Homogenization of systems of wave equations and ring solutions with dispersive profiles

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Abstract: We consider systems of wave equations such as the time-dependent Lamé system or elasticity. When the coefficients are periodic in space, the classical task in homogenization theory is to describe limits of solutions when the periodicity tends to zero. The effective equation is a system with constant coefficients, typically of the same structure as the original system. Instead, when long time intervals are considered, new dispersive terms can appear in the effective system. We derive such dispersive effective systems of wave equations using the Bloch method of homogenization. The method yields approximate representation formulas for solutions in Fourier space. These also allow to describe solutions as superpositions of ring waves, expanding with constant speed, with profiles that change on a slow time scale according to the dispersive terms.

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1 Introduction

We study wave systems of the form

$$\partial_t^2 u^\varepsilon(x,t) = \nabla \cdot (A(x/\varepsilon)\nabla u^\varepsilon(x,t)) \tag{1.1}$$

with initial data $u^\varepsilon(x,0) = u_0(x)$ and $\partial_t u^\varepsilon(x,0) = u_1(x)$. The independent variables are $(x,t) \in \mathbb{R}^d \times [0,\infty)$ for an arbitrary dimension $d \in \mathbb{N}$, the solution is a map $u^\varepsilon : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^p$ for some $p \in \mathbb{N}$. We are interested in the description of solutions in the limit $\varepsilon \to 0$. When solutions are studied on fixed time intervals $t \in [0, T_0]$, this is the classical task of homogenization theory, [1, 7, 9, 17]. The result is essentially that solutions u^ε converge to the solution u of the limit problem which is defined with the classically homogenized elliptic operator. To obtain strong homogenization limits, one has to be careful with initial conditions, we mention [10] for an analysis.

New effects appear when large time intervals $t \in [0, T_0\varepsilon^{-2}]$ are studied, the effective system then is a dispersive wave equation. While such large time effects

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are well understood for scalar equations, systems are more complicated and we are only aware of the large time homogenization results of [6, 14]; we compare the results below.

Our two main results are Theorem 2.1 and Theorem 3.3. The first is a homogenization result for large times: We derive an approximate representation formula in Fourier space, the approximation is valid for small $\varepsilon > 0$, uniformly in time on time-intervals of the form $[0, T_0/\varepsilon^2]$. In our second result, we conclude that waves expand in the form of p rings. In the special case of the two-dimensional Lamé system, we observe two rings, a larger ring of a pressure wave and a smaller ring of a shear wave. We quantify how dispersion affects the wave profiles in a slow time scale.

Let us clarify the notation that is used in the wave system (1.1). The spatial gradient of the solution is a map $\nabla u^\varepsilon : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^{p\times d}$. Accordingly, the coefficient of the elliptic operator is tensor-valued, $A : \mathbb{R}^d \ni y \mapsto A(y) \in \mathbb{R}^{d\times d\times p\times p}$, for every point y, the tensor $A(y)$ is a linear map $\mathbb{R}^{p\times d} \to \mathbb{R}^{p\times d}$. We assume that $y \mapsto A(y)$ is 2π-periodic in every direction, further symmetry and coercivity assumptions are specified after (2.1). With indices $\ell \le p$, the system can be written as $\partial_t^2 u_\ell^\varepsilon(x,t) = \sum_{i,j=1}^d \sum_{k=1}^p \partial_i (A_{\ell,k}^{i,j}(x/\varepsilon)\partial_j u_k^\varepsilon(x,t))$. Another interesting case with $p = d$ is the Lamé system.

1.1 Main results

Theorem 2.1 is concerned with the wave system (1.1) and its solution u^ε on large time scales, $t \in [0, T_0/\varepsilon^2]$. We provide a function U^ε that is described with an explicit formula in Fourier space. We show that $\|u^\varepsilon(\cdot,t) - U^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}$ is small for small $\varepsilon > 0$, independent of $t \in [0, T_0/\varepsilon^2]$, see (2.5).

This first result can be compared with [16], it is a Bloch-expansion homogenization result. While [16] deals with the elliptic elasticity problem, we are interested in the wave equation. Related, but more relevant is a second difference: While only leading order terms are included in [16], we include corrections of order ε^2. This allows to derive also dispersive long-time effects. We note that a more classical two scale approach to long-time homogenization of systems of wave equations has been performed recently, see [5] and [6].

Our second main result is formulated in Theorem 3.3. This theorem deals with an analysis of the function U^ε that is described in Fourier space with an effective dispersion relation ω_m^*, compare (2.4). We show that U^ε can be described as a superposition of rings. Those rings expand with speed c_m (where m is the index for the m-th ring). In this result, we must assume that c_m is independent of the direction in space. The dispersion of the profile function of the m-th ring is dictated by the function $b_m(q)$, which is the next term in the expansion of ω_m^*. This dispersive effect may depend on the direction, $q \in S^{d-1}$ is a direction in space. Constructing a multiple ring solution $w^\varepsilon(x,t)$, we find that $\|w^\varepsilon(\cdot,t) - U^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}$ is small for small $\varepsilon > 0$ and for large times of order $1/\varepsilon^2$, see (3.11).

When the two theorems are combined, they yield the following. Let u^ε be the solution of (1.1). Let $\delta > 0$ be an arbitrary error quantifier. A function $w^\varepsilon(x,t)$ can be constructed as a superposition of rings, using the initial data u_0 and the system tensor $A = A(y)$, such that, for any $0 < t_0 < T_0 < \infty$, there holds

$$\limsup_{\varepsilon \to 0} \sup_{t \in [t_0/\varepsilon^2, T_0/\varepsilon^2]} \|u^\varepsilon(\cdot, t) - w^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^d)} \leq \delta \,. \tag{1.2}$$

Theorem 3.3 requires that the speeds c_m are direction independent; we verify this property in Appendix C for a two-dimensional Lamé system where, for simplicity, we assume that only the second Lamé coefficient is periodically oscillatory in space. Therefore, a function w^ε can be defined as in (A.9)–(A.11) with profiles V_p and V_s such that w^ε approximates the solution u^ε of the oscillatory Lamé system as specified in (1.2). For non-constant coefficients, the profiles V_p and V_s will depend on the rescaled time variable $\tau = \varepsilon^2 t$. The dependence is given by the dispersive terms which can be calculated from the fourth order derivatives of the dispersion relation in $k = 0$, see (3.2) for b_m and (3.6)–(3.7) for $\hat{V}_m$.

1.2 Literature

Rigorous homogenization with the help of Bloch wave expansions started with [3, 4, 9, 11, 22]. The essential observation is that homogenized coefficients can be obtained from the dispersion relation of the Floquet-Bloch description. Even simpler: The second derivatives of the eigenvalue $\lambda(k = 0) = 0$ yield the homogenized coefficients. In the case of systems, one has a multiple eigenvalue $\lambda = 0$ and one must study additionally the eigenvectors of this reduced problem, see [16]. A comparison of the Bloch method with more classical methods was given in [2].

Regarding the analysis of long time spans in the context of homogenization, the theory started with [23], again based on a formal Bloch analysis. A first rigorous proof for dispersive effects in long-time homogenization of the wave equation appeared in [20], treating the one-dimensional case, and in [12, 13] for arbitrary dimension. The long-time homogenization was extended to an analysis of equations on a discrete lattice in [24] and to a stochastic scalar setting in [8, 14]. Even though this is not explicitly carried out, the approach of [14] is capable of treating also wave systems provided that the initial data vanish.

Except for [20], all the above approaches are based on Bloch waves (obviously with deep adaptations in the stochastic setting), it is also possible to obtain dispersive effects in the more classic framework by a derivation of higher correctors in the analysis of the elliptic operator. This was performed in [5] for equations and in [6] for systems. We note that [6] considers wave systems with vanishing initial data, whereas the initial value problem is studied here. Another difference with [6] is the error norm. In [6], classical energy methods were used to obtain L^2-estimates of the spatial and temporal gradients, while this article is based on spectral methods resulting in L^2-estimates of the functions themselves.

Regarding our second main result, namely the approximation of the solution with rings, we mention [24] for such a result in space dimension $d = 2$ and [21] for a result in $d \leq 3$. Both of these results treat only the scalar case. To the best of our knowledge, so far, there are no results on multiple ring solutions as they appear in systems of wave equations.

Illustrations of our results on ring solutions for the Lamé system (with constant coefficients) are given in Appendix A. The calculations clearly show that solutions exhibit a multiple ring structure for large times.

2 Approximation in Fourier space

We consider the wave system $\partial_t^2 u^\varepsilon = -L_\varepsilon u^\varepsilon$ of (1.1), with the elliptic operator

$$L_\varepsilon u^\varepsilon(x,t) := -\nabla \cdot (A(x/\varepsilon)\nabla u^\varepsilon(x,t)) \,. \tag{2.1}$$

The coefficient $A : \mathbb{R}^d \ni y \mapsto A(y) \in \mathbb{R}^{d\times d\times p\times p}$ is assumed to be 2π-periodic in every direction y_j, $j = 1, ..., d$. We may also say that A is Y-periodic for the periodicity cell $Y := [0, 2\pi)^d$. We demand that, for every y, the map $A(y) : \mathbb{R}^{p\times d} \to \mathbb{R}^{p\times d}$ is self-adjoint in the sense that $\zeta : A(y)\xi = A(y)\zeta : \xi$ for matrices $\zeta, \xi \in \mathbb{R}^{d\times p}$. We furthermore assume the coercivity $\zeta : A(y)\zeta \geq \gamma\|\zeta\|^2$ with $\gamma > 0$ independent of $y \in \mathbb{R}^d$ and independent of $\zeta \in \mathbb{R}^{d\times p}$.

To cover also the system of elasticity, we introduce also a weaker coercivity requirement in the case $d = p$. We say that A is coercive in the sense of elasticity when $A(y)\zeta = 0$ holds for every y and every skew-symmetric matrix ζ (i.e.: $\zeta^T = -\zeta$), and for some $\gamma > 0$ and every $y \in \mathbb{R}^d$ holds $\zeta : A(y)\zeta \geq \gamma\|\zeta\|^2$ for symmetric matrices (i.e.: $\zeta^T = \zeta$).

Our analysis is based on the Fourier transform. We use the following notation: For $u \in L^2(\mathbb{R}^n, \mathbb{C}^p)$, the Fourier transform and its inverse are given by

$$\mathcal{F}(u)(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} u(x)e^{-ik\cdot x}\,dx \quad \text{and} \quad u(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathcal{F}(u)(k)e^{ik\cdot x}\,dk\,.$$

The factor is chosen to have the isometry property $\|u\|_{L^2(\mathbb{R}^d,\mathbb{C}^p)} = \|\mathcal{F}(u)\|_{L^2(\mathbb{R}^d,\mathbb{C}^p)}$.

Our aim is to find an approximate representation formula for the solution u^ε of (1.1). To simplify formulas, we restrict ourselves to the following initial data:

$$u^\varepsilon(x,0) = u_0(x)\,, \quad \partial_t u^\varepsilon(x,0) = 0\,. \tag{2.2}$$

Furthermore, it is convenient to assume that $u_0 \in L^1(\mathbb{R}^d, \mathbb{R}^p) \cap L^2(\mathbb{R}^d, \mathbb{R}^p)$ has a Fourier transform that has a compact support $K \subset \mathbb{R}^d$. This assumption allows to write

$$u_0(x) = \frac{1}{(2\pi)^{d/2}} \int_K \mathcal{F}(u_0)(k)e^{ik\cdot x}\,dk\,. \tag{2.3}$$

We note that, under this assumption, $u_0 \in H^\ell(\mathbb{R}^d, \mathbb{R}^p)$ holds for every $\ell \in \mathbb{N}$, and the Fourier transform $\mathcal{F}(u_0)$ is bounded.

Our aim is to define a function U^ε such that the difference $u^\varepsilon - U^\varepsilon$ vanishes in the limit $\varepsilon \to 0$, uniformly in t. We construct U^ε as an inverse Fourier transform of a finite sum. The construction of U^ε is completed with formula (2.37), which provides

$$U^\varepsilon(x,t) := (2\pi)^{d/2}\mathcal{F}^{-1}\left(\sum_{m=1}^{p} B_m(k)\xi_m(\hat{k})\ \mathrm{Re}\left(\exp\left(i\omega_m^*(k)t/\varepsilon\right)\right)\right)\,. \tag{2.4}$$

Here, the inverse Fourier transform is applied to the given function with the argument $k \in \mathbb{R}^d$. Two terms of the formula, ξ_m and ω_m^*, are obtained from the coefficient function A of the elliptic operator L_ε of (2.1). The p functions $k \mapsto \omega_m^*(k)$ are constructed from the Taylor expansion of the dispersion relation corresponding to

A. The p functions $(\xi_m)_{m\le p}$ are approximations of Bloch functions, the argument $\hat{k} = k/|k|$ is the direction of the wave-vector k and is defined almost everywhere. The coefficient functions $(B_m(k))_{m\le p}$ are constructed from the initial data u_0 (using projections that depend on the vectors ξ_m).

Theorem 2.1 (Homogenization). *Let* $u_0 \in L^1(\mathbb{R}^d, \mathbb{R}^p) \cap L^2(\mathbb{R}^d, \mathbb{R}^p)$ *have compact support* K *in Fourier space and let* $T_0 > 0$ *be fixed. Let* L_ε *be as in* (2.1) *with the properties listed there, and let* u^ε *be the solution of* $\partial_t^2 u^\varepsilon = -L_\varepsilon u^\varepsilon$ *with initial condition* (2.2)*. Let* U^ε *be defined by the Fourier representation formula* (2.4)*. Then there holds, as* $\varepsilon \to 0$*,*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|u^\varepsilon(\cdot,t) - U^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)} \to 0\,. \tag{2.5}$$

It is quite restrictive to demand that the initial data have a compact support in Fourier space. We therefore comment on two possible generalizations.

Remark 2.2 (More general initial data). *For arbitrary initial data* $u_0 \in L^2(\mathbb{R}^d)$ *one can choose a sequence* $u_{0,\delta} \to u_0$ *in* $L^2(\mathbb{R}^d)$ *as* $\delta \to 0$ *such that every Fourier transform* $\mathcal{F}(u_{0,\delta})$ *has compact support. The corresponding solutions* u^ε_δ *to the initial data* $u_{0,\delta}$ *are close to the original solution* u^ε*, at least in energy norms, e.g.:* $\sup_t \|\partial_t u^\varepsilon_\delta(\cdot,t) - \partial_t u^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)} \to 0$ *for* $\delta \to 0$*, uniformly in* ε*. Theorem 2.1 provides a function* U^ε_δ *with an explicit formula such that* $u^\varepsilon_\delta - U^\varepsilon_\delta$ *is small for fixed* $\delta > 0$ *as* $\varepsilon \to 0$*. These two results provide smallness of* $u^\varepsilon - U^\varepsilon_\delta$ *in appropriate function spaces.*

It remains to discuss the term $\sup_t \|(U^\varepsilon - U^\varepsilon_\delta)(\cdot,t)\|_{L^2(\mathbb{R}^d)}$*. We have to inspect formula* (2.4) *which is used to define* U^ε *from* u_0 *and to define* U^ε_δ *from* $u_{0,\delta}$*. The functions* $\xi_m(\hat{k})$ *and* $\omega^*_m(k)$*, introduced in* (2.15) *and* (2.36)*, are independent of* u_0*. The function* $B_m(k)$*, by its definition in* (2.31)*, is a projection of* $\mathcal{F}(u_0)$*. When* $\|u_{0,\delta} - u_0\|_{L^2(\mathbb{R}^d)}$ *is small, the corresponding difference* $\|B^\delta_m - B_m\|_{L^2(\mathbb{R}^d)}$ *is also small and we find the desired result, smallness of* $u^\varepsilon - U^\varepsilon$*.*

We note that the above post-processing of the result requires to use different metrics to study the errors. Another approach would be to perform the entire proof for more general initial data u_0*. Even though formulas* (2.27)*,* (2.29) *and* (2.30) *are not valid for general initial data, we expect them to be true up to small error terms, at least for* u_0 *in the Schwartz class.*

In this contribution, in order to avoid excessive technicalities, we restrict ourselves to the proof of the result as stated in Theorem 2.1.

2.1 Floquet-Bloch expansion of the solution

We use the Floquet-Bloch transformation to analyze the elliptic operator L_ε with periodic coefficients. We use the periodicity cell εY and the quasiperiodicity parameter $\varepsilon k \in Y' := [-1/2, 1/2)^d$.

2.1.1 Floquet-Bloch transformation

For an arbitrary function $u = u(x)$ for $x \in \mathbb{R}^d$ (with sufficient decay), we can define its periodic Floquet transformation. The variables for the transformed functions are

$x \in \varepsilon Y$ for the position in the periodicity cell and $k \in Y'/\varepsilon$ for the dual variable. For fixed $\varepsilon > 0$, for an arbitrary function $u \in L^2(\mathbb{R}^d)$, we define

$$\hat{u}(x,k) := \varepsilon^d e^{-ik\cdot x} \sum_{z\in\mathbb{Z}^d} u(x + 2\pi\varepsilon z) e^{-2\pi i k\varepsilon\cdot z} . \tag{2.6}$$

We use this periodic version of the Floquet-Bloch transformation. It has the property that $\hat{u}(\cdot,k)$ is a periodic function on εY. The scaling is chosen such that, for a smooth function u and for small $\varepsilon > 0$, the transformation is like the Fourier transform: $\hat{u}(x,k) \approx (2\pi)^{-d/2}\mathcal{F}(u)(k)$ independent of $x \in \varepsilon Y$. When the support of $\mathcal{F}(u)$ is bounded, then, for sufficiently small $\varepsilon > 0$, there even holds $\hat{u}(x,k) = (2\pi)^{-d/2}\mathcal{F}(u)(k)$ by the Nyquist–Shannon sampling theorem, see Lemma 2.7 below. The following lemma collects the standard properties of the periodic Floquet-Bloch transform.

Lemma 2.3 (Floquet-Bloch transform)**.** *The transformation $u \mapsto \hat{u}$ of* (2.6) *has the following properties.*

a) The function $\hat{u}(\cdot,k)$ is εY-periodic: For every $j \le d$, almost every $x \in \varepsilon Y$ and every $k \in Y'/\varepsilon$ holds

$$\hat{u}(x + 2\pi\varepsilon e_j, k) = \hat{u}(x,k) .$$

b) The transformation is an L^2-isometry in the sense that

$$\|u\|^2_{L^2(\mathbb{R}^d)} = \frac{1}{\varepsilon^d}\int_{Y'/\varepsilon}\int_{\varepsilon Y} |\hat{u}(x,k)|^2\, dx\, dk = \int_{Y'/\varepsilon}\int_{Y} |\hat{u}(\varepsilon y,k)|^2\, dy\, dk . \tag{2.7}$$

The transformation is also unitary up to the same multiplicative factor.

c) The inverse transformation is given by

$$u(x) = \int_{Y'/\varepsilon} \hat{u}(x,k) e^{ik\cdot x}\, dk , \tag{2.8}$$

where $\hat{u}$ is identified with its εY-periodic extension to $\mathbb{R}^d$.

2.1.2 Bloch eigenvalues

The transformed function $\hat{u}(\cdot,k)$ is periodic and can therefore be expanded in the periodic eigenfunctions of the elliptic operator L_ε. This is the main idea of the Floquet-Bloch expansion. With the space $H^1_\sharp(Y,\mathbb{C}^p)$ of periodic functions of class H^1 on the cube Y we consider eigenfunctions $\phi \in H^1_\sharp(Y,\mathbb{C}^p)$ and eigenvalues $\lambda \in \mathbb{C}$ of the problem

$$-(\nabla + i\varepsilon k)\cdot(A(y)(\nabla + i\varepsilon k)\phi(y,\varepsilon k)) = \lambda(\varepsilon k)\phi(y,\varepsilon k) . \tag{2.9}$$

The left-hand side is a family of self-adjoint elliptic operators, indexed by the parameter $\varepsilon k \in \mathbb{R}^d$. We therefore find, for every parameter $\varepsilon k \in \mathbb{R}^d$, a family of eigenvalues $\mu_m(\varepsilon k) \in \mathbb{R}$, $m \in \mathbb{N}$ and eigenfunctions $\phi_m(\cdot,\varepsilon k) \in \mathbb{R}$, $m \in \mathbb{N}$. When we denote eigenvalues with μ, we always assume that they are ordered, $\mu_m(\varepsilon k) \le \mu_{m+1}(\varepsilon k)$ for all $m \in \mathbb{N}$.

Our first observation is that there is a spectral gap between the first p eigenvalues and the higher Bloch eigenvalues.

Lemma 2.4 (Spectral gap for Bloch eigenvalues)**.** *Let eigenvalues μ_m and eigenfunctions ϕ_m for the tensor field A be as described above. Let $K \subset \mathbb{R}^d$ be a compact set. Then, there exist constants $C, \mu_*, \varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$ and every $k \in K$,*

$$\mu_m(\varepsilon k) > \mu_* \qquad \text{for } m > p\,, \tag{2.10}$$

$$\mu_m(\varepsilon k) \le C\varepsilon^2 |k|^2 \quad \text{for } m \le p\,. \tag{2.11}$$

Proof. We use the Courant-Fisher characterization of eigenvalues. With the notation $|v|^2_{A(y)} := \bar{v} : A(y)v$ for matrices $v \in \mathbb{C}^{d\times p}$, it provides

$$\mu_m(\varepsilon k) = \inf_{\substack{V\subset H^1_\sharp(Y,\mathbb{C}^p)\\ \dim\, V=m}} \ \sup_{u\in V\setminus\{0\}} \frac{\int_Y |(\nabla + i\varepsilon k)u(y)|^2_{A(y)}\,dy}{\|u\|^2_{L^2(Y,\mathbb{C}^p)}}\,. \tag{2.12}$$

It is well known that (2.12) implies the Lipschitz-continuity of the eigenvalues μ_m. Moreover, for $k = 0$, it provides that the first p eigenvalues vanish (one can use a space V of constant functions), and that $\mu_{p+1}(0) > 0$ is satisfied (Poincaré estimate for periodic functions with vanishing average).

We set $\mu_* := \mu_{p+1}(0)/2 > 0$. Using the Lipschitz-continuity of the eigenvalues, we can choose $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$ and $k \in K$: (i) $\mu_{p+1}(\varepsilon k) > \mu_*$, and (ii) $\mu_p(\varepsilon k) < \mu_*/2$.

To show the upper bound (2.11), we exploit once more the Courant-Fisher characterization. For $m \le p$, we consider $V_m := \operatorname{span}\{e_1, \dots e_m\}$, where e_i denotes the constant function with value corresponding to the i-th canonical basis vector of $\mathbb{C}^d$. Then $\dim V_m = m$ and $\nabla u = 0$ for every $u \in V_m$. By the boundedness of $A(\cdot)$, (2.12) provides, for all $m \le p$,

$$\mu_m(\varepsilon k) \le \sup_{u\in V_m\setminus\{0\}} \frac{\int_Y |(\nabla + i\varepsilon k)u|^2_{A(y)}\,dy}{\|u\|^2_{L^2(Y,\mathbb{C}^p)}} \le C\varepsilon^2|k|^2\,.$$

This was the claim. □

2.1.3 Eigenvalues with smooth dependence

For the subsequent analysis it is important that eigenvalues depend smoothly on ε (in full generality they cannot be expected to be smooth in the vector-valued parameter k). In order to have smooth dependence, it might be necessary to number the eigenvalues in a new way. We denote the eigenvalues with new index as $\lambda_m(\varepsilon k)$, but we will not change the set of the first p eigenvalues: Our construction will imply that

$$\{\mu_1(\varepsilon k), \dots, \mu_p(\varepsilon k)\} = \{\lambda_1(\varepsilon k), \dots, \lambda_p(\varepsilon k)\} \tag{2.13}$$

for every small $\varepsilon \ge 0$ and every $k \in K$. We still denote eigenfunctions by ϕ_m even though the selection might now be different.

We construct the smooth eigenvalues λ_m as follows. The higher eigenvalues are copied, we set $\lambda_m(\varepsilon k) := \mu_m(\varepsilon k)$ for every $m > p$. As observed above, the eigenvalue $\lambda = 0$ for $k = 0$ has the multiplicity p. For each direction $\hat{k} \in S^{d-1} \subset \mathbb{R}^d$, we consider the eigenvalue problem with the real parameter $\varepsilon \in \mathbb{R}$. By a theorem of Rellich, there are p analytic eigenvalue branches $\varepsilon \mapsto \lambda_m(\varepsilon\hat{k})$ that branch from the multiple eigenvalue $\lambda = 0$, labeled with $m \in \{1, ..., p\}$. We choose the same branches for $\hat{k}$ and $-\hat{k}$. The corresponding eigenfunctions $\phi_m(\cdot, \varepsilon\hat{k})$ can also be chosen as analytic functions in ε and as an orthonormal basis of $L^2_\sharp(Y, \mathbb{C}^p)$. Of course, this new labelling depends on the chosen direction $\hat{k}$.

Our construction implies (2.13), at least after an appropriate choice of $\varepsilon_0 > 0$. Indeed, when $C\varepsilon_0^2|k|^2 < \mu_*$ holds for all $k \in K$, there is a gap in the two conditions (2.10) and (2.11). All the functions $\varepsilon \to \lambda_m(\varepsilon k)$ for $m \leq p$ satisfy $\lambda_m(0) = 0$ by construction and hence (2.11) for $\varepsilon = 0$. Since they do not have jumps, they must remain below μ_*.

The above facts on eigenvalues and eigenfunctions have also been exploited in [15, 16]. The facts are classical, we refer to [18], Theorem II, 6.1 for the formulation in the finite-dimensional case, to the subsequent discussion (starting on page 121) regarding analytic families of orthonormal basis of eigenfunctions, and the explanations in [18], VII Paragraph 3 on page 386 regarding the case of general Hilbert spaces. It is used that the operator is self-adjoint and holomorphic, it is important that the family of operators depends only on a single real parameter (this is why we keep $\hat{k}$ fixed and let only ε vary). We also refer to the appendix of [19], where the case of self-adjoint Fredholm operators is reduced to the finite-dimensional case.

2.1.4 Bloch eigenfunctions

The smallness of the first p eigenvalues implies that the first p Bloch eigenfunctions $\phi_1(\cdot, \varepsilon k), ..., \phi_p(\cdot, \varepsilon k)$ are approximately constant. This is shown in the next lemma. As a preparation, we define the limiting objects.

We fix $k \in Y'/\varepsilon$ with $k \neq 0$ and $m \leq p$. We introduce the average

$$\xi_m^\varepsilon(k) := \frac{1}{(2\pi)^d} \int_Y \phi_m(y, \varepsilon k)\, dy \in \mathbb{C}^p\,. \tag{2.14}$$

For $\hat{k} \in S^{d-1}$, we introduce the direction-dependent limits

$$\xi_m(\hat{k}) := \lim_{\varepsilon \to 0} \xi_m^\varepsilon(\hat{k})\,. \tag{2.15}$$

For every $\hat{k} \in S^{d-1}$, the limit exists by continuity of ϕ_m in ε. Since we have chosen the same branch for the two directions $\hat{k}$ and $-\hat{k}$, there holds $\xi_m(\hat{k}) = \xi_m(-\hat{k})$ and the limit $\varepsilon \to 0$ in (2.15) exists for arbitrary sequences $\varepsilon \to 0$ (independent of the sign). Due to the L^2-normalization of ϕ_m, the averages ξ_m^ε and the limits ξ_m are uniformly bounded.

Lemma 2.5 (Convergence of the lowest Bloch eigenfunctions). *Let the situation be as in Lemma 2.4. We consider* $0 \neq k \in K$, $m \leq p$ *and* $\xi_m(\hat{k})$ *as defined in* (2.15).

a) There holds $\phi_m(\cdot, \varepsilon k) \to \xi_m(\hat{k})$ *in* $L^2(Y, \mathbb{C}^p)$ *as* $\varepsilon \to 0$.

b) The vectors $\xi_1(\hat{k}), ..., \xi_p(\hat{k})$ *form an orthogonal basis of* $\mathbb{C}^p$. *They are normalized as* $|\xi_m(\hat{k})|^2 = |Y|^{-1} = (2\pi)^{-d}$ *for every* $m \leq p$.

Proof. We multiply the Bloch eigenvalue problem (2.9) with $\overline{\phi_m(\cdot, \varepsilon k)}$. Using that A is coercive with constant $\gamma > 0$ and exploiting the normalization $\|\phi_m\|_{L^2(Y,\mathbb{C}^p)} = 1$, we find

$$\begin{aligned}\gamma\,\|(\nabla+i\varepsilon k)\phi_m(\cdot,\varepsilon k)\|^2_{L^2(Y,\mathbb{C}^p)} &\le \int_Y |(\nabla+i\varepsilon k)\phi_m(y,\varepsilon k)|^2_{A(y)}\,dy\\ &= \lambda_m(\varepsilon k)\|\phi_m(\cdot,\varepsilon k)\|^2_{L^2(Y,\mathbb{C}^p)} = \lambda_m(\varepsilon k) \le C\varepsilon^2|k^2|\,,\end{aligned} \tag{2.16}$$

where the last inequality follows from Lemma 2.4. The normalization implies $\|i\varepsilon k\phi_m(\cdot,\varepsilon k)\|_{L^2(Y,\mathbb{C}^p)} = \varepsilon|k|$; together with (2.16), this yields $\nabla\phi_m(\cdot,k\varepsilon)\to 0$ in $L^2(Y,\mathbb{C}^p)$. The averages ξ^ε_m from (2.14) converge to ξ_m. This allows to apply Poincaré's inequality for functions with vanishing average with constant $C_P>0$ to find

$$\begin{aligned}\|\phi_m(\cdot,\varepsilon k)-\xi_m(\hat{k})\|_{L^2(Y,\mathbb{C}^p)} &\le \|\phi_m(\cdot,\varepsilon k)-\xi^\varepsilon_m(k)\|_{L^2(Y,\mathbb{C}^p)} + \|\xi^\varepsilon_m(k)-\xi_m(\hat{k})\|_{L^2(Y,\mathbb{C}^p)}\\ &\le C_P\|\nabla\phi_m(\cdot,\varepsilon k)\|_{L^2(Y,\mathbb{C}^p)} + \sqrt{|Y|}|\xi^\varepsilon_m(k)-\xi_m(\hat{k})| \to 0\end{aligned}$$

as $\varepsilon\to 0$. This proves a).

The claims of b) follow easily from a) and the fact that, for every $\varepsilon>0$ and every $k\in K$, the Bloch basis functions $\phi_m(\cdot,\varepsilon k)$ are orthonormal in $L^2(Y,\mathbb{C}^p)$,

$$\delta_{ml} = \int_Y \phi_m(y,\varepsilon k)\cdot\overline{\phi_l(y,\varepsilon k)}\,dy \to |Y|\xi_m(\hat{k})\cdot\overline{\xi_l(\hat{k})} \quad \text{as } \varepsilon\to 0\,.$$

This yields orthogonality of the $\xi_m(\hat{k})$ and the claimed normalization.

The same proof can be performed also when A is coercive in the sense of elasticity. We use a symmetrization in the calculation (2.16) and Korn's inequality in the L^2-estimate. □

2.1.5 Ordinary differential equations for the coefficients

Let $u^\varepsilon(x,t)$ be the solution to the wave system $\partial_t^2 u^\varepsilon = -L_\varepsilon u^\varepsilon$, with L_ε as in (2.1). For every t, we can perform the Floquet transformation for $x\in\varepsilon Y$ and $k\in Y'/\varepsilon$,

$$\hat{u}^\varepsilon(x,k,t) := \varepsilon^d e^{-ik\cdot x}\sum_{z\in\mathbb{Z}^d} u^\varepsilon(x+2\pi\varepsilon z,t)e^{-2\pi ik\varepsilon\cdot z}\,. \tag{2.17}$$

For fixed k and t, the function $x\mapsto\hat{u}^\varepsilon(x,k,t)$ is εY-periodic; it can therefore be expanded in rescaled Bloch eigenfunctions. We write it as

$$\hat{u}^\varepsilon(x,k,t) = \sum_{m\in\mathbb{N}} b^\varepsilon_m(k,t)\,\phi_m(x/\varepsilon,\varepsilon k)\,. \tag{2.18}$$

Our next aim is to derive an ordinary differential equation for the coefficients b^ε_m.

Time derivatives commute with the transformation, hence

$$\widehat{\partial_t^2 u^\varepsilon}(x,k,t) = \partial_t^2\hat{u}^\varepsilon(x,k,t) = \sum_{m\in\mathbb{N}}\partial_t^2 b^\varepsilon_m(k,t)\,\phi_m(x/\varepsilon,\varepsilon k)\,. \tag{2.19}$$

Spatial gradients transform differently. The product rule implies for spatial gradients that

$$\nabla_x(\phi_m(x/\varepsilon,\varepsilon k)e^{ik\cdot x}) = \frac{1}{\varepsilon}e^{ik\cdot x}\left(\nabla_y + ik\varepsilon\right)\phi_m(x/\varepsilon,\varepsilon k)\,.$$

The elliptic operator commutes with the transformation only up to a multiplication with the spatial phase factor. Using (2.18) and the product rule in the second equality, we find, by the eigenfunction property (2.9),

$$\widehat{L_\varepsilon u^\varepsilon}(x,k,t) = e^{-ik\cdot x} L_\varepsilon(\hat{u}^\varepsilon e^{ik\cdot x}) = \sum_{m\in\mathbb{N}} b^\varepsilon_m(k,t)\, \varepsilon^{-2}\lambda_m(\varepsilon k)\phi_m(x/\varepsilon, \varepsilon k)\,. \tag{2.20}$$

The original equation (1.1) implies that the expressions in (2.19) and (2.20) coincide up to their sign. Comparing factors of ϕ_m yields the ordinary differential equation for the coefficients b^ε_m. Taking into acount the initial condition $\partial_t u^\varepsilon(x,0) = 0$ we find, for every $m \in \mathbb{N}$ and every $k \in Y'/\varepsilon$:

$$\partial_t^2 b^\varepsilon_m(k,t) = -\varepsilon^{-2}\lambda_m(\varepsilon k)\, b^\varepsilon_m(k,t)\,, \tag{2.21}$$

$$\partial_t b^\varepsilon_m(k,0) = 0\,. \tag{2.22}$$

The fact that all eigenvalues of the elliptic eigenvalue problem are real-valued and non-negative allows to introduce $\omega_m(k) := \sqrt{\lambda_m(k)}$. As solutions of (2.21)–(2.22), the coefficients b^ε_m have the form

$$b^\varepsilon_m(k,t) = \mathrm{Re}\left(\exp\left(i\omega_m(\varepsilon k)t/\varepsilon\right)\right) b^\varepsilon_m(k,0)\,. \tag{2.23}$$

Inserting (2.23) into (2.18), we obtain an expansion of the solution $\hat{u}^\varepsilon$:

$$\hat{u}^\varepsilon(x,k,t) = \sum_{m\in\mathbb{N}} b^\varepsilon_m(k,0)\,\mathrm{Re}\left(\exp\left(i\omega_m(\varepsilon k)t/\varepsilon\right)\right)\phi_m(x/\varepsilon,\varepsilon k)\,. \tag{2.24}$$

The function U^ε of Theorem 2.1 is obtained by a simplification of the right-hand side of (2.24). Accordingly, the goal of the next subsections is to simplify this expression.

The following remark is included to illustrate the above theory. We determine the newly introduced functions for a system with constant coefficients.

Remark 2.6 (Constant coefficient two-dimensional Lamé system)**.** *Let us consider the Lamé system with constant coefficients in dimension $p = d = 2$, $A(y) = A_0$ for constants λ and μ. Then, for every k and $\hat{k} = k/|k|$, there exists a basis $(v_m(\hat{k}))_{m\le d}$ such that $A_0(\hat{k}\otimes v_m) = c_m^2 \hat{k}\otimes v_m$. Indeed, we can choose $v_2(\hat{k}) = \hat{k}$ and $v_1(\hat{k}) = \hat{k}^\perp$ with $c_2^2 = 2\mu+\lambda$ and $c_1^2 = 2\mu$. The eigenfunctions are constant functions, $\phi_m(y,\varepsilon\hat{k}) := v_m(\hat{k})$. They satisfy*

$$\begin{aligned} -(\nabla + i\varepsilon k)\cdot\Big(A(y)(\nabla+i\varepsilon k)\phi_m(y,\varepsilon\hat{k})\Big) &= -(\nabla+i\varepsilon k)\cdot A_0 i\varepsilon k\otimes v_m(\hat{k}) \\ = -i\varepsilon k\cdot c_m^2\, i\varepsilon k\otimes v_m(\hat{k}) = c_m^2\,|\varepsilon k|^2\, v_m(\hat{k}) &= c_m^2\,|\varepsilon k|^2\,\phi_m(y,\varepsilon k)\,. \end{aligned}$$

We therefore have the homogeneity $\lambda_m(\varepsilon k) = \varepsilon^2\lambda_m(k) = \varepsilon^2 c_m^2|k|^2$ and find $\omega_m(\varepsilon k) = \varepsilon c_m|k|$. The formula for b^ε_m reads

$$b^\varepsilon_m(k,t) = \mathrm{Re}\left(\exp\left(ic_m|k|t\right)\right) b^\varepsilon_m(k,0)\,. \tag{2.25}$$

The number c_m is the speed of the m-th wave form, $m = 1$ for shear waves, $m = 2$ for pressure waves.

2.2 Simplification of expansion (2.24)

In this subsection, we make two simplifications: 1. We express the initial data $b^\varepsilon_m(k,0)$ with the Fourier transform of u_0. 2. We determine the limit $B_m(k) = \lim_{\varepsilon\to 0} b^\varepsilon_m(k,0)$, exploiting that eigenfunctions are approximately constant.

Initial data for the coefficients b^ε_m. We evaluate the Bloch expansion (2.18) in the time instance $t=0$, in which the transformed initial values $\hat{u}^\varepsilon(x,k,0) = \hat{u}_0(x,k)$ appear on the left-hand side. We have

$$\hat{u}_0(x,k) = \sum_{m\in\mathbb{N}} b^\varepsilon_m(k,0)\phi_m(x/\varepsilon, k\varepsilon)\,. \tag{2.26}$$

This relation determines the initial values $b^\varepsilon_m(k,0)$ of the coefficients.

We assumed that u_0 has compact support $K\subset\mathbb{R}^d$ in Fourier-space. As indicated before, in this case, the transform $\hat{u}_0(\cdot,k)$ can be compared with the Fourier transform $\mathcal{F}(u_0)(k)$. More precisely, for $\varepsilon>0$ sufficiently small, the two quantities coincide up to a prefactor. This follows by the Nyquist-Shannon sampling theorem. For the sake of completeness we provide a proof of this well-known result, applied to the Floquet-Bloch transform.

Lemma 2.7 (Variant of Nyquist-Shannon). *Let $u\in L^2(\mathbb{R}^d,\mathbb{C}^p)$ be such that its Fourier transform has compact support in $B_R(0)$ with $R>0$ and let $\varepsilon>0$ satisfy $\varepsilon<1/(2R)$. The choice guarantees $\mathcal{F}(u)(k)=0$ for every k with $|k|>1/(2\varepsilon)$. Then, the Floquet-Bloch transform and the Fourier transform of u coincide in the sense that, for all $k\in Y'/\varepsilon$ and almost all $x\in\varepsilon Y$,*

$$\hat{u}(x,k) = (2\pi)^{-d/2}\mathcal{F}(u)(k)\,. \tag{2.27}$$

Proof. We keep the function u with Fourier transform $\mathcal{F}(u)$ fixed and choose $\varepsilon>0$ with the properties of the lemma. We consider the restriction of the Fourier transform $\mathcal{F}(u)$ to the cube $K_\varepsilon := \left(-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}\right)^d$. This function can be written as a Fourier series,

$$\mathcal{F}(u)(k) = \sum_{z\in\mathbb{Z}^d} c_{-z} e^{-2\pi i\varepsilon k\cdot z}\,.$$

We can calculate the Fourier coefficients c_{-z}, exploiting that $\mathcal{F}(u)(k)$ vanishes for $k\notin K_\varepsilon$:

$$c_{-z} = \varepsilon^d\int_{K_\varepsilon}\mathcal{F}(u)(k)\,e^{2\pi i\varepsilon k\cdot z}\,dk = \varepsilon^d\int_{\mathbb{R}^d}\mathcal{F}(u)(k)e^{2\pi i\varepsilon k\cdot z}\,dk = (2\pi)^{d/2}\varepsilon^d\,u(2\pi\varepsilon z)\,.$$

Inserting above, we obtain

$$(2\pi)^{-d/2}\mathcal{F}(u)(k) = (2\pi)^{-d/2}\sum_{z\in\mathbb{Z}^d} c_{-z}e^{-2\pi i\varepsilon k\cdot z} = \varepsilon^d\sum_{z\in\mathbb{Z}^d} u(2\pi\varepsilon z)e^{-2\pi i\varepsilon k\cdot z} = \hat{u}(0,k)\,. \tag{2.28}$$

This proves the claim for $x=0$.

We now fix a non-vanishing $x \in \mathbb{R}^d$. We consider the shifted function $v(\cdot) := u(x + \cdot)$. The Fourier transform of v satisfies $\mathcal{F}(v)(k) = e^{ik\cdot x}\mathcal{F}(u)(k)$ and the Floquet-Bloch transform of v satisfies

$$\begin{aligned}\hat{v}(0,k) &= \varepsilon^d \sum_{z\in\mathbb{Z}^d} v(2\pi\varepsilon z)e^{-2\pi i k\varepsilon\cdot z}\\ &= e^{ik\cdot x}\left[\varepsilon^d e^{-ik\cdot x}\sum_{z\in\mathbb{Z}^d} u(x+2\pi\varepsilon z)e^{-2\pi i k\varepsilon\cdot z}\right] = e^{ik\cdot x}\hat{u}(x,k)\,.\end{aligned}$$

We can apply (2.28) to v and obtain

$$e^{ik\cdot x}(2\pi)^{-d/2}\mathcal{F}(u)(k) = (2\pi)^{-d/2}\mathcal{F}(v)(k) = \hat{v}(0,k) = e^{ik\cdot x}\hat{u}(x,k)\,.$$

This provides the claim, $\hat{u}(x,k) = (2\pi)^{-d/2}\mathcal{F}(u)(k)$. □

Lemma 2.7 implies for the expansion (2.26) that, for small $\varepsilon > 0$, the left-hand side coincides with the Fourier transform of u_0 (up to a factor): The initial data $b^\varepsilon_m(k,0)$ are characterized by

$$(2\pi)^{-d/2}\mathcal{F}(u_0)(k) = \sum_{m\in\mathbb{N}} b^\varepsilon_m(k,0)\,\phi_m(x/\varepsilon,\varepsilon k)\,. \tag{2.29}$$

We use this to find an approximation for the coefficients $b^\varepsilon_m(k,0)$.

Approximation of eigenfunctions in the initial data. In what follows, for any $k \in Y'/\varepsilon$ with $k \neq 0$, we denote the corresponding direction by $\hat{k} := k/|k| \in S^{d-1}$.

Our next aim is to find, starting from (2.29), for arbitrary index $l \leq p$, a formula for the initial datum $b^\varepsilon_l(k,0)$. With this aim, we multiply (2.29) with the complex conjugate of $\phi_l(x/\varepsilon,\varepsilon k)$ and integrate over $x \in \varepsilon Y$. We obtain

$$(2\pi)^{-d/2}\mathcal{F}(u_0)(k)\int_Y \overline{\phi_l}(y,\varepsilon k)\,dy = \sum_{m\in\mathbb{N}} b^\varepsilon_m(k,0)\left(\int_Y \phi_m(y,\varepsilon k)\overline{\phi_l}(y,\varepsilon k)\,dy\right)\,.$$

Since Bloch eigenfunctions $\phi_m(\cdot,\varepsilon k)$ are orthonormal in $L^2(Y)$, this simplifies to

$$b^\varepsilon_l(k,0) \;=\; \mathcal{F}(u_0)(k)\cdot(2\pi)^{d/2}\,\overline{\xi^\varepsilon_l(k)}\,. \tag{2.30}$$

By the convergence $\xi^\varepsilon_l(k) \to \xi_l(\hat{k})$ as $\varepsilon \to 0$, we conclude, for every $0 \neq k \in Y'/\varepsilon$,

$$\lim_{\varepsilon\to 0} b^\varepsilon_m(k,0) \;=\; (2\pi)^{d/2}\mathcal{F}(u_0)(k)\cdot\overline{\xi_m(\hat{k})} \;=:\; B_m(k)\,. \tag{2.31}$$

Loosely speaking, the functions $B_m(k)$ are projections of the Fourier transform of the initial data. Formula (2.31) ensures that they are a good approximation of the initial values $b^\varepsilon_m(k,0)$.

Relation (2.30) has another important consequence regarding the support and boundedness of the coefficients b_m.

Remark 2.8 (Support of b^ε_m)**.** *Let the Fourier transform of* u_0 *have compact support* $K \subset \mathbb{R}^n$*. For sufficiently small* $\varepsilon > 0$ *(depending only on* K*),* (2.30) *implies that each coefficient* $b^\varepsilon_m(\cdot, 0)$ *is also supported in* K*. By the evolution equation* (2.23)*, also for every* $t \geq 0$ *and every* $m \in \mathbb{N}$ *holds:*

$$b^\varepsilon_m(k,t) = 0 \qquad \text{for every } k \notin K\,. \tag{2.32}$$

Moreover, for $u_0 \in L^1(\mathbb{R}^d, \mathbb{R}^p) \cap L^2(\mathbb{R}^d, \mathbb{R}^p)$*, the function* $\mathcal{F}(u_0)$ *is bounded; this implies that also the coefficients* b^ε_m *and* B_m *are uniformly bounded.*

The results of this section are the basis for the construction of the approximation U^ε used in Theorem 2.1.

2.3 From Floquet-Bloch to a Fourier representation

Theorem 2.1 makes a comparison of $u^\varepsilon = u^\varepsilon(x,t)$ with $U^\varepsilon = U^\varepsilon(x,t)$. In order to structure the proof, we introduce two intermediate approximations, namely the functions $v^\varepsilon = v^\varepsilon(x,t)$ and $V^\varepsilon = V^\varepsilon(x,t)$.

- The function u^ε is the solution of problem (1.1) with oscillatory coefficient $A = A(x/\varepsilon)$. It has the Floquet-Bloch representation (2.18), the coefficients are given by (2.23).

- We define a function v^ε by neglecting terms with $m > p$ in the sum: The Floquet-Bloch transformation of v^ε is given, for $x \in \varepsilon Y$ and $k \in Y'/\varepsilon$, by

$$\hat{v}^\varepsilon(x,k,t) := \sum_{m=1}^{p} b^\varepsilon_m(k,0)\phi_m(x/\varepsilon, \varepsilon k)\, \mathrm{Re}\left(\exp\left(i\omega_m(\varepsilon k)t/\varepsilon\right)\right)\,. \tag{2.33}$$

- We define a function V^ε with a Fourier transform. By (2.31) for every $k \in Y'/\varepsilon$ with $k \neq 0$ one has that $b^\varepsilon_m(k,0) \to B_m(k)$ as $\varepsilon \to 0$. Moreover, by Lemma 2.5, the first p Bloch eigenfunctions are approximately constant, $\phi_m(x/\varepsilon, \varepsilon k) \approx \xi_m(\hat{k})$ for $m \in \{1, ..., p\}$. Taking into account the formula for the inverse transformation (2.8), we define V^ε as the function with the Fourier transform

$$\mathcal{F}(V^\varepsilon)(k,t) := (2\pi)^{d/2} \sum_{m=1}^{p} B_m(k)\xi_m(\hat{k})\, \mathrm{Re}\left(\exp\left(i\omega_m(\varepsilon k)t/\varepsilon\right)\right)\,. \tag{2.34}$$

The formula for V^ε leads to the following definition of the final function U^ε. The remaining step is to perform a Taylor expansion of the dispersion relation ω_m. We use speeds $c_m(\hat{k}) \in \mathbb{R}$ and dispersive factors $b_m(\hat{k}) \in \mathbb{R}$. They are given by the Taylor expansion of the dispersion relation ω_m for a fixed direction $\hat{k} \in S^{d-1}$ in the single parameter $\varepsilon \in (0, \varepsilon_0)$,

$$\omega_m(\varepsilon k) = \omega_m(\varepsilon |k| \hat{k}) = c_m(\hat{k})\, |\varepsilon k| + b_m(\hat{k})\, |\varepsilon k|^3 + O(|\varepsilon k|^5)\,. \tag{2.35}$$

We emphasize that the error term is depending on the direction; the meaning of (2.35) is: For every $\hat{k} \in S^{d-1}$, there exist $b_m(\hat{k}), c_m(\hat{k}), C(\hat{k}) \in \mathbb{R}$ such that, for all $k \in$

$\mathbb{R}\,\hat{k}$, there holds $\left|\omega_m(\varepsilon k) - c_m(\hat{k})\,|\varepsilon k| - b_m(\hat{k})\,|\varepsilon k|^3\right| \leq C(\hat{k})\,|\varepsilon k|^5$. The truncated Taylor expansion is

$$\omega_m^*(\varepsilon k) := c_m(\hat{k})\,|\varepsilon k| + b_m(\hat{k})\,|\varepsilon k|^3\,, \tag{2.36}$$

and we use it in the definition of U^ε as in (2.4),

$$\mathcal{F}(U^\varepsilon)(k,t) = (2\pi)^{d/2} \sum_{m=1}^{p} B_m(k)\xi_m(\hat{k})\ \mathrm{Re}\left(\exp\left(i\omega_m^*(\varepsilon k)t/\varepsilon\right)\right)\,. \tag{2.37}$$

In the formula appears the argument $\omega_m^*(\varepsilon k)/\varepsilon = c_m(\hat{k})|k| + \varepsilon^2 b_m(\hat{k})|k|^3$. The leading order term $c_m(\hat{k})$ encodes the effective wave speed in direction $\hat{k}$, while the ε^2-term characterizes a weak dispersive effect.

From now on we will always assume that $\varepsilon > 0$ is sufficiently small to satisfy the assumptions of Lemma 2.4 and Lemma 2.7. Step by step, we show that all functions are close to each other.

2.3.1 Contributions of high modes

We compare u^ε and v^ε. We have to analyze the contribution from modes $m > p$. Smallness of their contributions relies on the spectral gap estimate in Lemma 2.4.

Lemma 2.9 (Comparison of u^ε with v^ε)**.** *There exists a constant $C > 0$ such that*

$$\sup_{t\in(0,\infty)} \|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon\,. \tag{2.38}$$

Proof. We consider an arbitrary $t > 0$. As derived in (2.18), the Floquet-Bloch transform of u^ε is

$$\hat{u}^\varepsilon(x,k,t) = \sum_{m\in\mathbb{N}} b_m^\varepsilon(k,t)\,\phi_m(x/\varepsilon, \varepsilon k)\,. \tag{2.39}$$

The Floquet-Bloch transform of v^ε is, by its definition in (2.33),

$$\hat{v}^\varepsilon(x,k,t) = \sum_{m=1}^{p} b_m^\varepsilon(k,t)\,\phi_m(x/\varepsilon, \varepsilon k)\,. \tag{2.40}$$

In order to prove the lemma, we have to show smallness of terms with $m > p$.

Step 1: Estimate for the coefficients. The first step is to derive an estimate for the higher mode coefficients $(b_m^\varepsilon)_{m>p}$. An energy estimate for $u^\varepsilon(x,t)$ is obtained by multiplying the evolution equation (1.1) with $\partial_t \bar{u}^\varepsilon(x,t)$, integrating over $x \in \mathbb{R}^d$ and over $t \in (0, t_0)$ with arbitrary $t_0 > 0$. Denoting the arbitrary time instance t_0 again by t, we find an $L^2(\mathbb{R}^d)$-bound for $\partial_t u^\varepsilon(\cdot,t)$ and the energy estimate

$$\int_{\mathbb{R}^d} L_\varepsilon u^\varepsilon(x,t) \cdot \bar{u}^\varepsilon(x,t)\,dx \leq C_0\,, \tag{2.41}$$

where C_0 depends only on u_0. We now use that the Floquet-Bloch transform is a unitary isometry (compare (2.7)), express u^ε with its Floquet-Bloch representation, exploit the eigenfunction property of the basis functions ϕ_m and their orthonormality, and apply (2.20):

$$\begin{aligned} C_0 &\geq \int_{\mathbb{R}^d} L_\varepsilon u^\varepsilon(x,t) \cdot \bar{u}^\varepsilon(x,t)\,dx \\ &= \frac{1}{\varepsilon^d} \int_{\varepsilon Y} \int_{Y'/\varepsilon} \widehat{L_\varepsilon u^\varepsilon}(x,k,t) \cdot \overline{\hat{u}^\varepsilon(x,k,t)}\,dk\,dx \end{aligned}$$

$$= \frac{1}{\varepsilon^d} \sum_{m,l\in\mathbb{N}} \int_{\varepsilon Y} \int_{Y'/\varepsilon} \varepsilon^{-2} \lambda_m(\varepsilon k) b^\varepsilon_m(k,t)\, \phi_m(x/\varepsilon, \varepsilon k) \bar{b}^\varepsilon_l(k,t)\, \bar{\phi}_l(x/\varepsilon,\varepsilon k)\, dk\, dx$$

$$= \varepsilon^{-2} \sum_{m\in\mathbb{N}} \int_{Y'/\varepsilon} \lambda_m(\varepsilon k)\, |b^\varepsilon_m(k,t)|^2\, dk\,.$$

Lemma 2.4 provides $\lambda_m(\varepsilon k) > \mu_*$ for every $m > p$ and every $k \in K$. We obtain for the higher modes the estimate

$$\sum_{m>p} \|b^\varepsilon_m(\cdot,t)\|^2_{L^2(Y'/\varepsilon)} \le \frac{C_0}{\mu_*}\varepsilon^2\,. \tag{2.42}$$

Step 2: Calculation of the norm of the difference. We can now calculate the norm of the error. Once more, we use that the Floquet-Bloch transform is an isometry and the orthogonality of the basis functions:

$$\|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d)} = \frac{1}{\varepsilon^d} \int_{\varepsilon Y} \int_{Y'/\varepsilon} \left| \sum_{m>p} b^\varepsilon_m(k,t)\, \phi_m(x/\varepsilon,\varepsilon k)\right|^2 dk\, dx$$

$$= \int_{Y'/\varepsilon} \sum_{m>p} |b^\varepsilon_m(k,t)|^2\, dk \le \frac{C_0}{\mu_*}\varepsilon^2\,.$$

Since t was arbitrary, we obtain (2.38) with $C := \sqrt{C_0/\mu_*}$. □

2.3.2 Simplification of the initial data

The next goal is to compare v^ε with V^ε, based on the approximation of initial data.

Lemma 2.10 (Comparison of v^ε with V^ε)**.** *As $\varepsilon \to 0$, the following convergence holds:*

$$\sup_{t\in(0,\infty)} \|v^\varepsilon(\cdot,t) - V^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)} \to 0\,. \tag{2.43}$$

Proof. The coefficient B_m has compact support $K \subset \mathbb{R}^n$, see (2.31) and (2.32). Accordingly, also V^ε has compact support K in Fourier space and, by Lemma 2.7, has the Floquet-Bloch transform

$$\hat{V}^\varepsilon(x,k,t) = (2\pi)^{-d/2}\mathcal{F}(V^\varepsilon)(k) = \sum_{m=1}^p B_m(k)\xi_m(\hat{k})\, \mathrm{Re}\left(\exp\left(i\omega_m(\varepsilon k)t/\varepsilon\right)\right)\,.$$

We use the isometry property (2.7) of the Floquet-Bloch transform to estimate the L^2-error between v^ε and V^ε:

$$\|v^\varepsilon(\cdot,t) - V^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d)} = \frac{1}{\varepsilon^d} \int_K \int_{\varepsilon Y} |\hat{v}^\varepsilon(x,k,t) - \hat{V}^\varepsilon(x,k,t)|^2\, dx\, dk$$

$$= \int_K \int_Y \left| \sum_{m=1}^p (b^\varepsilon_m(k,0)\phi_m(y,\varepsilon k) - B_m(k)\xi_m(\hat{k}))\, \mathrm{Re}\left(\exp\left(i\omega_m(\varepsilon k)t/\varepsilon\right)\right)\right|^2 dy\, dk$$

$$\le C(p) \sum_{m=1}^p \int_K \int_Y |b^\varepsilon_m(k,0)\phi_m(y,\varepsilon k) - B_m(k)\xi_m(\hat{k})|^2\, dy\, dk\,,$$

independent of t. In order to show (2.43), it remains to verify, for every $m \leq p$, that

$$F_m^\varepsilon := \int_K \int_Y |b_m^\varepsilon(k,0)\phi_m(y,\varepsilon k) - B_m(k)\xi_m(\hat{k})|^2 \, dy \, dk \to 0$$

as $\varepsilon \to 0$. Adding and subtracting $b_m^\varepsilon(k,0)\xi_m(\hat{k})$, we find

$$\begin{aligned} F_m^\varepsilon &\leq 2\int_K \int_Y |b_m^\varepsilon(k,0)|^2 |\phi_m(y,\varepsilon k) - \xi_m(\hat{k})|^2 + |b_m^\varepsilon(k,0) - B_m(k)|^2 |\xi_m(\hat{k})|^2 \, dy \, dk \\ &=: 2\int_K (F_A^\varepsilon(k) + F_B^\varepsilon(k)) \, dk \, . \end{aligned}$$

We now argue with dominated convergence. For a given $k \neq 0$, Lemma 2.5 yields $\|\phi_m(\cdot,\varepsilon k) - \xi_m(\hat{k})\|_{L^2(Y)} \to 0$. The factors b_m^ε are uniformly bounded by Remark 2.8. These two properties imply that the first error term converges pointwise: For almost every k, there holds $F_A^\varepsilon(k) \to 0$. Moreover, the ξ_m are bounded and there holds $|F_A^\varepsilon(k)| \leq C_A$, where the constant C_A does not depend on k or ε. The dominated convergence theorem yields $\int_K F_A^\varepsilon(k) \, dk \to 0$ (we use that the support is bounded, independent of ε).

The function F_B^ε is treated similarly. For fixed $k \neq 0$, by (2.31), the coefficient $b_m^\varepsilon(k,0)$ converges to $B_m(k)$, which implies $F_B^\varepsilon(k) \to 0$. Morover, all involved quantities are uniformly bounded, which yields $|F_B^\varepsilon(k)| \leq C_B$ for some constant C_B, independent of ε and k. We obtain $\int_K F_B^\varepsilon(k) \, dk \to 0$ and thus (2.43). □

2.3.3 Taylor expansion of the dispersion relation

We finally compare V^ε and U^ε using that the dispersion relation is well approximated by its Taylor expansion. We recall that (2.35) provided $|\omega_m(\varepsilon k) - \omega_m^*(\varepsilon k)| \leq C(\hat{k}) \, |\varepsilon k|^5$. Even though the error is depending on the direction, we will obtain strong convergence.

Lemma 2.11 (Comparison of V^ε with U^ε)**.** *For fixed $T_0 > 0$, there holds*

$$\sup_{t \in [0, T_0/\varepsilon^2]} \|V^\varepsilon(\cdot,t) - U^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)} \to 0 \, . \tag{2.44}$$

Proof. The two functions V^ε and U^ε are defined with the help of $k \mapsto B_m(k)\xi_m(\hat{k})$, which is a bounded function with compact support K, the other factors in the definition are also bounded. This implies that both functions $\mathcal{F}(U^\varepsilon)(\cdot,t)$ and $\mathcal{F}(V^\varepsilon)(\cdot,t)$ are bounded in $L^2(\mathbb{R}^d)$. We calculate the L^2-difference between V^ε and U^ε using the isometry property of the Fourier transform with the constant $C_1 = (2\pi)^{d/2}$ and a further constant C that is independent of ε and t:

$$\begin{aligned} &\|V^\varepsilon(\cdot,t) - U^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d,dx)} = \|\mathcal{F}(V^\varepsilon)(\cdot,t) - \mathcal{F}(U^\varepsilon)(\cdot,t)\|_{L^2(\mathbb{R}^d,dk)} \\ &\leq C_1 \sum_{m=1}^p \left\| B_m(k)\xi_m(\hat{k}) \left[\mathrm{Re}\left(\exp\left(i\omega_m(\varepsilon k)t/\varepsilon\right)\right) - \mathrm{Re}\left(\exp\left(i\omega_m^*(\varepsilon k)t/\varepsilon\right)\right) \right] \right\|_{L^2(\mathbb{R}^d,dk)} \\ &\leq C \sum_{m=1}^p \|\exp\left(i\omega_m(\varepsilon k)t/\varepsilon\right) - \exp\left(i\omega_m^*(\varepsilon k)t/\varepsilon\right)\|_{L^2(K,dk)} \, . \end{aligned}$$

It remains to show that the supremum $\sup_{t\in[0,T_0/\varepsilon^2]}$ of the right-hand side vanishes in the limit $\varepsilon \to 0$.

For a contradiction argument, we assume that, for some $m \le p$ and along two sequences $\varepsilon \to 0$ and $t_\varepsilon \in [0, T_0/\varepsilon^2]$, the norm on the right-hand side remains larger than $\eta > 0$. We consider the corresponding sequence of functions $h_\varepsilon(k) := \exp(i\omega_m(\varepsilon k)t_\varepsilon/\varepsilon) - \exp(i\omega_m^*(\varepsilon k)t_\varepsilon/\varepsilon)$.

For fixed $k \neq 0$, the sequence $h_\varepsilon(k)$ of complex numbers converges to zero as $\varepsilon \to 0$ by the Taylor expansion in (2.35). This is the pointwise convergence $h_\varepsilon \to 0$. On the other hand, the functions h_ε are bounded by 2, which is an integrable function over the compact domain K. By Lebesgue's dominated convergence, we obtain convergence in $L^2(\mathbb{R}^d, dk)$. □

Proof of Theorem 2.1. The triangle inequality yields

$$\begin{aligned}\|u^\varepsilon(\cdot,t) - U^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)} &\le \|u^\varepsilon(\cdot,t) - v^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}\\ &\quad + \|v^\varepsilon(\cdot,t) - V^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)} + \|V^\varepsilon(\cdot,t) - U^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}\,.\end{aligned}$$

Lemmas 2.9, 2.10 and 2.11 provide smallness of these three expressions. □

2.4 Bounds for the Taylor expansion

We will need boundedness properties of the function $\omega_m^*(\varepsilon k)$ in order to approximate by a smooth function. Boundedness properties can be obtained from the defining relations for eigenvalues $\lambda_m(k)$ near $k = 0$.

Lemma 2.12 (Boundedness of derivatives of λ_m). *For a fixed index $m \le p$, a fixed direction $\hat{k} \in S^{d-1}$ and a fixed $\varepsilon_0 > 0$, we consider the function $\Lambda : (-\varepsilon_0, \varepsilon_0) \ni \varepsilon \mapsto \lambda_m(\varepsilon\hat{k})$. As noted before, this function is analytic with $\Lambda(0) = 0$. There holds: The odd derivatives in $\varepsilon = 0$ vanish, $\partial_\varepsilon\Lambda(0) = \partial_\varepsilon^3\Lambda(0) = \partial_\varepsilon^5\Lambda(0) = 0$. The even derivatives $\partial_\varepsilon^2\Lambda(0)$ and $\partial_\varepsilon^4\Lambda(0)$ are bounded, independent of m and $\hat{k} \in S^{d-1}$. There exists a lower bound $c_0 > 0$ such that $\partial_\varepsilon^2\Lambda(0) \ge c_0$ holds independent of m and $\hat{k}$.*

Proof. We start from the eigenvalue problem (2.9),

$$-(\nabla + i\varepsilon k)\cdot(A(y)(\nabla + i\varepsilon k)\phi_m(y,\varepsilon k)) \;=\; \lambda_m(\varepsilon k)\phi_m(y,\varepsilon k)\,. \tag{2.45}$$

Differentiating, for fixed $k \neq 0$, with respect to $\varepsilon \in \mathbb{R}$, we find

$$\begin{aligned}&- ik\cdot(A(y)(\nabla + i\varepsilon k)\phi_m(y,\varepsilon k)) - (\nabla + i\varepsilon k)\cdot(A(y)(ik)\phi_m(y,\varepsilon k))\\ &- (\nabla + i\varepsilon k)\cdot(A(y)(\nabla + i\varepsilon k)\partial_\varepsilon\phi_m(y,\varepsilon k))\\ &\qquad = \partial_\varepsilon\lambda_m(\varepsilon k)\phi_m(y,\varepsilon k) + \lambda_m(\varepsilon k)\partial_\varepsilon\phi_m(y,\varepsilon k)\,.\end{aligned} \tag{2.46}$$

Evaluating in $\varepsilon = 0$, using $\nabla\phi_m(\cdot,0) \equiv 0$ and $\lambda_m(0) = 0$, we find

$$-\nabla\cdot(A(y)(ik)\phi_m(y,0)) - \nabla\cdot(A(y)\nabla\partial_\varepsilon\phi_m(y,0)) = \partial_\varepsilon\lambda_m(0)\phi_m(y,0)\,. \tag{2.47}$$

Integrating over the periodicity cell, we find $\partial_\varepsilon\lambda_m(0) = 0$.

Differentiating (2.46) with respect to $\varepsilon \in \mathbb{R}$, we find

$$\begin{aligned}
&- 2ik \cdot (A(y)(ik)\phi_m(y,\varepsilon k)) - 2ik \cdot (A(y)(\nabla + i\varepsilon k)\partial_\varepsilon \phi_m(y,\varepsilon k)) \\
&- 2(\nabla + i\varepsilon k) \cdot (A(y)(ik)\partial_\varepsilon \phi_m(y,\varepsilon k)) \\
&- (\nabla + i\varepsilon k) \cdot \big(A(y)(\nabla + i\varepsilon k)\partial_\varepsilon^2 \phi_m(y,\varepsilon k)\big) \\
&\quad = \partial_\varepsilon^2 \lambda_m(\varepsilon k)\phi_m(y,\varepsilon k) + 2\partial_\varepsilon \lambda_m(\varepsilon k)\partial_\varepsilon \phi_m(y,\varepsilon k) + \lambda_m(\varepsilon k)\partial_\varepsilon^2 \phi_m(y,\varepsilon k)\,. \qquad (2.48)
\end{aligned}$$

Evaluating in $\varepsilon = 0$, using $\partial_\varepsilon \lambda_m(0) = \lambda_m(0) = 0$ and $\nabla \phi_m(\cdot, 0) \equiv 0$, we find

$$\begin{aligned}
&- 2ik \cdot (A(y)(ik)\phi_m(y,0)) - 2ik \cdot (A(y)\nabla \partial_\varepsilon \phi_m(y,0)) \\
&- 2\nabla \cdot (A(y)(ik)\partial_\varepsilon \phi_m(y,0)) - \nabla \cdot \big(A(y)\nabla \partial_\varepsilon^2 \phi_m(y,0)\big) \\
&\quad = \partial_\varepsilon^2 \lambda_m(0)\phi_m(y,0)\,.
\end{aligned} \tag{2.49}$$

A multiplication with the constant function $\bar{\phi}_m(y,0)$ and integrating by parts some of the terms, we find

$$\begin{aligned}
&2\int_Y (A(y)k\phi_m(y,0)) : k\bar{\phi}_m(y,0) - 2\int_Y (A(y)\nabla \partial_\varepsilon \phi_m(y,0)) : (ik)\bar{\phi}_m(y,0) \\
&\quad = \partial_\varepsilon^2 \lambda_m(0) \int_Y |\phi_m(y,0)|^2\,.
\end{aligned} \tag{2.50}$$

The second integral can be replaced with the help of (2.47) (which has a vanishing right-hand side), using the self-adjointness of $A(y)$:

$$-2\int_Y (A(y)\nabla \partial_\varepsilon \phi_m(y,0)) : (ik)\phi_m(y,0) = 2\int_Y (A(y)\nabla \partial_\varepsilon \phi_m(y,0)) : \nabla \partial_\varepsilon \phi_m(y,0)\,.$$

From (2.50), we obtain the lower bound $\partial_\varepsilon^2 \lambda_m(0) \geq c_0|k|^2 > 0$. Furthermore, since $\partial_\varepsilon \phi_m(y,0)$ is characterized as the solution of (2.47), we obtain a bound for $|\partial_\varepsilon^2 \lambda_m(0)|$ that is uniform in the direction $\hat{k}$. This shows boundedness of c_m on the sphere S^{d-1}.

Continuing the calculations, one obtains formulas for higher derivatives $\partial_\varepsilon^\ell \lambda_m(0)$ and $\partial_\varepsilon^\ell \phi_m(y,0)$. These formulas imply boundedness of the derivatives. ☐

2.5 An approximate representation

As a preparation for the next section, we formulate a corollary to Theorem 2.1, which allows to approximate the function $\mathcal{F}(U^\varepsilon)$ of (2.37).

We use the functions $\hat{k} \mapsto \xi_m(\hat{k})$ from $S^{d-1} \to \mathbb{C}^p$ from (2.15). We recall that these functions are bounded and ε-independent, but there is no information about smoothness, we do not even have continuity guaranteed. We furthermore use the functions $B_m : \mathbb{R}^d \to \mathbb{C}^p$ of (2.31). They do not depend on ε, they are bounded, they depend on the initial data u_0, they have a compact support, but we have no smoothness property.

In the exponential function, we have the functions $\omega_m^* : \mathbb{R}^d \to \mathbb{C}$ of (2.36), which depend on the two functions $c_m(\hat{k})$ and $b_m(\hat{k})$ on the sphere. We have no smoothness property for c_m and b_m. However, by Lemma 2.12, we do have the information that the two functions are bounded on the sphere.

Upon approximating B_m, ξ_m and b_m with smooth functions B_m^δ, ξ_m^δ and b_m^δ, and defining $\omega_m^{*,\delta}$ with these approximations (and the original non-smooth function c_m), we can define another approximation:

$$U_\delta^\varepsilon(x,t) := (2\pi)^{d/2}\mathcal{F}^{-1}\left(\sum_{m=1}^{p} B_m^\delta(k)\xi_m^\delta(\hat{k})\ \mathrm{Re}\left(\exp\left(i\omega_m^{*,\delta}(\varepsilon k)t/\varepsilon\right)\right)\right). \tag{2.51}$$

We assume that the approximations are chosen such that, for $T_0 > 0$ fixed,

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\sup_{t\in[0,T_0/\varepsilon^2]}\|U^\varepsilon(\cdot,t)-U_\delta^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}\\&\quad=\limsup_{\varepsilon\to 0}\sup_{t\in[0,T_0/\varepsilon^2]}\|\mathcal{F}U^\varepsilon(\cdot,t)-\mathcal{F}U_\delta^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}\le\delta\,.\end{aligned}\tag{2.52}$$

This can be achieved by approximating B_m, ξ_m and b_m in sufficiently high L^p-spaces. Note that the approximation of B_m can be chosen in such a way that 0 is not contained in the support of B_m^δ.

Corollary 2.13 (Homogenization). *Let $u_0 \in L^1(\mathbb{R}^d,\mathbb{R}^p)\cap L^2(\mathbb{R}^d,\mathbb{R}^p)$ have compact support K in Fourier space and let T_0 be fixed. Let u^ε be the solution of* (1.1). *Let U_δ^ε be defined by* (2.51) *where the approximations imply* (2.52). *Then there holds, for any sequence $\varepsilon\to 0$,*

$$\limsup_{\varepsilon\to 0}\sup_{t\in[0,T_0/\varepsilon^2]}\|u^\varepsilon(\cdot,t)-U_\delta^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}\le\delta\,. \tag{2.53}$$

Proof. We use the triangle inequality and Theorem 2.1 to find

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\sup_{t\in[0,T_0/\varepsilon^2]}\|u^\varepsilon(\cdot,t)-U_\delta^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}\\&\quad\le\limsup_{\varepsilon\to 0}\sup_{t\in[0,T_0/\varepsilon^2]}\|u^\varepsilon(\cdot,t)-U^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}\\&\qquad+\limsup_{\varepsilon\to 0}\sup_{t\in[0,T_0/\varepsilon^2]}\|U^\varepsilon(\cdot,t)-U_\delta^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}\\&\quad=\limsup_{\varepsilon\to 0}\sup_{t\in[0,T_0/\varepsilon^2]}\|U^\varepsilon(\cdot,t)-U_\delta^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}\le\delta\,.\end{aligned}$$

This was the claim. □

3 Approximating Fourier representations with multiple ring solutions

In the previous section, we considered the solution u^ε of a partial differential equation with oscillatory coefficients. We approximated u^ε in $L^2(\mathbb{R}^d)$, uniformly in time on time intervals $[0,T_0/\varepsilon^2]$, with the function U^ε of (2.37). Another approximation is given with the function $U_\delta^\varepsilon(x,t)$ of (2.51), see Corollary 2.13.

Our next goal is to find an approximation of U_δ^ε in physical space. The new approximation is denoted as w^ε, we construct it as a superposition of ring solutions. Every ring has, in each direction, a different profile. The profile is determined by the initial data, but it is changing slowly in time as a result of the weak dispersion effects.

In comparison to the last section, we change the notation slightly. We suppress the index δ in the following and write U^ε instead of U^ε_δ. We do not use the Floquet-Bloch transform any more, which gives us the freedom to denote Fourier-transformed functions with a hat. We therefore write, e.g., $\hat{U}^\varepsilon(k,t) = \mathcal{F}U^\varepsilon(k,t)$. In the function $\hat{U}^\varepsilon$, we do not take the real part of the exponentials. This is no loss of generality, since all results hold also for the opposite sign in the exponents. Finally, we change the function u^ε in the sense that we suppress the prefactor $(2\pi)^{d/2}$.

With this new notation, our interest is to analyze a function $U^\varepsilon = U^\varepsilon(x,t)$ that has the Fourier transform

$$\hat{U}^\varepsilon(k,t) = \sum_{m=1}^{p} B_m(k)\xi_m(\hat{k}) \, \exp\left(i\omega_m^*(\varepsilon k)t/\varepsilon\right) . \tag{3.1}$$

We use the approximated dispersion relation, namely, with $\hat{k} = k/|k|$,

$$\omega_m^*(\varepsilon k) = c_m(\hat{k})|\varepsilon k| + b_m(\hat{k})|\varepsilon k|^3 . \tag{3.2}$$

We can assume that the functions B_m, ξ_m and b_m are smooth functions, that B_m has a compact support and that 0 is not contained in the support of B_m.

The aim of this section is to show that the function U^ε is close to a function that is constructed as the superposition of p rings. Closeness is measured uniformly in time on intervals $[t_0/\varepsilon^2, T_0/\varepsilon^2]$ with $0 < t_0 < T_0$ and in the sense of $L^2(\mathbb{R}^d)$ in space. We will define p profile functions $(V_m)_{1\le m\le p}$,

$$V_m : \mathbb{R} \times S^{d-1} \times (0,\infty) \to \mathbb{C}^p . \tag{3.3}$$

With the profile V_m, the single ring function is defined as

$$w_m^\varepsilon(x,t) := \frac{1}{(c_m t)^{(d-1)/2}} V_m\left(|x| - c_m t, \frac{x}{|x|}, \varepsilon^2 t\right) . \tag{3.4}$$

A reconstruction with p rings is obtained with the superposition

$$w^\varepsilon(x,t) := \sum_{m=1}^{p} w_m^\varepsilon(x,t) . \tag{3.5}$$

In general, the speed $c_m = c_m(x/|x|)$ in (3.4) may depend on the direction $x/|x|$. We actually have to assume that the principal speeds c_m do not depend on the direction.

Our main result is that w^ε of (3.5) provides a good approximation of U^ε of (3.1). A related statement for a single ring approximation has been derived in [21].

We next construct the profile functions. We start by introducing functions $\hat{G}_m$ that represent one part of the function $\hat{U}^\varepsilon(k,t)$. We do *not* include the phase factor $\exp(ic_m|k|t)$ and we consider time instances $t = \tau/\varepsilon^2$. Because of $ib_m(\hat{k})|\varepsilon k|^3\tau/\varepsilon^3 = ib_m(\hat{k})|k|^3\tau$, we obtain an ε-independent expression. We set

$$\hat{G}_m(k,\tau) := B_m(k)\, \xi_m(\hat{k}) \, \exp\left(ib_m(\hat{k})|k|^3\tau\right) . \tag{3.6}$$

With the help of $\hat{G}_m(k,\tau)$, we define the functions V_m and $\hat{V}_m$, where the latter is the one-dimensional Fourier transform of V_m in the first variable. We use the complex factor $z_* := (i)^{(d-1)/2}$, set $\hat{V}_m(\xi,q,\tau) := 0$ for $\xi \le 0$ and

$$\hat{V}_m(\xi,q,\tau) := \frac{1}{z_*}\,|\xi|^{(d-1)/2}\,\hat{G}_m(\xi q,\tau) \tag{3.7}$$

for $\xi > 0$. We recall that $\hat{V}_m$ is a function in $\xi \in \mathbb{R}$, $q \in S^{d-1}$ and $\tau > 0$. The one-dimensional inverse Fourier-transform in the variable $\xi \in \mathbb{R}$ defines the function $V_m(z,q,\tau)$, where we denote by z the variable that is dual to ξ.

We need a decay property of the profiles.

Assumption 3.1 (Decay of the profile functions)**.** *For $T_0 > 0$ we demand that every function V_m satifies: There exist $C,\alpha > 0$ such that, for every $z \in \mathbb{R}$, $\tau \in (0,T_0)$ and $q \in S^{d-1}$:*

$$|V_m(z,q,\tau)| \le C(1+|z|)^{-d-\alpha}\,. \tag{3.8}$$

Additionally, we assume for the derivatives

$$|\partial_z V_m(z,q,\tau)| + |\partial_q V_m(z,q,\tau)| \le C(1+|z|)^{-d/2-\alpha}\,. \tag{3.9}$$

We note that the approximation of Section 2.5 satisfies this condition. Indeed, for $\hat{G}_m$ having a compact support that does not contain $k=0$, it is sufficient to demand uniform boundedness of $\hat{G}_m \in H^\beta(\mathbb{R}^d)$ for some $\beta > d$ (uniform with respect to τ). Under these conditions, Assumption 3.1 is satisfied.

As a starting point for the subsequent considerations, we observe that the construction (3.4) provides $L^2(\mathbb{R}^d)$-bounded ring functions.

Lemma 3.2 (L^2-boundedness)**.** *We consider $m \le p$ and $T_0 > 0$. Let V_m satisfy Assumption 3.1 and let w_m^ε be given by (3.4). Then there exists $\tilde{C} > 0$ such that, for every $t \in (1/c_m, T_0/\varepsilon^2)$, there holds*

$$\|w_m^\varepsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)} \le \tilde{C}\,. \tag{3.10}$$

Proof. Condition (3.8) allows to calculate with polar coordinates $x = r\,q$ and the substitution $z = r - c_m\,t$:

$$\begin{aligned}
\|w_m^\varepsilon(\cdot,t)\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \frac{1}{(c_m t)^{d-1}} \left| V_m\left(|x| - c_m t, \frac{x}{|x|}, \varepsilon^2 t\right)\right|^2 dx \\
&= \int_0^\infty \int_{S^{d-1}} \frac{r^{d-1}}{(c_m t)^{d-1}} \left|V_m\left(r - c_m t, q, \varepsilon^2 t\right)\right|^2 dS(q)\,dr \\
&\le C^2 \int_{-c_m t}^\infty \int_{S^{d-1}} \frac{(c_m t + z)^{d-1}}{(c_m t)^{d-1}} (1+|z|)^{-2d-2\alpha}\, dS(q)\,dz \\
&\le C^2 \int_{\mathbb{R}} (1+|z|)^{-2d-2\alpha+d-1}\,dz \le \tilde{C}^2\,.
\end{aligned}$$

This was the claim. □

Lemma 3.2 provides uniform L^2-boundedness for w^ε for $t \in (1/c^*, T_0/\varepsilon^2)$ with $c^* := \min\{c_1,\ldots,c_p\}$. The subsequent theorem is the main result of this section. It states that the function U^ε is approximated well by the ring solution w^ε in L^2-sense.

Theorem 3.3 (Approximation with rings)**.** *Let* $p \in \mathbb{N}$ *be the number of rings and let the dimension be* $d = 2$ *or* $d = 3$, $T_0 > 0$. *We assume that the speeds* $c_m(q) = c_m$ *are independent of* q *and different from each other,* $c_m \neq c_\ell$ *for* $m \neq \ell$. *Let* $U^\varepsilon : \mathbb{R}^d \times (0,\infty) \to \mathbb{C}^p$ *have the Fourier-transform* (3.1) *with* ω_m^* *of* (3.2) *and let* $\hat{G}_m$ *be given by* (3.6). *We assume that all coefficients* $\hat{G}_m(\cdot,\tau) \in C^{d-1}(\mathbb{R}^d)$ *have compact support,* 0 *not contained in the support, and such that Assumption 3.1 is satisfied. Let* V_m *be as constructed in* (3.7) *and let* $w^\varepsilon : \mathbb{R}^d \times (0,\infty) \to \mathbb{C}^p$ *be given by* (3.4)–(3.5). *Then, for an arbitrary sequence* $\varepsilon \to 0$ *and an arbitrary time sequence* $t_\varepsilon \in (0, T_0/\varepsilon^2)$ *with* $\liminf_{\varepsilon\to 0} \varepsilon^2 t_\varepsilon > 0$, *there holds*

$$\|w^\varepsilon(x,t_\varepsilon) - U^\varepsilon(x,t_\varepsilon)\|_{L^2(\mathbb{R}^d,dx)} \to 0\,. \tag{3.11}$$

3.1 Fourier transform of ring functions and convergence

In order to show Theorem 3.3, we analyze the convergence of $\hat{w}^\varepsilon$ when multiplied with the oscillatory prefactor $e^{ic_m|k|t}$. The convergence relies on a stationary phase result. Essentially, we can use the stationary phase result of [21], but we need additionally some uniform bounds. In order to have this article self-contained, we provide the proof in Appendix B.

Proposition 3.4 (Approximation by p rings)**.** *Let* $d \in \{2,3\}$, $T_0 > 0$, $p \in \mathbb{N}$, $c_m(q) = c_m > 0$ *and* $\hat{G}_m : \mathbb{R}^d \times (0,\infty) \to \mathbb{C}^p$ *for* $m \in \{1,...,p\}$ *be so that all requirements of Theorem 3.3 hold. Let the function* $w^\varepsilon : \mathbb{R}^d \times (0,\infty) \to \mathbb{C}^p$ *be constructed as in* (3.4)–(3.5) *from the profile functions* V_m. *Let* $t_\varepsilon \in (0, T_0/\varepsilon^2]$ *be a time sequence with* $\tau := \lim_{\varepsilon\to 0} \varepsilon^2 t_\varepsilon > 0$. *Then:*

a) For every $0 \neq k \in \mathbb{R}^d$ *holds*

$$e^{ic_m|k|\,t_\varepsilon}\, \hat{w}_m^\varepsilon(k,t_\varepsilon) \to \hat{G}_m(k,\tau)\,. \tag{3.12}$$

b) The convergence (3.12) *is also a strong convergence in* $L^2(\mathbb{R}^d, dk)$.

c) For the combined ring solution w^ε, *weak convergence holds when the phase factor selects a specific ring: For arbitrary* $m \leq p$, *there holds the following weak convergence in* $L^2(\mathbb{R}^d, dk)$:

$$e^{ic_m|k|\,t_\varepsilon}\, \hat{w}^\varepsilon(k,t_\varepsilon) \rightharpoonup \hat{G}_m(k,\tau)\,. \tag{3.13}$$

We cannot expect strong convergence in (3.13), see Lemma 3.5 below.

Proof of Proposition 3.4. Our assumption $\lim_{\varepsilon\to 0} \varepsilon^2 t_\varepsilon > 0$ implies $t_\varepsilon \to \infty$ as $\varepsilon \to 0$. In the following, we drop the subscript ε in the time variable and write t instead of t_ε. All calculations are first done for dimension $d = 2$, then the necessary modifications for $d = 3$ are indicated.

Step 1: Calculation of the Fourier transform of w_m^ε. We have to calculate the d-dimensional Fourier transform of w_m^ε. In the following calculation we use, in this order: (i) The definition of w_m^ε in (3.4) and a Fourier transform with the variables $x = rq$ and $q \in S^{d-1}$. (ii) For every $q \in S^{d-1}$, the one-dimensional parameter transformation from the variable r to the variable z by setting $r = c_m t + z$. An error term $E^\varepsilon(k,\varepsilon,t)$ appears, since we replace an integration over $z \in (-c_m t, \infty)$ by an integration over $z \in (-\infty,\infty)$; furthermore, in dimension $d = 2$ we replace r by

$c_m t$ under the integral. The approximation in dimension $d = 3$ is slightly different, see (3.14) below. (iii) We exploit that there appears the one-dimensional Fourier transform of V_m with integration variable z and re-write the expression with $\hat{V}_m$. (iv) The definition of $\hat{V}_m$ in (3.7) and $z_0 := (2\pi i)^{(d-1)/2} = (2\pi)^{(d-1)/2} z_*$.

$$\begin{aligned}
\hat{w}^\varepsilon_m(k,t) &= \frac{1}{(2\pi)^{d/2}} \int_{S^{d-1}} \int_0^\infty e^{-ik\cdot qr} \frac{r^{d-1}}{(c_m t)^{(d-1)/2}} V_m\left(r - c_m t, q, \varepsilon^2 t\right) \, dr \, dS(q) \\
&= \frac{1}{(2\pi)^{d/2}} \int_{S^{d-1}} \int_{\mathbb{R}} e^{-ik\cdot q c_m t} e^{-ik\cdot qz} (c_m t)^{(d-1)/2} \, V_m\left(z, q, \varepsilon^2 t\right) \, dz \, dS(q) \, + \, E^\varepsilon(k,t) \\
&= \frac{1}{(2\pi)^{(d-1)/2}} \int_{S^{d-1}} e^{-ik\cdot q c_m t} (c_m t)^{(d-1)/2} \, \hat{V}_m\left(k\cdot q, q, \varepsilon^2 t\right) \, dS(q) \, + \, E^\varepsilon(k,t) \\
&= \frac{1}{z_0} \int_{S^{d-1}\cap\mathcal{H}_k} e^{-ik\cdot q c_m t} (c_m t)^{(d-1)/2} |k\cdot q|^{(d-1)/2} \hat{G}_m(k\cdot q\, q, \varepsilon^2 t) \, dS(q) + \, E^\varepsilon(k,t) \,,
\end{aligned}$$

where $\mathcal{H}_k := \{q \in \mathbb{R}^d \,|\, q\cdot k > 0\}$ is the half-space in which $q \mapsto \hat{V}_m\left(k\cdot q, q, \varepsilon^2 t\right)$ is supported. The error term $E^\varepsilon(k,t) = E^\varepsilon_1(k,t) + E^\varepsilon_2(k,t)$ can be written, with the exponent $\beta := (d-1)/2$, as

$$\begin{aligned}
E^\varepsilon_1 &:= -\frac{1}{(2\pi)^{d/2}} \int_{S^{d-1}} \int_{-\infty}^{-c_m t} e^{-ik\cdot q(c_m t + z)} \frac{(c_m t + z)^{2\beta}}{(c_m t)^\beta} \, V_m\left(z, q, \varepsilon^2 t\right) \, dz \, dS(q) \,, \\
E^\varepsilon_2 &:= \frac{1}{(2\pi)^{d/2}} \int_{S^{d-1}} \int_{\mathbb{R}} e^{-ik\cdot q(c_m t + z)} \left[\frac{(c_m t + z)^{2\beta}}{(c_m t)^\beta} - (c_m t)^\beta \right] V_m\left(z, q, \varepsilon^2 t\right) \, dz \, dS(q) \,.
\end{aligned}$$

We next show that both error terms are of order $(c_m t)^{-\beta}$ uniformly in k and ε. We exploit the decay condition (3.8) of Assumption 3.1 and the identity $-d - \alpha = -(d-1) - (\alpha+1) = -2\beta - (\alpha+1)$ to obtain

$$\begin{aligned}
|E^\varepsilon_1(k,t)| &\leq \frac{(2\pi)^{-d/2}}{(c_m t)^\beta} \int_{S^{d-1}} \int_{-\infty}^{-c_m t} |c_m t + z|^{2\beta} \, \left|V_m\left(z, q, \varepsilon^2 t\right)\right| dz \, dS(q) \\
&\leq \frac{C(2\pi)^{-d/2}}{(c_m t)^\beta} \int_{S^{d-1}} \int_{-\infty}^{-c_m t} \left(\frac{|c_m t + z|}{1 + |z|} \right)^{2\beta} (1 + |z|)^{-(\alpha+1)} \, dz \, dS(q) \\
&\leq \frac{1}{(c_m t)^\beta} \left(C(2\pi)^{-d/2} \, 2^{2\beta} \int_{S^{d-1}} \int_{-\infty}^{-c_m t} (1 + |z|)^{-(\alpha+1)} \, dz \, dS(q) \right) = \frac{\tilde{C}}{(c_m t)^\beta} \,.
\end{aligned}$$

In the last inequality we have used that $|z| \geq c_m t$, hence

$$\frac{|c_m t + z|}{1 + |z|} \leq \frac{c_m t}{1 + c_m t} + \frac{|z|}{1 + |z|} \leq 2$$

and in the last line we exploited that $(1 + |z|)^{-(\alpha+1)}$ is integrable over $z \in \mathbb{R}$. This shows that E^ε_1 is of order $(c_m t)^{-\beta}$ uniformly in k and ε.

In the error E^ε_2, the term in squared brackets is $\left[(c_m t + z)^{2\beta}/(c_m t)^\beta - (c_m t)^\beta\right] = \left[(c_m t + z)^{2\beta} - (c_m t)^{2\beta}\right]/(c_m t)^\beta$; in two space dimensions, i.e., $2\beta = d - 1 = 1$, the term is simply $z/(c_m t)^\beta$. Again by integrability of $z \mapsto (1 + |z|) V_m(z, q, \varepsilon^2 t)$ from (3.8), E^ε_2 is of order $(c_m t)^{-\beta}$ uniformly in k and ε.

Modifications in dimension $d = 3$. In the calculation of $\hat{w}^\varepsilon_m(k,t)$ we replace, in the second equality, $r^{d-1} = r^2 = (c_m t + z)^2$ by the approximation $(c_m t)^2 + 2z c_m t$ to obtain

$$
\begin{aligned}
&\hat{w}^\varepsilon_m(k,t) \qquad (3.14)\\
&= \frac{1}{(2\pi)^{d/2}} \int_{S^{d-1}} \int_{\mathbb{R}} e^{-ik\cdot q c_m t} e^{-ik\cdot qz} \left[(c_m t)^{(d-1)/2} + 2z\right] V_m\left(z,q,\varepsilon^2 t\right)\, dz\, dS(q)\\
&\quad + E^\varepsilon(k,t)\\
&= \frac{1}{(2\pi)^{(d-1)/2}} \int_{S^{d-1}} e^{-ik\cdot q c_m t} (c_m t)^{(d-1)/2} \left[\hat{V}_m + \frac{2i}{(c_m t)^{(d-1)/2}} \partial_\xi \hat{V}_m\right]\left(k\cdot q, q, \varepsilon^2 t\right)\, dS(q)\\
&\quad + E^\varepsilon(k,t)\,,
\end{aligned}
$$

where $E^\varepsilon(k,t) = E^\varepsilon_1(k,t) + E^\varepsilon_2(k,t)$ with E^ε_1 as in dimension $d = 2$ and

$$
\begin{aligned}
&E^\varepsilon_2(k,t)\\
&:= (2\pi)^{-d/2} \int_{S^{d-1}} \int_{\mathbb{R}} e^{-ik\cdot q(c_m t+z)} \left[\frac{(c_m t + z)^2}{(c_m t)} - (c_m t + 2z)\right] V_m\left(z,q,\varepsilon^2 t\right)\, dz\, dS(q)\,.
\end{aligned}
$$

In dimension $d = 3$, the term in the squared brackets is

$$
\frac{(c_m t + z)^2}{(c_m t)} - (c_m t + 2z) = \frac{z^2}{c_m t} = \frac{z^2}{(c_m t)^\beta}\,,
$$

since $\beta = (d-1)/2 = 1$. By integrability of $z \mapsto (1+|z|)^2 V_m(z,q,\varepsilon^2 t)$ from (3.8), E^ε_2 is of order $(c_m t)^{-\beta}$ uniformly in k and ε.

Step 2: Pointwise convergence of Claim a). Our aim is to prove (3.12). We therefore analyze the function $\hat{w}^\varepsilon_m$ with a phase factor. Using the result of Step 1 we find, for $d = 2$,

$$
\begin{aligned}
e^{ic_m|k|\,t} \hat{w}^\varepsilon_m(k,t) &= O(1/(c_m t)^\beta) + \\
&+ \frac{1}{z_0} \int_{S^{d-1}\cap \mathcal{H}_k} e^{ic_m|k|(1-k\cdot q/|k|)t} (|k\cdot q| c_m t)^{(d-1)/2} \hat{G}_m(k\cdot q\, q, \varepsilon^2 t)\, dS(q)\,.
\end{aligned} \qquad (3.15)
$$

For later use we note that the error term $O(1/(c_m t)^\beta)$ is bounded and small for $t \to \infty$, independent of k and ε in compact sets.

The stationary phase result of Proposition B.1 in the appendix provides, for $d \le 3$,

$$
A^N_\phi := \frac{1}{z_0} \int_{S^{d-1}\cap\mathcal{H}_\kappa} N^{(d-1)/2}\, e^{i(1-q\cdot\kappa)\,N}\, \phi(q)\, dS(q) \to \phi(\kappa) \qquad (3.16)
$$

as $N \to \infty$ for test-functions $\phi \in C^1(S^{d-1})$. We use this relation with the large number $N = c|k|\,t$ for the wave-speed $c = c_m$, with the direction $\kappa = k/|k|$, and the test-function $\phi(q) = (|q\cdot k|/|k|)^{(d-1)/2} \hat{G}_m(k\cdot q\, q, \varepsilon^2 t)$. We note that the pre-factor of $\hat{G}_m$ does not effect the smoothness $\phi \in C^1(S^{d-1})$, since $\hat{G}_m$ has a support that does not contain $k = 0$. The application of stationary phase convergence (3.16) to the expression in (3.15) yields

$$
e^{ic_m|k|\,t} \hat{w}^\varepsilon_m(k,t) \to \hat{G}_m(k,\tau)\,.
$$

We have obtained the pointwise convergence (3.12).

Modifications in dimension $d = 3$. In dimension $d = 3$ one has

$$\begin{aligned} e^{ic_m|k|\,t}\hat{w}^\varepsilon_m(k,t) \;&=\; O(1/(c_m t)^\beta) \\ &+ \frac{1}{z_0}\int_{S^{d-1}\cap\mathcal{H}_k} e^{ic_m|k|(1-k\cdot q/|k|)t}(c_m t)^{(d-1)/2}\times \\ &\qquad\times\left[\hat{V}_m + \frac{2i}{(c_m t)^{(d-1)/2}}\partial_\xi \hat{V}_m\right]\left(k\cdot q, q, \varepsilon^2 t\right)\, dS(q)\,. \end{aligned} \tag{3.17}$$

By our assumptions, $\hat{V}_m(\xi, q, \tau) = \frac{1}{z_*}|\xi|^{(d-1)/2}\hat{G}_m(\xi q, \tau)$ is of class $C^{d-1} = C^2$. The stationary phase result is applicable to the C^1-functions $\hat{V}_m$ and $\partial_\xi V_m$. The first term in the squared brackets generates the same limit as in $d = 2$, while the second term vanishes in the limit $t \to \infty$ due to the small prefactor $1/(c_m t)^{(d-1)/2}$.

A note on the above argument: For sequences $t = \tau\varepsilon^{-2}$ with fixed $\tau > 0$, one can directly apply (3.16) above, the test function is $\phi(q) = (|q\cdot k|/|k|)^{(d-1)/2}\hat{G}_m(k\cdot q\, q, \tau)$. For a more general sequence $t \to \infty$, one has to use a triangle inequality after inserting a term containing $\hat{G}_m(k\cdot q\, q, \varepsilon^2 t) - \hat{G}_m(k\cdot q\, q, \tau)$. This latter term is small by the fact that $\hat{G}_m$ is of order C^1 and by the uniform bound (B.3) of the stationary phase result.

Step 3: Strong convergence of Claim b). The following proof for $d = 2$ requires only one adaptation for dimension $d = 3$: We have to replace (3.15) by (3.17).

In order to show the strong convergence, we divide $\mathbb{R}^d$ into three different domains of integration using radii $0 < \rho < R < \infty$. We will show that, by choosing ρ small, the integral over $k \in B_\rho(0)$ is small. Furthermore, by choosing R large, the integral over $k \in \mathbb{R}^d \setminus B_R(0)$ is small. Finally, for fixed $\rho < R$, we have strong convergence for $k \in B_R(0) \setminus B_\rho(0)$.

Step 3a: The domain $B_\rho(0)$. For small $\rho > 0$, the functions $\hat{G}_m(k\cdot q\, q, \tau)$ are vanishing for all m, q and τ by the assumption on the support of $\hat{G}_m$. Hence the second term in (3.15) vanishes, while the $O(1/(c_m t)^\beta)$-term is integrated over the small ball $B_\rho(0)$ and vanishes in the limit $t \to \infty$.

Step 3b: The domain $B_R(0) \setminus B_\rho(0)$. We use dominated convergence. Since pointwise convergence is already verified, it is sufficient to show uniform boundedness of the integrand.

A uniform bound for $e^{ic_m|k|\,t}\hat{w}^\varepsilon_m(k,t)$ for such k follows from (3.15)-(3.16) and the uniform boundedness statement of Proposition B.1. We note that Proposition B.1 provides uniform boundedness only for all k with $|k| = 1$, but scaling k with a factor in the compact interval $[\rho, R]$ leaves the expression in (3.16) bounded.

Pointwise convergence together with uniform boundedness for $k \in B_R(0)\setminus B_\rho(0)$ implies strong L^2-convergence on this set.

Step 3c: The domain $\mathbb{R}^d \setminus B_R(0)$. We claim that, for an arbitrary error quantifier $\eta > 0$, we can choose R large such that

$$\int_{\mathbb{R}^d\setminus B_R(0)} \left|e^{ic_m|k|\,t}\hat{w}^\varepsilon_m(k,t)\right|^2\, dk = \int_{\mathbb{R}^d\setminus B_R(0)} |\hat{w}^\varepsilon_m(k,t)|^2\, dk \le \eta \tag{3.18}$$

is satisfied for all ε and all t. With (3.18), the strong convergence in (3.12) is shown.

We choose a small radius $\sigma > 0$ and a C^1-cut-off function $\chi_\sigma : \mathbb{R}^d \to [0,1]$ that is identical to 1 on $B_{\sigma/2}(0)$ and that is supported in $B_\sigma(0)$. Upon choosing σ small, we achieve that

$$\int_{\mathbb{R}^d \setminus B_R(0)} |\mathcal{F}(w_m^\varepsilon \chi_\sigma)(k,t)|^2 \, dk \le \int_{\mathbb{R}^d} |(w_m^\varepsilon \chi_\sigma)(x,t)|^2 \, dx \le \eta/2 \,, \tag{3.19}$$

since w_m^ε is uniformly bounded by Assumption 3.1 and, effectively, the domain of integration is a small ball.

The remaining part is $[w_m^\varepsilon(1-\chi_\sigma)](x,t)$. This function is bounded in $L^2(\mathbb{R}^d)$ since w_m^ε is bounded, compare Lemma 3.2. Furthermore, $\nabla[w_m^\varepsilon(1-\chi_\sigma)](x,t)$ is bounded in $L^2(\mathbb{R}^d)$. Indeed, the radial derivative of w_m^ε is $(c_m t)^{-(d-1)/2} \partial_z V_m \left(|x| - c_m t, \frac{x}{|x|}, \varepsilon^2 t \right)$, and the calculation in the proof of Lemma 3.2 with the decay assumption (3.9) provides boundedness. The q-derivative is $(c_m t)^{-(d-1)/2} |x|^{-1} \partial_q V_m \left(|x| - c_m t, \frac{x}{|x|}, \varepsilon^2 t \right)$. Because of the cut-off function $(1-\chi_\sigma)(x,t)$, we do not consider small values of x. The calculation of Lemma 3.2 with the decay assumption (3.9) provides L^2-boundedness for all derivatives.

Switching from physical space with independent variable x to Fourier space with independent variable k, we find that the functions $(1+|k|)\, \mathcal{F}[w_m^\varepsilon(1-\chi_\sigma)](k,t)$ satisfy uniform $L^2(\mathbb{R}^d)$-bounds. This guarantees that the L^2-norm of $\mathcal{F}[w_m^\varepsilon(1-\chi_\sigma)](k,t)$ on the domain $k \in \mathbb{R}^d \setminus B_R(0)$ is smaller than $\eta/2$ for $R > 0$ large. Together with (3.19), we obtain (3.18).

Step 4: Selection and Claim c). We turn to the weak convergence of (3.13). It treats a sum of rings on the left-hand side, the oscillatory pre-factor selects one profile. The claim follows from

$$e^{ic_m|k|t} \hat{w}^\varepsilon(k,t) = e^{ic_m|k|t} \hat{w}_m^\varepsilon(k,t) + \sum_{\ell \ne m} e^{i(c_m - c_\ell)|k|t} e^{ic_\ell|k|t} \hat{w}_\ell^\varepsilon(k,t) \rightharpoonup \hat{G}_m(k,\tau) \,.$$

The last convergence is obtained as follows: The strong convergences of the first term (with the "right" phase factor) was observed in the last step. The other terms are products of one strongly convergent sequence and one weakly convergent bounded sequence. This implies the weak convergence of the product. □

3.2 Strong convergence for separated rings

In this subsection we prove our main result Theorem 3.3 starting with an auxiliary statement about the convergence of the norm of $\hat{w}^\varepsilon$.

Lemma 3.5 (Separation of profiles)**.** *Let the setting be that of Proposition 3.4. Then there holds:*

$$\|\hat{w}^\varepsilon(k,t_\varepsilon)\|^2_{L^2(\mathbb{R}^d,dk)} \to \sum_{m=1}^p \|\hat{G}_m(k,\tau)\|^2_{L^2(\mathbb{R}^d,dk)} \,. \tag{3.20}$$

Proof. Without loss of generality we can assume that the wave-speeds are ordered, $c_1 < c_2 < \ldots < c_p$, and we can choose real numbers σ_j in each interval,

$$c_1 < \sigma_1 < c_2 < \sigma_2 < c_3 < \ldots < \sigma_{p-1} < c_p \,.$$

We calculate the norms with Plancherel, using $\sigma_0 := 0$ and $\sigma_p := +\infty$, writing t instead of t_ε,

$$\begin{aligned}
\|\hat{w}^\varepsilon(k,t)\|^2_{L^2(\mathbb{R}^d,dk)} &= \|w^\varepsilon(x,t)\|^2_{L^2(\mathbb{R}^d,dx)} \\
&= \sum_{j=1}^p \int_{B_{t\sigma_j}(0)\setminus B_{t\sigma_{j-1}}(0)} |w^\varepsilon(x,t)|^2\, dx \\
&= \sum_{j=1}^p \int_{S^{d-1}} \int_{t\sigma_{j-1}}^{t\sigma_j} |w^\varepsilon(rq,t)|^2\, r^{d-1}\, dr\, dS(q) \\
&= \sum_{m=1}^p \sum_{j=1}^p \int_{S^{d-1}} \int_{t\sigma_{j-1}}^{t\sigma_j} \frac{1}{(c_m t)^{d-1}} |V_m\left(r - c_m t, q, \varepsilon^2 t\right)|^2\, r^{d-1}\, dr\, dS(q) \\
&= \sum_{m=1}^p \int_{S^{d-1}} \int_{t\sigma_{m-1}}^{t\sigma_m} \frac{1}{(c_m t)^{d-1}} |V_m\left(r - c_m t, q, \varepsilon^2 t\right)|^2\, r^{d-1}\, dr\, dS(q) \\
&\quad + \sum_{j\neq m}^p \int_{S^{d-1}} \int_{t\sigma_{j-1}}^{t\sigma_j} \frac{1}{(c_m t)^{d-1}} |V_m\left(r - c_m t, q, \varepsilon^2 t\right)|^2\, r^{d-1}\, dr\, dS(q)\,.
\end{aligned}$$

Regarding the sum over $j \neq m$, we exploit that V_m is small whenever the argument $r - c_m t$ is large in absolute value, see (3.8). More precisely, for $j \neq m$ and $r \in (t\sigma_{j-1}, t\sigma_j)$, the quantity $|r - c_m t|$ is bounded from below by a positive multiple of t. The decay assumption (3.8) implies $|V_m|^2 \leq C t^{-2d-2\alpha}$ and $|V_m|^2 \leq C r^{-2d-2\alpha}$. This implies smallness of the last sum in the limit $t \to \infty$.

With the same argument, we can extend the integral in the other term to find

$$\begin{aligned}
&\|\hat{w}^\varepsilon(k,t)\|^2_{L^2(\mathbb{R}^d,dk)} \\
&= \sum_{m=1}^p \int_{S^{d-1}} \int_{t\sigma_{m-1}}^{t\sigma_m} \frac{1}{(c_m t)^{d-1}} |V_m\left(r - c_m t, q, \varepsilon^2 t\right)|^2\, r^{d-1}\, dr\, dS(q) \,+\, o(1) \\
&= \sum_{m=1}^p \int_{S^{d-1}} \int_0^\infty \frac{1}{(c_m t)^{d-1}} |V_m\left(r - c_m t, q, \varepsilon^2 t\right)|^2\, r^{d-1}\, dr\, dS(q) \,+\, o(1) \\
&= \sum_{m=1}^p \int_{\mathbb{R}^d} |w_m^\varepsilon(x,t)|^2\, dx \,+\, o(1)\,.
\end{aligned}$$

The strong convergence $e^{ic_m|k|t}\, \hat{w}_m^\varepsilon(k,t) \to \hat{G}_m(k,\tau)$ of (3.12) implies the convergence of the norms, by Plancherel also the convergence $\|w_m^\varepsilon(x,t)\|_{L^2(\mathbb{R}^d,dx)} \to \|\hat{G}_m(k,\tau)\|_{L^2(\mathbb{R}^d,dk)}$. This yields the claim of (3.20). □

Lemma 3.6 (Strong multiple-ring approximation)**.** *Let the setting be that of Proposition 3.4. Let $\hat{U}^\varepsilon$ be defined by* (3.1), *i.e.*

$$\hat{U}^\varepsilon(k,t) := \sum_{m=1}^p e^{-i|k|c_m t}\, \hat{G}_m(k, \varepsilon^2 t)\,. \tag{3.21}$$

The function U^ε is obtained by inverse Fourier transform. Then there holds

$$\|w^\varepsilon(x,t_\varepsilon) - U^\varepsilon(x,t_\varepsilon)\|_{L^2(\mathbb{R}^d,dx)} \to 0\,. \tag{3.22}$$

We note that the convergence (3.22) coincides with (3.11) of Theorem 3.3.

Proof. The result can be obtained with a direct calculation. We calculate the squared norm using (i) Plancherel, (ii) relation (3.21) defining $\hat{U}^\varepsilon$, exploiting with the term $o(1)$ that $\tau \mapsto \hat{G}_m(\cdot,\tau)$ is a continuous map into $L^2(\mathbb{R}^d)$ because of the C^1-requirement and the bounded support in k, (iii) formulas for calculating the scalar product of sums, (iv) the norm convergence (3.20), the weak convergence relations (3.13), the weak convergence of the exponential where the other factor is a fixed function. We find, as $\varepsilon \to 0$,

$$
\begin{aligned}
&\|w^\varepsilon(x,t_\varepsilon) - U^\varepsilon(x,t_\varepsilon)\|^2_{L^2(\mathbb{R}^d,dx)} = \left\|\hat{w}^\varepsilon(k,t_\varepsilon) - \hat{U}^\varepsilon(k,t_\varepsilon)\right\|^2_{L^2(\mathbb{R}^d,dk)}\\
&\quad= \left\|\hat{w}^\varepsilon(k,t_\varepsilon) - \sum_{m=1}^{p} e^{-i|k|c_m\tau/\varepsilon^2}\,\hat{G}_m(k,\tau)\right\|^2_{L^2(\mathbb{R}^d,dk)} + o(1)\\
&\quad= \|\hat{w}^\varepsilon(k,t_\varepsilon)\|^2_{L^2(\mathbb{R}^d,dk)} - 2\sum_{m=1}^{p}\mathrm{Re}\int_{\mathbb{R}^d}\hat{w}^\varepsilon(k,t_\varepsilon)\,e^{i|k|c_m\tau/\varepsilon^2}\,\overline{\hat{G}_m(k,\tau)}\,dk\\
&\qquad+\sum_{m=1}^{p}\int_{\mathbb{R}^d}|\hat{G}_m(k,\tau)|^2\,dk + 2\sum_{m\neq\ell}\mathrm{Re}\int_{\mathbb{R}^d}e^{-i|k|(c_m-c_\ell)\tau/\varepsilon^2}\,\hat{G}_m(k,\tau)\,\overline{\hat{G}_\ell(k,\tau)}\,dk\\
&\qquad\quad+ o(1)\\
&\quad\to \sum_{m=1}^{p}\left\|\hat{G}_m(k,\tau)\right\|^2_{L^2(\mathbb{R}^d,dk)}\quad\big(1-2+1\big)\;=\;0\,.
\end{aligned}
$$

This verifies (3.22). ☐

Proof of Theorem 3.3. Lemma 3.6 provides the statement of the theorem for time sequences t_ε with limit $\tau := \lim_{\varepsilon\to 0}\varepsilon^2 t_\varepsilon > 0$. For arbitrary time sequences $t_\varepsilon \in (0, T_0/\varepsilon^2)$ with $\liminf_{\varepsilon\to 0}\varepsilon^2 t_\varepsilon > 0$ one exploits that there exists a subsequence t_ε (not relabeled) such that $\lim_{\varepsilon\to 0}\varepsilon^2 t_\varepsilon = \tau > 0$. This follows directly from the boundedness of $\varepsilon^2 t_\varepsilon$ and the assumption on the lim inf. Lemma 3.6 provides convergence for all such subsequences and thus the statement of Theorem 3.3. ☐

A Ring solutions for the Lamé system

Our main results treat questions of periodic homogenization, which means, in the Lamé system, to treat the coefficients $\mu = \mu(x/\varepsilon)$ and $\lambda = \lambda(x/\varepsilon)$. But, even when constant coefficients are studied, our results yield interesting characterizations.

For an illustration of the results of Section 3, we consider the Lamé system with constant coefficients in two space dimensions, $d = 2$. For constant coefficients, we cannot observe dispersion, but rings will appear in the description of solutions u. For every $t > 0$, the solution is a map $u(\cdot,t) : \mathbb{R}^d \to \mathbb{R}^d$, it solves

$$\partial_t^2 u = -Lu \quad\text{ with }\quad L := -2\mu\Delta - \lambda\nabla\,\mathrm{div}\,. \tag{A.1}$$

In order to understand L, we apply it to plane waves. For arbitrary $v \in \mathbb{R}^2$ and arbitrary $k \in \mathbb{R}^2$, we calculate for $u(x) := v\,e^{ik\cdot x}$

$$Lu(x) = 2\mu|k|^2 u(x) + \lambda k\otimes k\cdot u(x)\,. \tag{A.2}$$

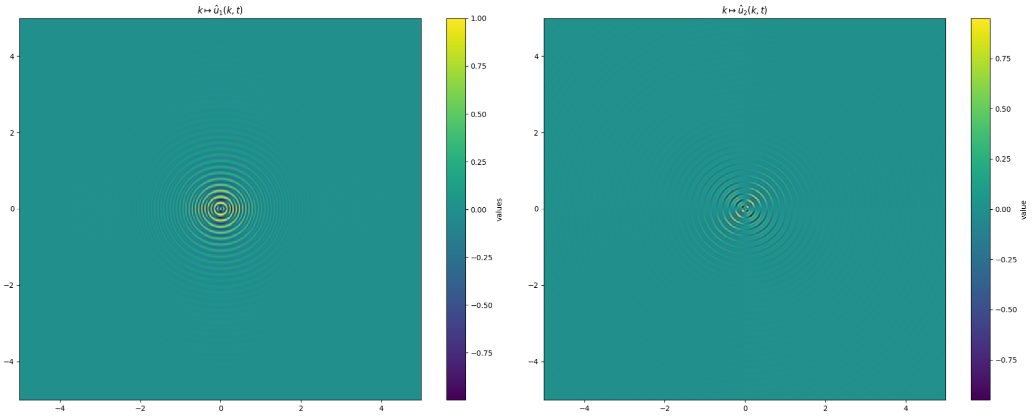

Figure 1: Fourier transform of the solution to the Lamé system, calculated with (A.8). The two (real valued) components $\hat{u}_1$ (left) and $\hat{u}_2$ (right) of the function $k \mapsto \hat{u}(k) = \mathcal{F}(u)(k, t_0)$.

In two space dimensions we use, for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, the rotated vector $\xi^\perp := (-\xi_2, \xi_1) \in \mathbb{R}^2$. For every $k \in \mathbb{R}^2$, we can define two special plane waves:

$$u_p(x) := k e^{ik\cdot x}\,, \qquad L u_p(x) = (2\mu + \lambda)|k|^2 u_p(x)\,, \tag{A.3}$$

$$u_s(x) := k^\perp e^{ik\cdot x}\,, \qquad L u_s(x) = 2\mu|k|^2 u_s(x)\,. \tag{A.4}$$

With $\omega^p(k) := \sqrt{2\mu + \lambda}|k|$ and $\omega^s(k) := \sqrt{2\mu}|k|$ we find two solutions to the time-dependent Lamé system:

$$u_p(x,t) := k e^{ik\cdot x} e^{-i\omega^p(k)t}\,, \qquad u_s(x,t) := k^\perp e^{ik\cdot x} e^{-i\omega^s(k)t}\,. \tag{A.5}$$

Fourier-representation of the general solution. Taking into account (A.2) we may write the equation $\partial_t^2 u = -Lu$ in Fourier space as

$$\partial_t^2 \mathcal{F}(u)(k,t) = -2\mu|k|^2 \mathcal{F}(u)(k,t) - \lambda k \otimes k \cdot \mathcal{F}(u)(k,t)\,. \tag{A.6}$$

For $k \neq 0$, there are two projections $\Pi_p(k), \Pi_s(k) : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\Pi_p(k)\, v := |k|^{-2}\, k \otimes k \cdot v\,, \qquad \Pi_s(k)\, v := \left(\mathrm{id} - |k|^{-2}\, k \otimes k\right) \cdot v\,. \tag{A.7}$$

They allow to find solutions of (A.6) for fixed $k \neq 0$:

$$\mathcal{F}(u)(k,t) = \mathrm{Re}\left(e^{-i\omega^p(k)t}\right) \Pi_p(k)\, \mathcal{F}(u_0)(k) \;+\; \mathrm{Re}\left(e^{-i\omega^s(k)t}\right) \Pi_s(k)\, \mathcal{F}(u_0)(k)\,. \tag{A.8}$$

By taking the real part of the exponential function, (A.8) provides solutions to arbitrary initial data $\hat{u}(k,0) = \hat{u}_0(k)$ and $\partial_t \hat{u}(k,0) = 0$.

Profile approximation. Our second main result applies to any function that has a representation in Fourier space as in (A.8) (compare the characterization of $\hat{U}^\varepsilon(k,t)$ in (3.1)). We show that the solution can be approximated, for large times t, as a superposition of rings, see (3.4) and (3.5). In order to describe large times, we proceed as follows: We fix a value $T_0 > 0$ and consider a rescaled time variable

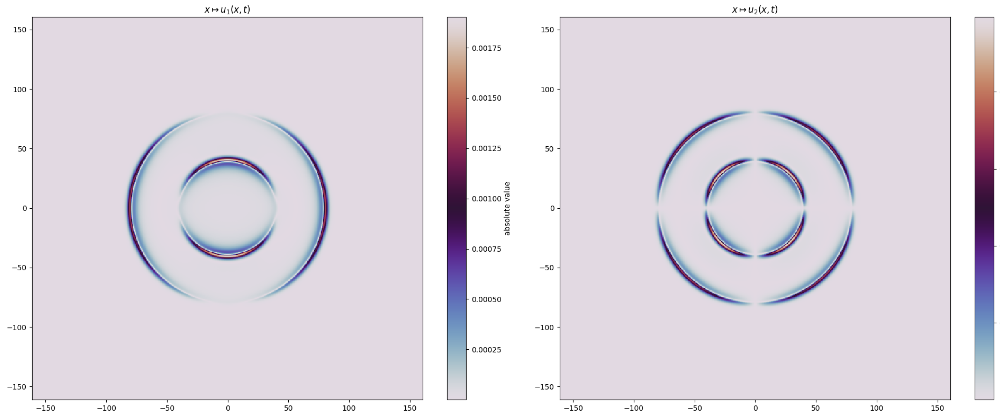

Figure 2: The solution u to the Lamé system, calculated by applying the inverse Fourier transform to $\hat{u}$. The absolute values of the two real components u_1 (left) and u_2 (right) of the solution $x \mapsto u(x, t_0)$ in physical space. The pressure wave is concentrated along the outer ring, the shear wave along the inner ring.

$\tau \in [0, T_0]$. The original time variable is written as $t = \tau/\varepsilon^2$ such that a small parameter $\varepsilon > 0$ allows to investigate large t.

In the case of the Lamé system, the approximation is

$$w(x,t) := w_p(x,t) + w_s(x,t) \tag{A.9}$$

with

$$w_p(x,t) := \frac{1}{(c_p t)^{(d-1)/2}} V_p \left(|x| - c_p t, \frac{x}{|x|}, \varepsilon^2 t \right) , \tag{A.10}$$

$$w_s(x,t) := \frac{1}{(c_s t)^{(d-1)/2}} V_s \left(|x| - c_s t, \frac{x}{|x|}, \varepsilon^2 t \right) . \tag{A.11}$$

The fact that constant coefficients are treated leads to profile functions V_p and V_s that are independent of the last parameter, the rescaled time.

Numerical illustration. For an illustration of the ring superposition result, we calculated an approximate solution of the Lamé system with constant coefficients. We used the initial values $\hat{u}_0(k) := (1 + |k|^2)^{-2}\, e_1$ in Fourier-space, the time instance $t_0 = 40$, the elasticity parameters $\mu = 0.5$ and $\lambda = 3.0$ such that the speeds are $c_s = 1$ and $c_p = 2$. We evaluated the initial values on the square $M := (-5, 5)^2$, the independent variable k was discretized with 512^2 points. The result for the two components of $k \mapsto \hat{u}(k, t_0)$ is plotted in Figure 1. The inverse Fourier-transform of this function provides $x \mapsto \hat{u}(x, t_0)$, the two components are plotted in Figure 2. The profile map in diagonal direction $r \mapsto u(r(1,1)/\sqrt{2}, t_0)$ is shown in Figure 3. We can see that the solution is a superposition of two rings. The profiles V_s and V_p can be determined from the initial data, see the constructions leading to (3.7). In the present example of the Lamé system with constant coefficients, there is no dispersion, the profiles do not even change slowly in time. On the other hand, the profiles do depend on the direction $x/|x|$ as can be seen clearly in Figure 2.

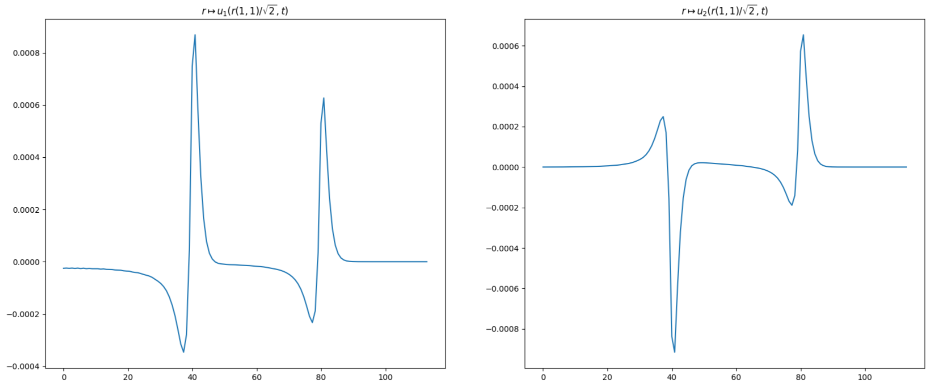

Figure 3: Numerically determined profiles in a 45°-direction, showing the first and the second component of the solution u. Left: The map $r \mapsto u_1(r(1,1)/\sqrt{2}, t_0)$. Right: The map $r \mapsto u_2(r(1,1)/\sqrt{2}, t_0)$. The profile of the shear-wave is centered at $c_s t_0 = 40$, we see essentially a shifted version of V_s, the profile of the pressure-wave is centered at $c_p t_0 = 80$, we see a shifted version of V_p.

B The stationary phase result

We recall a result of [21] and extend it with some uniform estimates.

Proposition B.1 (A singular integral over oscillatory functions)**.** *Let the dimension be* $d \in \{1, 2, 3\}$. *Let* $\kappa \in S^{d-1}$ *be a point on the sphere and let* $\phi \in C^1(S^{d-1}, \mathbb{R})$ *be supported in* $\{q \in S^{d-1} \,|\, q \cdot \kappa \geq 0\}$. *We investigate the expression*

$$A_\phi^N := (2\pi i)^{-(d-1)/2} \int_{S^{d-1}} N^{(d-1)/2} \, e^{i(1-q\cdot\kappa)\, N} \, \phi(q) \, dS(q) \, . \tag{B.1}$$

There holds, as $N \to \infty$,

$$A_\phi^N \to \phi(\kappa) \, . \tag{B.2}$$

Furthermore, the expression A_ϕ^N *is uniformly bounded: There exists* $C > 0$ *such that*

$$|A_\phi^N| \leq C \|\phi\|_{C^1(S^{d-1})} \, , \tag{B.3}$$

independently of N, κ *and* ϕ.

In the following, we recall the proof of (B.2) in dimension $d = 2$ and extend it by the estimates. Extending the proof of [21] for $d = 3$ by the bound (B.3) can be done along the same lines.

Proof of Proposition B.1. Consider $d = 2$. By radial symmetry it is sufficient to consider the case $\kappa := e_1$.

We use the coordinates $q(\theta) := (\cos(\theta), \sin(\theta))$ with $\theta \in (-\pi, \pi)$, the line element is $J = 1$. The expression of (B.1) can be written, using that ϕ is supported on

$\{q|\ q\cdot e_1>0\}\subset S^{d-1}$,

$$\begin{aligned}A_\phi^N &= (2\pi i)^{-1/2}\int_{S^1} N^{1/2}\, e^{i(1-q\cdot e_1)\,N}\,\phi(q)\,dS(q)\\ &= (2\pi i)^{-1/2}\int_{-\pi/2}^{\pi/2} N^{1/2}\, e^{i(1-\cos(\theta))\,N}\,\phi(q(\theta))\,d\theta\\ &= (2\pi i)^{-1/2}\int_{0}^{\pi/2} N^{1/2}\, e^{i(1-\cos(\theta))\,N}\,\tilde\phi(\theta)\,d\theta\,,\end{aligned}$$

where $\tilde\phi(\theta):=\phi(q(\theta))+\phi(q(-\theta))$ denotes a symmetrized version of ϕ. We split the integral into two parts, $\theta\in(0,\delta)$ and $\theta\in(\delta,\pi)$, where the small parameter δ is chosen N-dependent, $\delta:=N^{-\beta}$ with $\beta=3/10$.

First integral. We treat the first part of A_ϕ^N, the integral over $\theta\in(\delta,\pi)$. We do a trivial rewriting and then use an integration by parts:

$$\begin{aligned}&\int_{N^{-3/10}}^{\pi/2} N^{1/2}\, e^{i(1-\cos(\theta))\,N}\,\tilde\phi(\theta)\,d\theta\\ &= \frac{1}{\sqrt N}\int_{N^{-3/10}}^{\pi/2}\sin(\theta) i\,N\,e^{i(1-\cos(\theta))\,N}\,\frac{\tilde\phi(\theta)}{i\sin(\theta)}\,d\theta\\ &= \frac{1}{\sqrt N}\left[e^{i(1-\cos(\theta))\,N}\,\frac{\tilde\phi(\theta)}{i\sin(\theta)}\right]_{\theta=N^{-3/10}}^{\pi/2} - \frac{1}{\sqrt N}\int_{N^{-3/10}}^{\pi/2} e^{i(1-\cos(\theta))\,N}\,\frac{\frac{d}{d\theta}\tilde\phi(\theta)}{i\sin(\theta)}\,d\theta\\ &\quad+\frac{1}{\sqrt N}\int_{N^{-3/10}}^{\pi/2} e^{i(1-\cos(\theta))\,N}\,\frac{\tilde\phi(\theta)\cos(\theta)}{i\sin^2(\theta)}\,d\theta\\ &=: I_N + II_N + III_N\,.\end{aligned}$$

The terms I_N, II_N vanish in the limit as $N\to\infty$. Indeed, as $N\to\infty$,

$$\begin{aligned}|I_N| &= \frac{1}{\sin(N^{-3/10})\sqrt N}|\tilde\phi(N^{-3/10})| \le C\frac{N^{3/10}}{\sqrt N}\|\phi\|_{C^1(S^{d-1})}\\ &= CN^{3/10-1/2}\|\phi\|_{C^1(S^{d-1})}\to 0\end{aligned}$$

and, since the sine-function is monotonically increasing in $(0,\pi/2)$, as $N\to\infty$,

$$\begin{aligned}|II_N| &\le \frac{1}{\sqrt N}\int_{N^{-3/10}}^{\pi/2}\left|\frac{\frac{d}{d\theta}\tilde\phi(\theta)}{\sin(\theta)}\right|\,d\theta \le C\frac{N^{3/10}}{\sqrt N}\int_{N^{-3/10}}^{\pi/2}\left|\frac{d}{d\theta}\tilde\phi(\theta)\right|\,d\theta\\ &\le \tilde C N^{3/10-1/2}\|\phi\|_{C^1(S^{d-1})}\to 0\,.\end{aligned}$$

These calculations also show that $|I_N|$ and $|II_N|$ satisfy the uniform bound (B.3).

To treat III_N, we re-write the term and integrate by parts once more:

$$III_N = -N^{-3/2}\int_{N^{-3/10}}^{\pi/2} iNe^{i(1-\cos(\theta))\,N}\sin(\theta)\frac{\tilde\phi(\theta)\cos(\theta)}{\sin^3(\theta)}\,d\theta$$

$$
\begin{aligned}
&= -N^{-3/2}\left[e^{i(1-\cos(\theta))\,N}\frac{\tilde\phi(\theta)\cos(\theta)}{\sin^3(\theta)}\right]_{\theta=N^{-3/10}}^{\pi/2}\\
&\quad + N^{-3/2}\int_{N^{-3/10}}^{\pi/2} e^{i(1-\cos(\theta))\,N}\,\frac{\frac{d}{d\theta}\tilde\phi(\theta)\cos(\theta)-\tilde\phi(\theta)\sin(\theta)}{\sin^3(\theta)}\,d\theta\\
&\quad - N^{-3/2}\int_{N^{-3/10}}^{\pi/2} e^{i(1-\cos(\theta))\,N}\,\frac{3\cos^2(\theta)\tilde\phi(\theta)}{\sin^4(\theta)}\,d\theta\,.
\end{aligned}
$$

Since $1/\sin^3(N^{-3/10}) \le CN^{9/10}$, the first term is of order $N^{-3/2}N^{9/10}\|\phi\|_{C^1(S^{d-1})} = N^{-3/5}\|\phi\|_{C^1(S^{d-1})} \to 0$ as $N\to\infty$. For the integral expressions we note that $1/\sin^4(N^{-3/10}) \le CN^{6/5}$ and by assumption $\tilde\phi \in C^1([0,\pi])$. We conclude that the last integral scales as $N^{-3/2}N^{6/5}\|\phi\|_{C^1(S^{d-1})} = N^{-3/10}\|\phi\|_{C^1(S^{d-1})} \to 0$ as $N\to\infty$. The first integral is of lower order, namely $N^{-3/5}\|\phi\|_{C^1(S^{d-1})}$. This proves that $III_N \to 0$. The calculations also show that $|III_N|$ satisfies the uniform bound (B.3).

Second integral. We now treat the other part of A^N_ϕ, the integral over $\theta \in (0, N^{-3/10})$. We first note that, since $\tilde\phi$ is Lipschitz-continuous, for $\theta \in (0, N^{-3/10})$, one has

$$
|\tilde\phi(\theta)-\tilde\phi(0)| \le C\|\phi\|_{C^1(S^{d-1})}\theta \le C\|\phi\|_{C^1(S^{d-1})}N^{-3/10} \tag{B.4}
$$

and thus

$$
\begin{aligned}
&\left|\int_0^{N^{-3/10}} N^{1/2}\,e^{i(1-\cos(\theta))\,N}\,\tilde\phi(\theta)\,d\theta - \int_0^{N^{-3/10}} N^{1/2}\,e^{i(1-\cos(\theta))\,N}\,\tilde\phi(0)\,d\theta\right|\\
&\qquad \le C\,\|\phi\|_{C^1(S^{d-1})}\,N^{1/2}N^{-3/10}N^{-3/10} = C\,\|\phi\|_{C^1(S^{d-1})}\,N^{-1/10}\,.
\end{aligned}
$$

This integral is uniformly bounded in the sense of (B.3) and it vanishes in the limit as $N\to\infty$.

It remains to investigate the integral

$$
\tilde\phi(0)\,(2\pi i)^{-1/2}\int_0^{N^{-3/10}} N^{1/2}\,e^{i(1-\cos(\theta))\,N}\,d\theta\,. \tag{B.5}
$$

Using Lemma B.2 below and recalling that $\tilde\phi(0) = 2\phi(e_1)$, we obtain boundedness in the sense of (B.3) and, as $N\to\infty$,

$$
\tilde\phi(0)(2\pi i)^{-1/2}\int_0^{N^{-3/10}} N^{1/2}\,e^{i(1-\cos(\theta))\,N}\,d\theta \to \tilde\phi(0)(2\pi i)^{-1/2}\frac{1}{2}(2\pi i)^{1/2} = \phi(e_1)\,.
$$

This shows the claims (B.2) and (B.3) in dimension $d=2$. □

An oscillatory one-dimensional integral. It remains to treat the integral that appeared in (B.5). The following lemma was already shown in [21]; its proof is omitted here.

Lemma B.2 (One dimensional integral)**.** *Let $\beta\in(1/6,1/2)$. Then, as $N\to\infty$,*

$$
I_N := \int_0^{N^{-\beta}} N^{1/2}\,e^{i(1-\cos(\theta))\,N}\,d\theta \to \frac{1}{2}\sqrt{\pi}(1+i) = \frac{1}{2}(2\pi i)^{1/2}\,. \tag{B.6}
$$

C Homogenization of a Lamé system

This appendix provides a non-trivial example to which our main theorems can be applied. We consider a 2-dimensional Lamé system with periodically oscillatory second Lamé constant λ. We study $\partial_t^2 u^\varepsilon(x,t) = -L_\varepsilon u^\varepsilon(x,t)$ for the elliptic operator

$$L_\varepsilon u^\varepsilon(x,t) := -2\mu\Delta u^\varepsilon(x,t) + \nabla(\lambda(x/\varepsilon)\operatorname{div} u^\varepsilon(x,t))\,. \tag{C.1}$$

As the notation indicates, $\mu > 0$ is a constant. The second parameter is given by $\lambda : \mathbb{R}^2 \to [\lambda_0, \infty)$ with $\lambda_0 > 0$. We assume that λ is 2π-periodic in every direction.

Our aim is to show that the homogenized system is again a Lamé system with an effective Lamé constant λ^*. This implies that, as demanded in Theorem 3.3, the effective speeds $c_1(\hat{k})$ and $c_2(\hat{k})$ of (2.35) are independent of the direction $\hat{k}$ and different from each other. As a consequence, the Lamé system with the operator (C.1) permits the approximation of solutions with two rings for large times.

To obtain the homogenized system (at least formally), one makes the classical two-scale-ansatz

$$u^\varepsilon(x,t) = u^0(x,y,t) + \varepsilon u^1(x,y,t) + \varepsilon^2 u^2(x,y,t) + \dots\,, \tag{C.2}$$

where, for every $i \in \mathbb{N}_0$, the functions $u^i : \mathbb{R}^2 \times \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}^2$ are smooth and Y-periodic in the second variable y. Using the ansatz (C.2) in the Lamé system provides, at order ε^{-2},

$$2\mu\Delta_y u^0(x,y,t) + \nabla_y(\lambda(y)\operatorname{div}_y u^0(x,y,t)) = 0\,, \tag{C.3}$$

where the subscript y indicates that derivatives are taken with respect to the y-variable. This yields that u^0 does not depend on the fast variable y, i.e. $u^0(x,y,t) = \tilde{u}^0(x,t)$.

At order ε^{-1} one finds, exploiting that $\nabla_y \cdot (\nabla_x u^0) = 0$,

$$2\mu\Delta_y u^1(x,y,t) + \nabla_y(\lambda(y)(\operatorname{div}_x \tilde{u}^0(x,t) + \operatorname{div}_y u^1(x,y,t))) = 0\,. \tag{C.4}$$

The solution u^1 to the above cell problem depends linearly on $\operatorname{div}_x \tilde{u}^0(x,t)$ and can thus be written as

$$u^1(x,y,t) = \operatorname{div}_x \tilde{u}^0(x,t)\Psi(y) + \tilde{u}_1(x,t)\,, \tag{C.5}$$

where $y \mapsto \Psi(y) \in \mathbb{R}^2$ is periodic and solves the cell problem

$$2\mu\Delta_y\Psi(y) + \nabla_y(\lambda(y)(1 + \operatorname{div}_y \Psi(y))) = 0\,. \tag{C.6}$$

In particular, for a fixed time instance t,

$$\begin{aligned}\nabla u^\varepsilon(x,t) &= \nabla_x\tilde{u}^0(x,t) + \operatorname{div}_x \tilde{u}^0(x,t)\nabla_y\Psi(x/\varepsilon) + O(\varepsilon)\,,\\ \operatorname{div} u^\varepsilon(x,t) &= \operatorname{div}_x \tilde{u}^0(x,t) + \operatorname{div}_x \tilde{u}^0(x,t)\operatorname{div}_y \Psi(x/\varepsilon) + O(\varepsilon)\,,\end{aligned}$$

and the following weak convergence results hold in $L^2(\mathbb{R}^2 \times (0,T))$ for arbitrary $T > 0$:

$$\nabla u^\varepsilon(x,t) \rightharpoonup \nabla_x\tilde{u}^0(x,t) + (\operatorname{div}_x \tilde{u}^0(x,t))\int_Y \nabla_y\Psi(y)\,dy = \nabla_x\tilde{u}^0(x,t)\,,$$

$$\lambda(x/\varepsilon)\operatorname{div} u^\varepsilon(x,t) \rightharpoonup (\operatorname{div}_x \tilde{u}^0(x,t))\int_Y \lambda(y)(1 + \operatorname{div}_y \Psi(y))\,dy =: \lambda^*\operatorname{div}_x \tilde{u}^0(x,t)\,.$$

This provides the homogenized system for $\tilde{u}^0$,

$$\partial_t^2 \tilde{u}^0(x,t) \;=\; 2\mu\Delta\tilde{u}^0(x,t) + \lambda^*\nabla\operatorname{div}\tilde{u}^0(x,t)) \;=:\; -L^*\tilde{u}^0(x,t)\,. \tag{C.7}$$

This is a Lamé system with the constant coefficients μ and λ^*. As noted above, this system has two different speed values for pressure and shear waves, and both wave speeds are independent of the direction.

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