Aufgabe: $\frac{2}{x-1}+1 = x+1 \overset{x\neq 1}{\iff} 2+(x-1) = (x+1)(x-1)$

$\iff 2+x-\cancel{1} = x^2-\cancel{1} \iff x^2-x-2=0$

$\iff x^2 - x + (\frac{1}{2})^2 \underbrace{-(\frac{1}{2})^2-2}_{-\frac{9}{4}} = 0$

$\iff (x-\frac{1}{2})^2 = \frac{9}{4}$

$\iff x = \frac{1}{2} \pm \sqrt{\frac{9}{4}} \iff x=\frac{1}{2}+\frac{3}{2}=2 \lor x = \frac{1}{2}-\frac{3}{2}=-1$

zu 9.7: $(F(g(x)))' = F'(g(x))\cdot g'(x) = f(g(x))\cdot g'(x)$

Bsp: $\int \frac{1}{\sqrt{x+1}}\,dx = \int \frac{1}{\sqrt{x+1}}\cdot 1\,dx = 2\cdot\sqrt{x+1}+C$

$\frac{1}{\sqrt{x+1}} = f(g(x))$ mit $f(y)=\frac{1}{\sqrt{y}}$, $g(x)=1+x$ und $g'(x)=1$.

Stammfkt von f ist $F(y)=2\sqrt{y}$

Bsp: $\int\frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int\frac{1}{\sqrt{x^2+1}}\cdot 2x\,dx = \frac{1}{\cancel{2}}\cdot\cancel{2}\sqrt{x^2+1}+C = \sqrt{x^2+1}+C$

$\frac{1}{\sqrt{x^2+1}} = f(g(x))$ mit $f(y)=\frac{1}{\sqrt{y}}$, $g(x)=x^2+1$ und $g'(x)=2x$

Alternativ: Substituiere $y=x^2+1$. Dann:

$\frac{dy}{dx} = (x^2+1) = 2x \Rightarrow x\,dx = \frac{dy}{2}$

$\Rightarrow \int \frac{x}{\sqrt{x^2+1}}\,dx \overset{y=x^2+1}{=} \int\frac{1}{\sqrt{y}}\frac{1}{2}\,dy = \frac{1}{2}\int\frac{1}{\sqrt{y}}\,dy = \frac{1}{\cancel{2}}\cdot\cancel{2}\sqrt{y} + C = \sqrt{x^2+1}+C$

zu Folgerung 9.8

1. $f(x+a) = f(g(x))$ mit $g(x)=x+a$ und $g'(x)=1$

$\Rightarrow \int f(x+a)\,dx = \int f(g(x))\cdot g'(x) = F(\underbrace{x+a}_{g(x)})+C$

2. $f(ax) = f(g(x))$ mit $g(x)=ax$ und $g'(x)=a$

$\Rightarrow \int f(ax)\,dx = \frac{1}{a}\int \underbrace{f(ax)}_{f(g(x))}\cdot\underbrace{a}_{g'(x)}\,dx = \frac{1}{a} F(ax)+C$

3. $g(x)\cdot g'(x) = f(g(x))\cdot g'(x)$ mit $f(y)=y$ und $F(y)=\frac{1}{2} y^2$

$\Rightarrow \int g(x)g'(x)\,dx = \int f(g(x))\cdot g'(x)\,dx = F(g(x))+C = \frac{1}{2}(g(x))^2+C$

Bsp: $\int \underbrace{\sin x}_{g(x)}\cdot\underbrace{\cos x}_{g'(x)}\,dx = \frac{1}{2}(\sin x)^2+C$ Alternativ: $\int \sin x\cdot\cos x\,dx$

$= -\int \underbrace{(-\sin x)}_{g'(x)}\cdot\underbrace{\cos x}_{g(x)}\,dx = -\frac{1}{2}\cos^2 x+\tilde{C}$

Bem: $\sin^2 x+\cos^2 x=1 \Rightarrow \frac{1}{2}\sin^2 x+\frac{1}{2}\cos^2 x=\frac{1}{2}$

$\Rightarrow \frac{1}{2}\sin^2 x = \frac{1}{2}-\frac{1}{2}\cos^2 x$

$\Rightarrow \frac{1}{2}\sin^2 x + C = -\frac{1}{2}\cos^2 x + \overbrace{\frac{1}{2}+C}^{=:\tilde{C}}$

Alternativ mit Produktregel:

$$\int \sin x \cdot \cos x \, dx \overset{\text{Partielle Int.}}{=} \sin x \cdot \sin x - \int \underbrace{(\sin x)'}_{\cos x} \sin x \, dx = \sin^2 x - \int \cos x \cdot \sin x \, dx$$

$$\Rightarrow 2 \cdot \int \sin x \cdot \cos x \, dx = \sin^2 x + \tilde{C}$$

$$\Rightarrow \int \sin x \cdot \cos x \, dx = \frac{1}{2}\sin^2 x + \underbrace{\frac{\tilde{C}}{2}}_{=:C} = \frac{1}{2}\sin^2 x + C$$

Vorgehen analog zu:
Finde y mit
$y = C - y$
$\Leftrightarrow 2y = C \Leftrightarrow y = \frac{1}{2}C$
Hier: $y \hat{=} \int \sin x \cdot \cos x \, dx$
$C \hat{=} \sin^2 x$

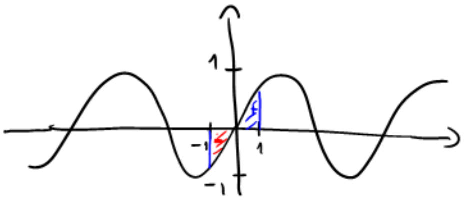
Bsp zu 9.9

$$\int_0^3 x^2 dx = \frac{1}{3}x^3 \Big|_0^3 = \frac{1}{\cancel{3}} 3^{\cancel{3}\,2} - \frac{1}{3}0^3 = 9$$

$$\int_3^0 x^2 dx = \frac{1}{3}x^3 \Big|_3^0 = \frac{1}{3} \cdot 0^3 - \frac{1}{3}3^3 = -9$$

Insbesondere $\int_a^b f(x)dx = -\int_b^a f(x)dx$

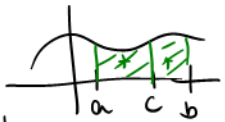
Frage: $\int_{-1}^{1} \sin x \, dx = 0$, da sich wegen der Punktsymmetrie des sin die positive und negative Fläche gegenseitig aufheben.



Rechnerisch: $\int_{-1}^{1} \sin x \, dx = -\cos x \Big|_{-1}^{1} = -\cos(1) + \cos(-1) \overset{\cos(-x) = \cos x}{=} -\cos(1) + \cos(1) = 0$

zu Satz 9.10

1. $\int_a^a f(x)dx = F(a) - F(a) = 0$

2. 

4. $\int_a^b f(g(x)) \cdot g'(x) dx \overset{F(g(x)) \text{ ist Stammfkt}}{=} F(g(x)) \Big|_a^b = F(g(b)) - F(g(a))$

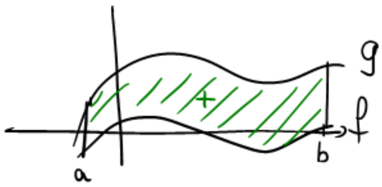
$$= F(t) \Big|_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(t) dt$$

Bsp: $\int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot 2x \, dx = \int_{g(0)}^{g(1)} \frac{1}{\sqrt{t}} dt = \int_1^2 \frac{1}{\sqrt{t}} dt = 2\sqrt{t} \Big|_1^2 = 2 \cdot \sqrt{2} - 2$

Hier: $f(t) = \frac{1}{\sqrt{t}}$, $g(x) = 1 + x^2$

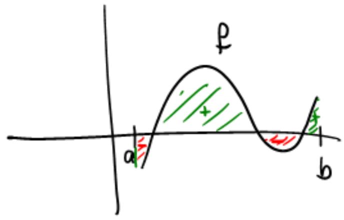
Alternativ: Erstmal Stammfkt zu $\frac{1}{\sqrt{1+x^2}} \cdot 2x$ ausrechnen, (Stammfkt ist $2\sqrt{1+x^2}$) und danach die Grenzen 0, 1 einsetzen, d.h.

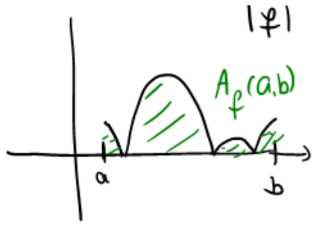
$$\int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot 2x \, dx = 2\sqrt{1+x^2} \Big|_0^1 = 2 \cdot \sqrt{1+1} - 2 \cdot \sqrt{1+0} = 2\sqrt{2} - 2$$

5. 

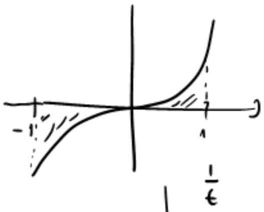
$$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b (g(x) - f(x)) dx \geq 0$$

zu 9.11

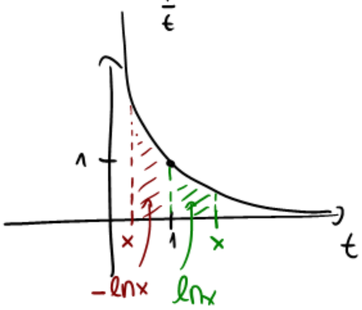




zu 9.12



zu 10.1:



Für $x<1$: $\displaystyle\int_1^x \frac{1}{t}\,dt = -\underbrace{\int_x^1 \frac{1}{t}\,dt}_{\text{Fläche unter } t\mapsto\frac{1}{t} \text{ in den Grenzen } x \text{ bis } 1}$