All the Way with Pythagoras

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$$a^2 + b^2 = c^2$$

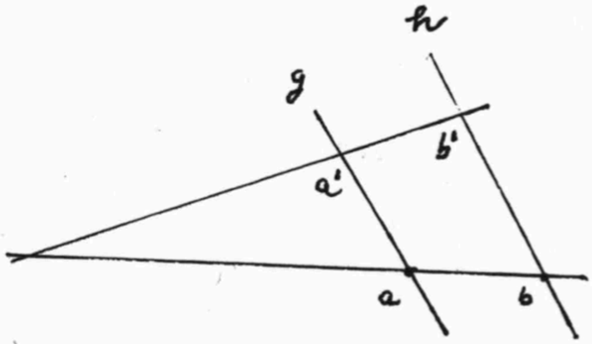
Πυθαγορας 570 - 510


$A^2+B^2=C^2$

There is a problem: In elementary geometry the numbers you have at disposal are length of line segments.

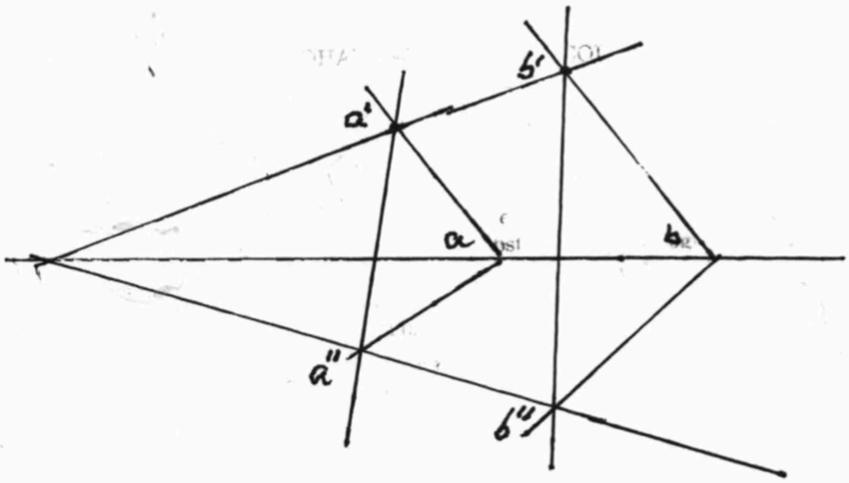
How can you get a measure for areas. Is that well defined?

Try another method. Use ratios:



a:b=a' : b' iff the lines g and h are parallel

again that has to be well defined:



Desargues

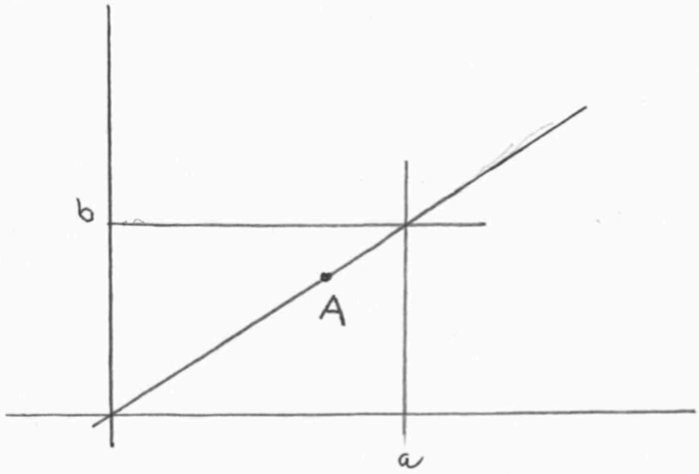
Both problems can be overcome by elementary methods, that is, only by use of axioms for congruence, ordering and parallelism, but without use of calculus.

But that is tricky

Hilbert: Grundlagen der Geometry

If you accept these axioms, you may find orthonormal coordinates with respect to the standard scalar product over an ordered field P.

Now look at the line g connecting 0 and (a,b) :



On g we should find a point A such that the line segment between 0 and A has length 1.
So we have to solve the equation

$s^2 (a^2+b^2) = 1$. That means P has the

Property: For all $a,b \in P$ there exists $c \in P$ such that $c^2 = a^2 + b^2$.

Definition: A field P is called *pythagorean,* if it is formally real and if the sum of two squares in P is again a square.

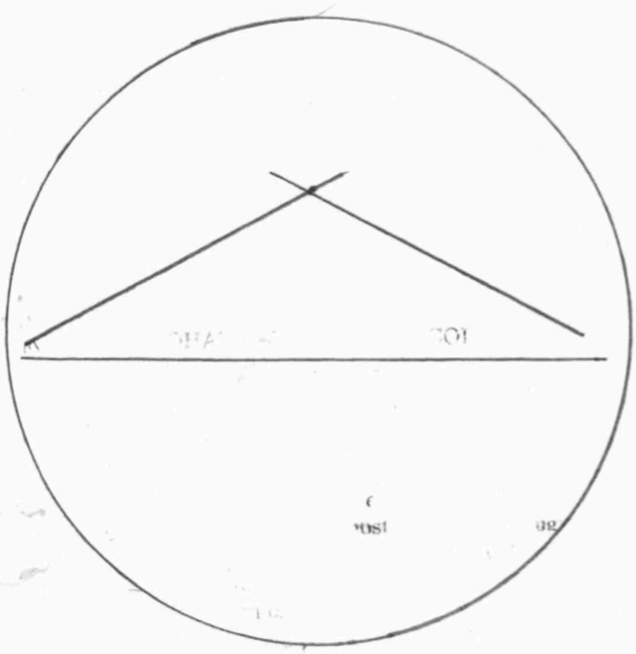
Note, that a in a field of characteristic 2 one has always $a^2+ b^2= (a+b)^2$ and a non formally real field with this property is quadratically closed because of the identity: $a= ½ (a+1)^2-(a-1)^2$.

Definition: A field E is called *euclidean*, if it has a unique ordering such that the positive elements are squares.

Paradox: For basic euclidean geometry the field of scalers is pythagorean (with a fixed ordering).

For basic non euclidean geometry the field of scalars is euclidean.

Note that Pythagoras law does not hold in non euclidean geometry.



Fields which are close to the rational numbers or function fields are never pythagorean. So let us look at the other end. Look at the Galois theory.

What can we say about the Galois theory of pythagorean fields?

After what I have told so far it is not surprising, that an early paper is due to two mathematicians, who worked on foundations of geometry at that time.

Zur Galoistheorie pythagoreischer Körper

Professor KARL STRUBECKER zum 60. Geburtstag gewidmet

Von

J. DILLER und A. DRESS

ARCH. Math. XVI, 1965

Here is the first result:

Proposition. *Let K be a field of Char. $\neq 2$*
a) In K is every sum of squares a square, if K admits no cyclic extension of degree 4.
In particular, K is pythagorean, iff K admits no cyclic extension of degree 4, but extensions of degree 2.

b) If K is formally real and not pythagorean, the pythagorean closure of K admits a cyclic extension of degree 4.

Corollary. If E is pythagorean and finite over K, then K is pythagorean too.

I will call a property p of a class of fields genetic, if:

Let L/K be a finite extension. Then if L has property p, then K has property p too.

So genetic properties are:

To be pythagorean
To be euclidean
To be henselean

Also, no proper finite extension C/K can be algebraically closed (quadratically closed) unless K is real closed (euclidean).

Also they show

Proposition: *Let P be formally real,* $q \notin -P^2$*. Then* $P(\sqrt{q})$ *is pythagorean, iff*

$P^2 + q P^2 = P^2 \cup q P^2$.

In that case P is called q- pythagorean.

Proposition: *For a formally real field P the following properties are equivalent:*

a) $P(\sqrt{q})$ *is pythagorean for all* $q \notin -P^2$.

b) $P^2 + qP^2 = P^2 \cup qP^2$ *for all* $q \notin -P^2$.

c) $P(i)$ *is the only non-real quadratic extension of P.*

$i := \sqrt{-1}$.

If these properties hold for P, then they hold for every real quadratic extension of P too.
All this is shown by direct calculation (Diller-Dress, 1965 loc. cit), may be a little bit hidden.
Pythagorean fields with these properties came up again about 1972 under different names:

Strictly pythagorean fields (Br.)

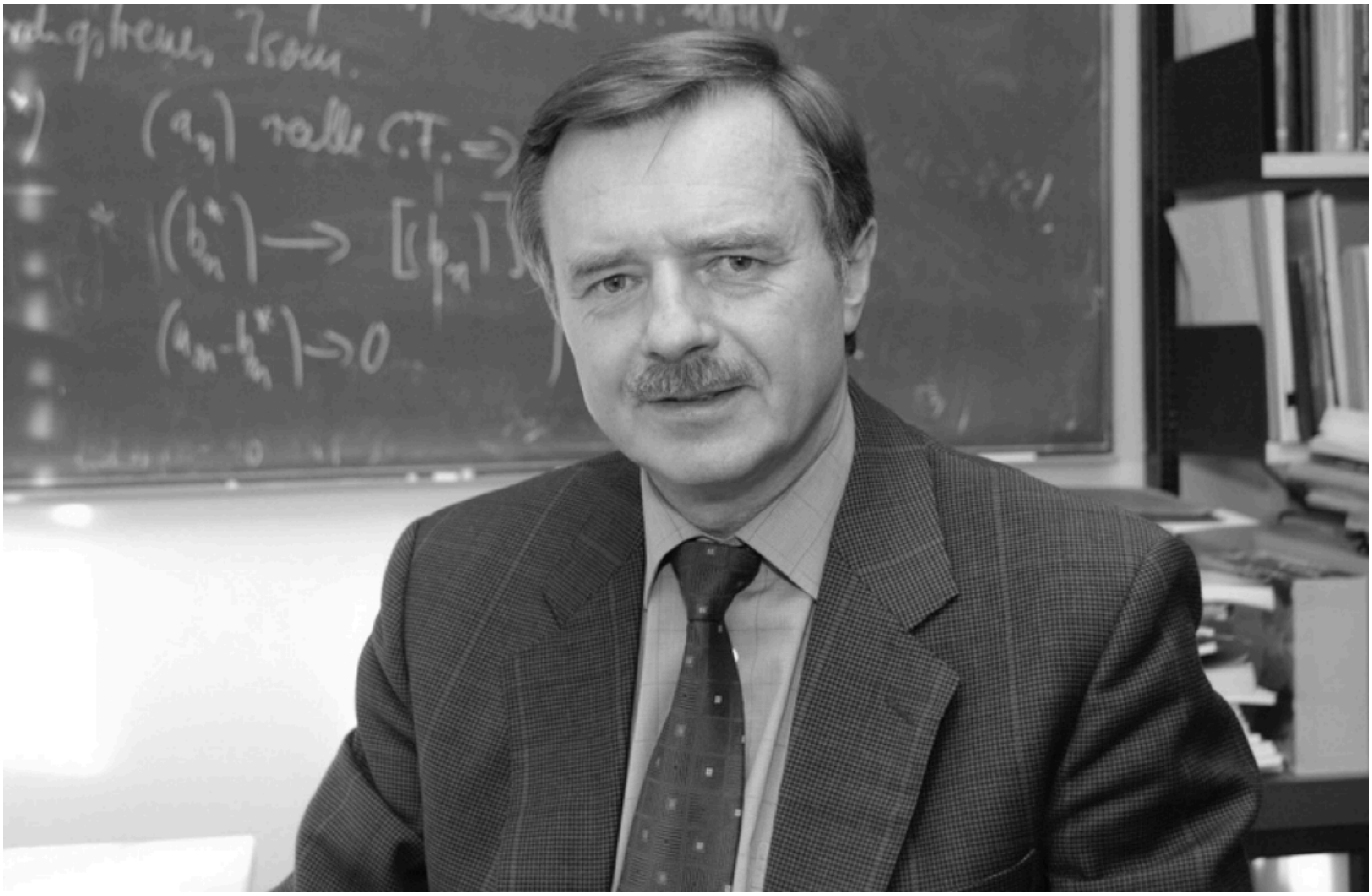
Superpythagorean fields (Elman,Lam)

2- hereditarily Pythagorean fields (Becker)

Eberhards goal was, to understand the Galois group of these fields.

More generally he showed the following result which I like very much.

For this let P be a formally real field, C its algebraic closure and $i = \sqrt{-1}$.



Theorem (Becker 1976): *For a formally real field P the following conditions are equivalent:*

a) P is hereditarily pythagorean, that is,

every real algebraic extension of P is pythagorean.

b) Every non real extension of P contains i.

c) Gal (C/ P (i)) is abelean.

d) Every real extension of P is the intersection of its real closures.

Under these conditions $b^2 = 1$ for all $b \in$ Gal (C/P), $b \notin$ Gal (C/ P(i)).

Here b) --> a) and d) --> a) are obvious, using the property (p): If a finite extension of a field K is pythagorean, than K is pythagorean too

b) --> c) is interesting: Let $j: i \to -i$ and $u \in Gal(C/P(i))$. Since $j(iu) = -iu$ we have $i \notin F := Fix(ju)$. So F is real and Gal(C/F) is abelean. => Gal(C/F) is of order 2. => $(ju)^2 = id$. This for all $u \in Gal(C/P(i))$. => Gal(C/P(i)) is abelean.

Here is a further equivalent condition I found shortly later:

e) The Haar- measure of Inv(Gal(C/P))) is 0, where Inv (G) is the set of involutions in a group G.

The proof does not use Beckers theorem, but Diller-Dress, in particular property p.

Question*: Let G be a profinite group having no other finite subgroups but Z_2 such that the Haar-measure of Inv (G) 0. Does that imply, that Haar-measure of Inv (G) = ½ ?*

Such a group is known to be abelean py finite. (Levai-Pyber, ARCH.MATH 75,2000).

Here is a

<u>Counterexample</u>. Let $B := \varprojlim_n \mathbb{Z}/2^n\mathbb{Z}$

the additive group of 2-adic numbers, which are integers. So B is not 2-divisible, which is important. Also, let $A := B \times B$

Let V be the 4-element group

$V = \{1, \sigma, \tau, \rho\}$ where $\sigma^2 = \tau^2 = \rho^2 = 1$, $\sigma\tau = \rho = \tau\sigma$.

A is a V-module under the action

$(x,y)^\sigma = (-x,-y)$, $(x,y)^\tau = (y,x)$ hence $(x,y)^\rho = (-y,-x)$.

Consider the map $V \times V \to A$;

$f(\alpha,\beta) = (0,0)$ if $\alpha = 1$ or $\beta = 1$

Also $f(\sigma,\sigma) = f(\sigma,\tau) = f(\sigma,\rho) = (0,0)$

$f(\tau,\tau) = f(\rho,\tau) = a := (1,1)$

$f(\rho,\rho) = f(\tau,\rho) = b := (1,-1)$

$f(\tau,\sigma) = f(\rho,\sigma) = c := (-2,0)$.

One verifies that f is a non-trivial 2-cocycle for the V-module A which defines a non split extension of A by V.

$1 \to A \to G \xrightarrow[\pi]{} V \to 1$.

Among the subgroups of V only $\{1,\sigma\}$ split,

and $\pi^{-1}(\sigma)$ consist of involutions.

Thus the Haar-measure of $\mathrm{Inv}(G) = 1/4$.

Beckers Theorem generalizes the characterization of hereditariy euclidean fields

Definition: A field E is called hereditarily euclidean, if every finite real extension of E is euclidean.

Theorem (Prestel-Ziegler 1975). *For a field E the following conditions are equivalent:*

a) E is hereditarily Euclidean.

b) Every real algebraic extension of E is of odd degree.

c) Every non-real algebraic extension of E is quadratically closed.

d) E is the intersection of its real closures where any two of them are conjugate over their intersection..

This follows either from Beckers Theorem or by use of similar methods.

There is a completely different approach to hereditarily pythagorean fields. The first step was the following

Proposition (Br. 1972) . *Let K be a superpythagorean field. If K admits an archimedean ordering, then K admits at most one further ordering.*

For the short proof I surprisingly used Pfisters local-global principle for ternary forms, which contradicts the condition $K^2+qK^2= K^2 \cup qK^2$ if you have more than 1 further ordering.

Inspired by Beckers Theorem I generalized this to

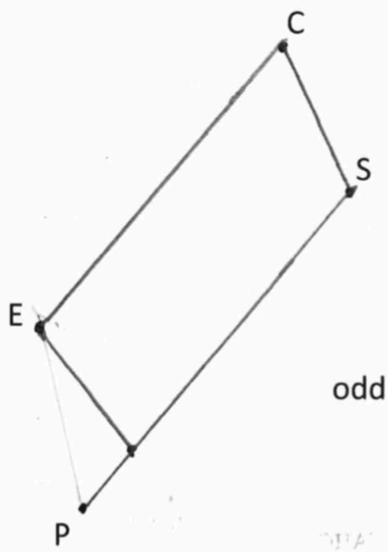
Theorem (Br. 1976). *Let P be a heretiarily pythagorean field. Then P admits a henselean valuation such that the residue field admits at most 2 orderings, hence 4 square classes.*

I was asking me, if there was a short way from this to Beckers theorem, but I didn't find it except for this:

Theorem (Becker 1976). *Let P be hereditarily pythagorean bearing only one ordering. Then P is hereditarily euclidean.*

Proof: It is suffices to consider the residue fields. Let E/P a real extension and let C be the Galois closure of E/P.

Also let S be the fixed field under the 2-Sylow subgroup of C/P.



C=S or S is euclidean

odd but at most 2 extensions of the ordering

=> E is euclidean.

Pythagoras number

It is generally told that Pythagoras claimed:

Everything is number.

An expert told me that there exists no original citation. However his followers

(hundred years later) referred to him claiming, for instance:

Παντα τα γιγνωσκομενα αριθμον εχοντι

wich means: *Every insight includes a number.*

Sometimes this number is hard to find.

Definition: The Pythagoras number p(K) of a field K is the smallest p such that every sum of squares in K is a sum of p squares (it can be ∞).

Definition: The level ℓ(K) of a non real field K is the smallest number ℓ such that -1 can be written as a sum of ℓ squares in K.

Well known: ℓ(K) is always a power of 2 , and p(K)=ℓ(K) or ℓ(K)+1, if K is not real.

Examples: p(K) = 1 iff K is Pythagorean or quadratically closed.
p($\mathbb{Q}$)= 4 , (Euler Lagrange). Also p(L) = 4 for finite extensions L/$\mathbb{Q}$ (Hasse Minkowski)

Proposition (Becker 1976): *Let P be a hereditarily pythagorean field. Then p(P(t)) =2. P(t) = field of rational functions.*

Proof: Let ϱ Be a quadratic form over P(t) . Then $\varrho \sim 0$, if $\varrho \sim 0$ over every K_p , where K_p is the discrete complete valued field induced by p.
(Milnor sequence). Now let a be a sum of squares in P. Then
(1,1, -a, -a) $\cong$ 0 $\cong$ (1,-1, a, -a), hence by cancelation (1,1, -a) = (-1, 1, a).

Couldn't we repeat this procedure ?

Problem: We do not know p(K) and s(K) for algebraic extensions of $\mathbb{Q}$ (t).

Let

$K/\mathbb{Q}$ be finitely generated.

What is $p(K)$, depending on $tdg(K/\mathbb{Q})$?

That is a challenge for professionals. So far one has just bounds

Our topic are pythagorean fields. Here is a recent result, the proof of which is not minor subtle:

Theorem (Becher et al.): *Let P be hereditarily pythagorean and K a finite extension of P (t). Then* $p(K) \leq 5$.

Corollary : *Let P be hereditarily pythagorenan. Then* $p(P((x),(y))) \leq 8$.

The End