)|^2\,dy &\le \frac{1}{|\mathcal{K}_\varepsilon|}\sum_{k_1\in\mathcal{K}_\varepsilon}\int_{Y_U}|u^\varepsilon(\varepsilon(y_1+k_1),\varepsilon y_2)|^2\,dy\\
&= \frac{1}{\varepsilon^2|\mathcal{K}_\varepsilon|}\sum_{k_1\in\mathcal{K}_\varepsilon}\int_{(k_1\varepsilon,k_1\varepsilon+\varepsilon)\times(-\varepsilon,0)}|u^\varepsilon(x)|^2\,dx\\
&\le \frac{1}{\varepsilon^2}\left(\frac{\varepsilon}{d-c}+O(\varepsilon^2)\right)\int_{(c,d)\times(-\varepsilon,0)}|u^\varepsilon(x)|^2\,dx\\
&\le \left(\frac{1}{d-c}+O(\varepsilon)\right)\left(\frac{1}{\varepsilon}\int_{(c,d)\times(-\varepsilon,0)}|u^\varepsilon(x)|^2\,dx\right)\le C\,,
\end{aligned}$$

where in the last step we exploited the boundedness of u^ε in $H^1(\Omega_0)$: The second bracket in the last line converges to the L^2-norm of the trace of u^ε on Γ_0. We have obtained that the sequence $U^\varepsilon|_{Y_U}$ is uniformly bounded in $L^2(Y_U)$.

Regarding $U^\varepsilon|_{Y_V}$ we perform the same calculation and use, in the last step,

$$\begin{aligned}
\int_{Y_V}|U^\varepsilon(y)|^2\,dy &\le C\frac{1}{\varepsilon}\int_c^d\int_{L\varepsilon}^{(L+V)\varepsilon}|u^\varepsilon(x)|^2\,dx\\
&\le 2C\frac{1}{\varepsilon}\int_c^d\int_{L\varepsilon}^{(L+V)\varepsilon}|u^\varepsilon(x)-v^\varepsilon(x_1)|^2\,dx + 2C\frac{1}{\varepsilon}\int_c^d\int_{L\varepsilon}^{(L+V)\varepsilon}|v^\varepsilon(x_1)|^2\,dx\\
&\le O(\varepsilon)+2CV\int_c^d|v^\varepsilon(x_1)|^2\,dx_1 \le C\,,
\end{aligned}$$

where we used the one-dimensional Poincaré (also called Poincaré-Wirtinger) inequality with averages $v^\varepsilon(x_1)$ for the first integral, exploiting that the domain size is $V\varepsilon$. In the last estimate we used the boundedness of v^ε in $L^2((0,a))$ that was assumed in Assumption 1.2.

The above estimates allow to proceed with the weak L^2-compactness of bounded sequences. There exist limit functions U and V such that, up to a subsequence, $U^\varepsilon|_{Y_U}\rightharpoonup U$ in $L^2(Y_U)$ and $U^\varepsilon|_{Y_V}\rightharpoonup V$ in $L^2(Y_V)$ as $\varepsilon\to 0$. It is not difficult to verify that the limit functions U and V are constant functions. Indeed, the gradient of $U^\varepsilon|_{Y_U}$ satisfies, in the limit $\varepsilon \to 0$,

$$\begin{aligned}\int_{Y_U} |\nabla U^\varepsilon(y)|^2\,dy &\le \frac{\varepsilon^2}{|\mathcal{K}_\varepsilon|}\sum_{k_1\in\mathcal{K}_\varepsilon}\int_{Y_U}|\nabla u^\varepsilon(\varepsilon(y_1+k_1),\varepsilon y_2)|^2\,dy\\ &= \frac{1}{|\mathcal{K}_\varepsilon|}\sum_{k_1\in\mathcal{K}_\varepsilon}\int_{(k_1\varepsilon,k_1\varepsilon+\varepsilon)\times(-\varepsilon,0)}|\nabla u^\varepsilon(x)|^2\,dx\\ &\le \left(\frac{\varepsilon}{d-c}+O(\varepsilon^2)\right)\int_{(c,d)\times(-\varepsilon,0)}|\nabla u^\varepsilon(x)|^2\,dx \to 0\,,\end{aligned}$$

since u^ε is bounded in $H^1(\Omega_\varepsilon)$. Analogously, $\|\nabla U^\varepsilon\|_{L^2(Y_V)} \to 0$ and we obtain $\nabla U = \nabla V = 0$. As a consequence, for two real numbers $\xi_U, \xi_V \in \mathbb{R}$, the constant functions are $U \equiv \xi_U$ and $V \equiv \xi_V$.

In our next step we identify the constants ξ_U and ξ_V. There holds

$$\begin{aligned}\xi_U \leftarrow \int_{Y_U} U^\varepsilon(y)\,dy &= \frac{1}{|\mathcal{K}_\varepsilon|}\sum_{k_1\in\mathcal{K}_\varepsilon}\int_{Y_U} u^\varepsilon(\varepsilon(y_1+k_1,\varepsilon y_2)\,dy\\ =&\frac{1}{\varepsilon^2|\mathcal{K}_\varepsilon|}\sum_{k_1\in\mathcal{K}_\varepsilon}\int_{(k_1\varepsilon,k_1\varepsilon+\varepsilon)\times(-\varepsilon,0)} u^\varepsilon(x_1,x_2)\,dx\\ =&\left(\frac{1}{d-c}+O(\varepsilon)\right)\left(\frac{1}{\varepsilon}\int_{I^\varepsilon_{(c,d)}\times(-\varepsilon,0)} u^\varepsilon(x_1,x_2)\,dx\right)\\ \to&\frac{1}{d-c}\int_c^d u(x_1,0)\,dx_1\,.\end{aligned}$$

Analogously, using definition (1.7) of v^ε and the weak convergence $v_\varepsilon \rightharpoonup v$ in $L^2(I)$,

$$\begin{aligned}V\,\xi_V \leftarrow \int_{Y_V} U^\varepsilon(y)\,dy &= \frac{1}{|\mathcal{K}_\varepsilon|}\sum_{k_1\in\mathcal{K}_\varepsilon}\int_{Y_V} u^\varepsilon(\varepsilon(y_1+k_1,\varepsilon y_2)\,dy\\ =&\left(\frac{1}{d-c}+O(\varepsilon)\right)V\left(\frac{1}{V\varepsilon}\int_{I^\varepsilon_{(c,d)}\times(\varepsilon L,\varepsilon(L+V))} u^\varepsilon(x)\,dx\right)\\ =&\left(\frac{V}{d-c}+O(\varepsilon)\right)\int_{I^\varepsilon_{(c,d)}} v^\varepsilon(x_1)\,dx_1\\ \to&\frac{V}{d-c}\int_c^d v(x_1)\,dx_1\,.\end{aligned}$$

We have found

$$\xi_U = \frac{1}{d-c}\int_c^d u(x_1,0)\,dx_1 \quad\text{and}\quad \xi_V = \frac{1}{d-c}\int_c^d v(x_1)\,dx_1\,. \tag{3.12}$$

At this point, we identified the averages of U^ε (below and above the channel) with u and v. In order to check (3.10)–(3.11), it remains to relate averages of U^ε in the bulk areas to averages of U^ε in the ends of the channel. This can be done with a Lemma that was proved and used in [11].

We use Lemma A.1 of [11] with slightly adapted notation. The obstacle in the single cell Y is given by $\Sigma_Y^\varepsilon := Y \setminus (\overline{Y_U} \cup \overline{Y_V} \cup \overline{C_Y^\varepsilon})$. We furthermore assume only boundedness of the gradients of U^ε and not the L^2-boundedness everywhere (also in the channel). An inspection of the proof in [11] shows that this is sufficient.

The essential part of the proof is the following: With the tangential vector in each channel being e_2, one considers the functions $V^\varepsilon := \partial_2 U^\varepsilon$. These functions solve the same Helmholtz equation and they satisfy homogeneous boundary conditions: Dirichlet conditions on one part of the boundary, Neumann conditions on the other. This allows to multiply the equation for V^ε with V^ε. One finds uniform H^1-estimates for V^ε which yield uniform H^2-estimates for U^ε. The embedding $H^2(Y_U) \subset C^0(Y_U)$ (accordingly for Y_V) allows to compare point values of U^ε with averages of U^ε.

Lemma 3.2 (Adaption of Lemma A.1 from [11])**.** *Let $U^\varepsilon : Y \setminus \overline{\Sigma_Y^\varepsilon} \to \mathbb{R}$ be a family of H^1-functions such that the L^2-norms of ∇U^ε are bounded. We assume that every U^ε solves the Helmholtz equation*

$$\begin{aligned} -\Delta U^\varepsilon &= \omega^2 \varepsilon^2 U^\varepsilon \quad \text{in } Y \setminus \overline{\Sigma_Y^\varepsilon}\,, \\ \partial_n U^\varepsilon &= 0 \quad \text{on } \partial \Sigma_Y^\varepsilon\,. \end{aligned}$$

No boundary conditions are imposed on ∂Y. Assume that

$$\begin{aligned} U^\varepsilon|_{Y_U} &\rightharpoonup \xi_U \quad \text{in } L^2(Y_U)\,, \\ U^\varepsilon|_{Y_V} &\rightharpoonup \xi_V \quad \text{in } L^2(Y_V)\,, \end{aligned}$$

as $\varepsilon \to 0$. Then

$$⨍_{\Gamma_Y^{\varepsilon,U}} U^\varepsilon(y)\, d\mathcal{H}^1(y) \to \xi_U \quad \text{and} \quad ⨍_{\Gamma_Y^{\varepsilon,V}} V_\varepsilon(y)\, d\mathcal{H}^1(y) \to \xi_V\,. \tag{3.13}$$

The Lemma can indeed be applied. (a) U^ε solves the (rescaled) Helmholtz equation with Neumann boundary condition since u^ε satisfies the (non-rescaled) system. (b) The L^2-boundedness of ∇U^ε follows easily from H^1-boundedness of u^ε (compare the calculations above in this proof). (c) The weak L^2 limits ξ_U and ξ_V have been verified above.

With the result of the lemma at hand, it only remains to compare the limits in (3.10)–(3.11) with the limits in (3.13). We calculate with $\Gamma_Y^{\varepsilon,U} := \overline{C_Y^\varepsilon} \cap \overline{Y_U} = [0, \alpha\varepsilon^2] \times \{0\}$:

$$\begin{aligned} ⨍_{\Gamma_Y^{\varepsilon,U}} U^\varepsilon(y)\, d\mathcal{H}^1(y) &= \frac{1}{\alpha\varepsilon^2} \int_{\Gamma_Y^{\varepsilon,U}} \frac{1}{|\mathcal{K}_\varepsilon|} \sum_{k_1 \in \mathcal{K}_\varepsilon} u^\varepsilon(\varepsilon(y_1 + k_1, \varepsilon y_2)\, d\mathcal{H}^1(y) \\ &= \frac{1}{\alpha\varepsilon^2} \frac{1}{|\mathcal{K}_\varepsilon|} \frac{1}{\varepsilon} \int_{\Gamma_\varepsilon^U} u^\varepsilon(x)\, d\mathcal{H}^1(x) = \left(\frac{1}{\varepsilon^2 \alpha (d-c)} + O\left(\frac{1}{\varepsilon}\right)\right) \int_{\Gamma_\varepsilon^U} u^\varepsilon(x)\, d\mathcal{H}^1(x)\,. \end{aligned}$$

The results (3.13) and (3.12) thus imply

$$\frac{1}{\varepsilon^2}\int_{\Gamma_\varepsilon^U} u^\varepsilon(x)\,d\mathcal{H}^1(x) \to \alpha(d-c)\xi_U = \alpha\int_c^d u(x_1,0)\,dx_1\,,$$

which is the claim (3.11). The limit (3.10) is obtained in an analogous way. □

It remains to derive a further relation between u, v, and j. We obtain a relation from mass conservation in the resonator volume: The flux through the channels (and hence the density j) can be expressed in terms of v.

Proposition 3.3 (Mass conservation)**.** *Let j be as in Proposition 3.1 and v as in Assumption 1.2. Then*

$$j = V\partial_1^2 v + V\omega^2\, v \tag{3.14}$$

in the sense of distributions. Furthermore, for every $\psi \in C^1(\bar I)$,

$$\int_I j(x_1)\psi(x_1)\,dx_1 = -\int_I V\partial_1 v(x_1)\partial_1\psi(x_1)\,dx_1 + \int_I V\omega^2 v(x_1)\psi(x_1)\,dx_1\,, \tag{3.15}$$

and v has the regularity $\partial_1^2 v \in L^1(I)$.

Proof. We fix $\psi \in C^2(\bar I) = C^2([0,a])$ with $\partial_1\psi(0) = \partial_1\psi(a) = 0$. We use θ^ε of (3.4) and consider $\varphi^\varepsilon(x_1,x_2) := \psi(x_1)\theta^\varepsilon(x_2)$ in equation (2.2) for w^ε. The left hand side vanishes since φ^ε vanishes in Ω_0. There remains

$$\frac{1}{\varepsilon}\int_{S_\varepsilon\cup C_\varepsilon} \nabla u^\varepsilon\cdot\nabla\varphi^\varepsilon = \frac{\omega^2}{\varepsilon}\int_{S_\varepsilon\cup C_\varepsilon} u^\varepsilon\,\varphi^\varepsilon\,. \tag{3.16}$$

Left hand side of (3.16). We calculate

$$\frac{1}{\varepsilon}\int_{S_\varepsilon\cup C_\varepsilon}\nabla u^\varepsilon\cdot\nabla\varphi^\varepsilon = \frac{1}{\varepsilon}\int_{C_\varepsilon}\partial_1 u^\varepsilon(x_1,x_2)\partial_1\psi(x_1)\,\frac{x_2}{\varepsilon L} + \frac{1}{\varepsilon}\int_{S_\varepsilon}\partial_1 u^\varepsilon(x_1,x_2)\partial_1\psi(x_1)$$
$$+\frac{1}{\varepsilon}\int_{C_\varepsilon}\partial_2 u^\varepsilon\frac{1}{L\varepsilon}\psi(x_1)\,.$$

Since $\partial_1 u^\varepsilon$ satisfies (2.1), the first integral vanishes in the limit $\varepsilon\to 0$. For the second term we find

$$\frac{1}{\varepsilon}\int_{S_\varepsilon}\partial_1 u^\varepsilon(x_1,x_2)\partial_1\psi(x_1) = -\frac{1}{\varepsilon}\int_{S_\varepsilon} u^\varepsilon(x_1,x_2)\partial_1^2\psi(x_1) \to -V\int_I v(x_1)\partial_1^2\psi(x_1)\,dx_1$$

by the weak convergence $v^\varepsilon \rightharpoonup v$ in $L^1(I)$. For the last term we exploit Lemma 2.2, which ensures that

$$\frac{1}{\varepsilon}\int_{C_\varepsilon}\partial_2 u^\varepsilon\frac{1}{L\varepsilon}\psi(x_1) = \int_{\mathbb{R}^2} j^\varepsilon\psi(x_1) \to \int_{\Gamma_0}\psi(x_1)\,dj_*(x) = \int_I \psi(x_1)j(x_1)\,dx_1\,.$$

Right hand side of (3.16). We obtain

$$\frac{\omega^2}{\varepsilon}\int_{S_\varepsilon\cup C_\varepsilon} u^\varepsilon\,\varphi^\varepsilon = \frac{\omega^2}{\varepsilon}\left(\int_{C_\varepsilon} u^\varepsilon(x_1,x_2)\,\psi(x_1)\frac{x_2}{\varepsilon L} + \int_{S_\varepsilon} u^\varepsilon(x_1,x_2)\,\psi(x_1)\right)$$
$$\to V\omega^2\int_I v(x_1)\psi(x_1)$$

as $\varepsilon\to 0$, where we used the convergence of averages in S_ε to v and, for the first term, $\varepsilon^{-1}\int_{C_\varepsilon}|u^\varepsilon|\le\varepsilon^{-1}\|u^\varepsilon\|_{L^2}|C_\varepsilon|^{1/2}\le C\varepsilon^{1/2}\to 0$. We obtain from (3.16)

$$\int_I j(x_1)\psi(x_1)\,dx_1 = \int_I V v(x_1)\partial_1^2\psi(x_1)\,dx_1 + \int_I V\omega^2 v(x_1)\psi(x_1)\,dx_1\,. \tag{3.17}$$

Relation (3.17) provides (3.14). In particular, the distribution $\partial_1^2 v$ is expressed by the L^1-functions v and j (compare Proposition 3.1). We therefore find $v\in W^{2,1}(I)$.

Relation (3.15) follows with another integration by parts from (3.17). The set of test functions is dense in $H^1(I)$, hence (3.15) holds for all $\psi\in H^1(I)$ and, in particular, for all $\psi\in C^1(\bar I)$. □

We can now formally conclude the proof of our main theorem.

Proof of Theorem 1.4. Equation (1.12) for w was checked in Proposition 2.1. Due to Propositions 3.3 and 3.1 we find

$$V(\partial_1^2 v(x_1)+\omega^2 v(x_1)) = j(x_1) = \frac{\alpha}{L}(v(x_1)-u(x_1,0))$$

in the sense of distributions. Re-ordering terms, we may write this relation equivalently as

$$-\partial_1^2 v(x_1) + \left(\frac{\alpha}{LV}-\omega^2\right)v(x_1) = \frac{\alpha}{LV}u(x_1,0)\,,$$

which is relation (1.11). For a test function $\psi\in C^1(\bar I)$, the distribution $\partial_1^2 v$ can be integrated by parts once without boundary terms. We therefore have obtained also (1.13), which encodes additionally the homogeneous Neumann boundary condition for v. □

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