Positivstellensätze for Semirings in Real Algebraic Geometry

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Dedicated to the 80-th Birthday of EBERHARD BECKER

Contents

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Aims of the talk:

- Positivstellensätze for semirings without denominators
- New applications and examples

Bernstein's Theorem

Proposition 1: Bernstein (1915)

Let $p \in \mathbb{R}[x]$. If $p(x) > 0$ for all $x \in [-1, 1]$, then there are numbers $m \in \mathbb{N}$ and $a_{kj} \geq 0$ for $k, j = 1, \dots, m$, such that

$$p(x) = \sum_{k,j=0}^{m} a_{kj}(1 - x)^k (1 + x)^j. \tag{1}$$

More precisely: Suppose $p \neq 0$.
p **has a representation (1) with** $a_{kj} \geq 0$ **iff** $p(x) > 0$ **for** $x \in (-1, 1)$**.**

The smallest number m is called the **Lorentz degree** of p.

The set of polynomials on the right-hand side of (1) is a convex cone which is invariant under multiplication: $\Longrightarrow$ **semiring**.

Simplest version of the **Archimedean Positivstellensatz for semirings**.

A quadratic polynomial

An example: quadratic polynomial

Let $0 < \varepsilon < 1$ such that ε^{-2} is an integer. Set

$$p(x) = x^2 + \frac{(6\varepsilon^2 - 2)x + 8\varepsilon^4 - 5\varepsilon^2 + 1}{1 - \varepsilon^2}$$
$$= \left(x + \frac{3\varepsilon^2 - 1}{1 - \varepsilon^2}\right)^2 + C_\varepsilon^2.$$

As shown by Erdelyi and Szabados (1988), p has **Lorentz degree** ε^{-2}.
$p(1) = 8\varepsilon^4$ and the zeros of p are on the ellipse $y^2 + \varepsilon^2 x^2 = \varepsilon^2$.

Proof of Bernstein's theorem

Abbreviate $\mathrm{Pos}(K) := \{p \in \mathbb{R}[x] : p(x) \geq 0 \text{ on } K\}$

1. Let $p \in \mathbb{R}[x]$, $p \neq 0$, and $n = \deg(p)$. The **Goursat transform** of p is

$$\tilde{p}(x) = (1+x)^n p\left(\frac{1-x}{1+x}\right).$$

Then $p \in \mathrm{Pos}([-1,1])$ if and only if $\tilde{p} \in \mathrm{Pos}([0,+\infty))$.

2. **Polya's theorem**:
Suppose $p \in \mathbb{R}[x]$ and $p(x) > 0$ for $x \in [0,+\infty)$. Then there exists an $N \in \mathbb{N}$ such that $(1+x)^N p(x)$ has only positive coefficients.

3. Powers and Reznick (2000): estimate for the **Lorentz degree** m of $p(x) = \sum_{j=0}^{d} a_j x^j$: If $\lambda = \min\{p(x) : x \in [-1,1]\}$, $M = \max\{|a_j|\}$, then

$$m \leq 3d + 1 + 2d^2 M \lambda^{-1}.$$

Proposition 2: Markoff-Lukacs (1918)

$$\mathrm{Pos}([-1,1]))_{2n} = \{\, f_n^2 + (1-x^2)g_{n-1}^2 : f_n \in \mathbb{R}[x]_n, g_{n-1} \in \mathbb{R}[x]_{n-1} \,\}$$

$$\mathrm{Pos}([-1,1])) = \{\, f^2 + (1-x^2)g^2 : f, g \in \mathbb{R}[x] \,\}$$
$$= \{\, p + (1-x^2)q : p.q \in \sum \mathbb{R}[x] \,\}.$$

Simple version: **Archimedian Positivstellensatz for quadratic modules**

Proposition 3: Fejer-Riesz (1915)

Let $p(z) \in \mathbb{C}[z, z^{-1}]$. If $p(z) \geq 0$ for all $z \in \mathbb{T} := \{z \in \mathbb{C}, |z| = 1\,\}$, there exists $q(z) \in \mathbb{C}[z]$, such that

$$p(z) = |q(z)|^2 \quad \text{for} \quad z \in \mathbb{T}.$$

General Notions

Throughout: A - **commutative unital real algebra**.
C - **cone in** A (that is, $C + C \subseteq C$, $\lambda \cdot C \subseteq C$ for $\lambda \geq 0$) **s. t.** $1 \in C$.

Definition: C is called a

- **quadratic module** of A if $a^2 b \in C$ for all $b \in C$ and $a \in A$.
- **semiring** (or a **preprime**) of A if $C \cdot C \subseteq C$.
- **preordering** of A if C is a quadratic module and a semiring.

Smallest preordering of A: $\sum A^2 = \{ \sum_{i=1}^k a_i^2 : a_i \in A, k \in \mathbb{N} \}$.
The **semiring generated by** $f_1, \dots, f_r \in A$ is

$$\mathcal{S} = \Big\{ \sum\nolimits_{n_1,\dots,n_r=0}^{k} \alpha_{n_1,\dots,n_r} f_1^{n_1} \cdots f_r^{n_r} : \alpha_{n_1,\dots,n_r} \in [0,+\infty), k \in \mathbb{N}_0 \Big\}.$$

The **quadratic module generated by** $f_1, \dots, f_r \in A$ is

$$\mathcal{Q} = \Big\{ \sum\nolimits_{j=1}^{r} \sigma_j f_j : \sigma_j \in \sum A^2 \Big\}.$$

Definition:

The bounded part of A with respect to C is defined by

$$A^{\mathrm{bd}}(C) := \left\{ a \in A : \text{there exists } \lambda \in (0,\infty) \text{ such that } (\lambda 1 \pm a) \in C \right\}$$

C is called **Archimedean** if $A^{\mathrm{bd}}(C) = A$.

Lemma:

If C is a semiring or a quadratic module, $A^{\mathrm{bd}}(C)$ is a unital subalgebra.

It suffices to check the Archimedean condition for *algebra generators*.

Example:

$$\mathcal{S} := \left\{ \sum_{k,j=0}^{m} a_{kj}(1-x)^k(1+x)^j : a_{kj} \geq 0, m \in \mathbb{N}_0 \right\}$$

is an Archimedean semiring, since $1-x, 1+x \in \mathcal{S}$.

Dagger cones

Definition:

If C is Archimedean, define $C^\dagger := \{a \in A : a + \epsilon 1 \in C \text{ for } \epsilon \in (0, \infty)\}$.

Since $1 \in C$, we have $C \subseteq C^\dagger$.

Example:

Let A be a real algebra of bounded real-valued functions on a set X which contains the constant functions. Then

$$C := \{f \in A : f(x) > 0 \textit{ for all } x \in X\},$$
$$C^\dagger = \{f \in A : f(x) \geq 0 \text{ for all } x \in X\}.$$

Dagger cones were first defined by Kuhlmann and Marshall (2002). $C^\dagger$ is the sequential closure of C in the finest locally convex topology (Cimpric, Marshall, Netzer (2011)).

Definition

Let $\mathcal{S}$ be a semiring of A. A cone C in A s.t. $1 \in C$ is called an $\mathcal{S}$-**module** if $ac \in C$ for all $a \in \mathcal{S}$ and $c \in C$.

Setting $c = 1$ we get $\mathcal{S} \subseteq C$. Example: $C = g_1\mathcal{S} + \cdots g_k\mathcal{S}$, where $g_j \in A$.

Theorem 1:

Let $\mathcal{S}$ be an Archimedean semiring and C an $\mathcal{S}$-module. Then $C^\dagger$ and $S^\dagger$ are Archimedean preorderings.

Theorem 2:

If $\mathcal{Q}$ is an Archimedean quadratic module, then $\mathcal{Q}^\dagger$ is an Archimedean preordering.

We sketch a **crucial step in the proof of Theorem 1**:

We prove that $a^2 \in \mathcal{S}^\dagger$ for any $a \in A$. We have

$$\frac{1}{2^k k(k-1)} \sum_{j=0}^{k} \binom{k}{j} (k-2j)^2 \, (1+x)^{k-j}(1-x)^j = x^2 + \frac{1}{k-1}.$$

Since $\mathcal{S}$ is Archimedean, there is $\lambda > 0$ such that $(\lambda 1 \pm a) \in \mathcal{S}$.
Since the semiring $\mathcal{S}$ is closed under multiplication,

$$(a/\lambda)^2 + \frac{1}{k-1} = \frac{1}{2^k k(k-1)} \sum_{j=0}^{k} \binom{k}{j} (k-2j)^2 \, \big(1+(a/\lambda)\big)^{k-j} \big(1-(a/\lambda)\big)^j \in \mathcal{S}.$$

Hence $((a/\lambda)^2 + \varepsilon) \in \mathcal{S}$ for any $\varepsilon > 0$, so $a^2 \in \mathcal{S}^\dagger$.

$\hat{A}$: characters of A, unital algebra homomorphisms $\varphi : A \to \mathbb{R}$.
For $C \subseteq A$ define

$$\mathcal{K}(A; C) := \{\varphi \in \hat{A} : \varphi(a) \geq 0 \ \text{ for all } \ a \in C\}.$$

Standard example: $A = \mathbb{R}[x_1, \dots, x_d]$

$\hat{A}$ is the set of point evaluations $\varphi_t, t \in \mathbb{R}^d$: $\varphi_t(p) = p(t), p \in A$.
If C is the semiring or the quadratic module generated by $f_1, \dots, f_r \in A$, then $\mathcal{K}(A; C)$ is the **semi-algebraic set**

$$\mathcal{K}(f_1, \dots, f_r) := \{t \in \mathbb{R}^d : f_1(t) \geq 0, \dots, f_r(t) \geq 0\}.$$

If C is Archimedean, $\mathcal{K}(f_1, \dots, f_r)$ is compact (since $(\lambda 1 - x_j) \in C$).

$\mathcal{K}(f_1, \dots, f_r)$ compact does not imply that C is Archimedean.

(Prestels's counter-example: $d = 2$, quadratic module generated by $f_1 = x - \frac{1}{2}, f_2 = y - \frac{1}{2}, f_3 = 1 - xy$; $(\lambda - x^2 - y^2) \notin C$).

Archimedean Positivstellensätze

Theorem 3: Archimedean Positivstellensatz

Suppose

- C is an S-module of an **Archimedean semiring** S or
- C is an **Archimedean quadratic module** of A.

For any $a \in A$, the following are equivalent:

$(i)_C$ $\varphi(a) > 0$ for all $\varphi \in \mathcal{K}(A; C)$.

$(ii)_C$ There exists $\epsilon \in (0, \infty)$ such that $a \in \epsilon \cdot 1 + C$.

Each of the conditions $(i)_C$ and $(ii)_C$ holds for C iff it does for $C^\dagger$.

Therefore, since $C^\dagger$ is always an Archimedean preordering by Theorems 1 and 2, **it suffices to prove Theorem 3 for Archimedean preorderings.**

$(ii)_C \to (i)_C$ is trivial: If $a = \epsilon + c$ with $c \in C$, then $\varphi(a) \geq \epsilon > 0$:

Archimedean Positivstellensatz for semirings: Krivine (1964)

Archimedean Positivstellensatz for quadratic modules: Jacobi (2001)

Proof of Theorem 3 for semirings:

Assume that $c \notin C$. Separation of convex sets: There exist a linear functional $L \neq 0$ on A such that $L \geq 0$ on C such that $L(c) \leq 0$.

First Proof: Case of Semirings:

Separation: L is **extremal** in $C^\wedge$. Suffices to show that L is a **character**:

$$L(ab) = L(a)L(b) \quad \text{for} \quad a, b \in A. \tag{2}$$

Since C is Archimedean, $A = C - C$, w.l.o.g. $a \in C$. Then $L(a) \geq 0$.
Case 1: $L(a) = 0$.
Choose $\lambda > 0$ such that $\lambda - b \in C$. Then $(\lambda - b)a \in C$ and $ab \in C$, so $L((\lambda - b)a) = -L(ab) \geq 0$ and $L(ab) \geq 0$. Hence $L(ab) = 0$, so (2).
Case 2: $L(a) > 0$.
Choose $\lambda > 0$ such that $(\lambda - a) \in C$ and $L(\lambda - a) > 0$. Then $L_1(\cdot) := L(a)^{-1}L(a\cdot)$ and $L_2(\cdot) := L(\lambda - a)^{-1}L((\lambda - a)\cdot)$ are in $C^\wedge$ and

$$L = \lambda^{-1}L(a)\, L_1 + \lambda^{-1}L(\lambda - a)\, L_2.$$

Since L is extremal in $C^\wedge$, $L_1 = L$. Applied to b, this gives (2).

Proof of Theorem 3 for quadratic modules

Second Proof: Case of Quadratic Modules:

Since C is a quadratic module, $a^2 \in C$ and hence $L(a^2) \geq 0$ for each $a \in A$. Thus L is a **positive** linear functional on A.

GNS-construction: $\langle a, b \rangle := L(ab), a, b \in A$, is a semi-scalar product, since $\langle a, a \rangle = L(a^2) \geq 0$. Factoring out null space, complexification and completion: $\langle \cdot, \cdot \rangle$ becomes a scalar product of some Hilbert space.

$\pi_L(a)b = ab, a, b \in A$, defines a $*$-representation of A.

Fix $a \in A$. Choose $\lambda > 0$ such that $(\lambda - a^2) \in C$.
Then $(\lambda - a^2)b^2 \in C$, so $L((\lambda - a^2)b^2) \geq 0$,

$$\|\pi_L(a)b\|^2 = L(a^2 b^2) \leq \lambda L(b^2) = \lambda \|b\|^2.$$

Hence $\pi_L(a)$ is bounded. $\pi_L(A)$ is $*$-algebra of bounded operators.

Its completion is a **commutative** C^***-algebra**. Its character correspond to $\mathcal{K}(A; C)$, L is an integral of characters: Since $L(c) \leq 0$, $(i)_C$ cannot hold.

Example: Unit ball via quadratic module

Let $f(x) := 1 - x_1^2 - \cdots - x_d^2$. Then

$$\mathcal{K}(f) = \{x \in \mathbb{R}^d : x_1^2 + \cdots + x_d^2 \leq 1\}.$$

The quadratic module $\mathcal{Q}(f) = \{p + fq : p, q \in \sum \mathbb{R}_d[x]^2\}$ is Archimedean.

The Archimedean Positivstellensatz for quadratic modules says:

If $p \in \mathbb{R}_d[x]$ satisfies $p(x) > 0$ for $x \in \mathcal{K}(f)$, then $p \in Q(f)$.

Example: Unit ball via semiring

Let $\mathcal{S}$ denote the semiring of $A = \mathbb{R}[x_1, \ldots, x_d]$ generating by

$$f(x) := 1 - x_1^2 - \cdots - x_d^2, \; g_{j,\pm}(x) := (1 \pm x_j)^2, \; j = 1, \ldots, d. \tag{3}$$

Then $\mathcal{K}(A, \mathcal{S}) \cong \mathcal{K}(f) = \{x \in \mathbb{R}^d : x_1^2 + \cdots + x_d^2 \leq 1\}$. Since

$$d+1\pm 2x_k = (1-x_1^2-\cdots-x_d^2)+(1\pm x_k)^2+\frac{1}{2}\sum_{i=1, i\neq k}^{d}((1+x_j)^2+(1-x_j)^2) \in \mathcal{S},$$

$\mathcal{S}$ is Archimedean. By Theorem 3, if $p(x) > 0$ on $\mathcal{K}(f)$, p is a sum of

$$\alpha \; f^{2n}(1-x_1)^{2k_1}(1+x_1)^{2\ell_1}\cdots(1-x_d)^{2k_d}(1+x_d)^{2\ell_+d}$$
$$f\beta \; f^{2n}(1-x_1)^{2k_1}(1+x_1)^{2\ell_1}\cdots(1-x_d)^{2k_d}(1+x_d)^{2\ell_d},$$

where $\alpha \geq 0, \; \beta \geq 0, \; n, k_i, \ell_i \in \mathbb{N}_0$.

The following is a sharpening of the Archimedean Positivstellensatz for preorderings.

Theorem 4:

Suppose $A = \mathbb{R}_d[x] := \mathbb{R}[x_1, \dots, x_d]$ and $\mathcal{K}(f_1, \dots, f_r)$ is **compact**.

Then there exist $p_1, \dots, p_s \in \mathbb{R}_d[x], s \in \mathbb{N}$, such that the semiring $\mathcal{S}$ generated by $f_1, \dots, f_r, p_1^2, \dots, p_s^2$ is **Archimedean**.

If $q \in \mathbb{R}_d[x]$ satisfies $q(x) > 0$ for $x \in \mathcal{K}(f_1, \dots, f_n)$, then q is a finite sum of terms

$$\alpha\, f_1^{e_1} \cdots f_r^{e_r}\; f_1^{2n_1} \cdots f_r^{2n_r}\; p_1^{2k_1} \cdots p_s^{2k_s}, \tag{4}$$

where $\alpha \in (0, +\infty)$, $e_1, \dots, e_r \in \{0, 1\}$, $n_1, \dots, n_r, k_1, \dots, k_s \in \mathbb{N}_0$.

Sketch of proof of Theorem 4:

Let $g_1, \dots, g_m$, $m \in \mathbb{N}$, be a set of algebra generators of $\mathbb{R}_d[x]$.
Since $\mathcal{K}(f_1, \dots, f_r)$ is compact, there exist $\alpha_j > 0$ such that

$$\alpha_j + g_j(x) > 0 \;\text{ and }\; \alpha_j - g_j(x) > 0 \;\text{ for }\; x \in \mathcal{K}(f_1, \dots, f_r).$$

The Archimedean Positivstellensatz for preorderings implies:
$\alpha_j \pm g_j(x)$ are in the preordering generated by $f_1, \dots, f_r$.
Hence $\alpha_j \pm g_j$ is a finite sum of $f_1^{e_1} \cdots f_r^{e_r} p^2$ with $p \in A$, $e_i\{0, 1\}$.

Let $\mathcal{S}$ denote the semiring generated by $f_1, \dots, f_r$ and all p^2 occurring in the corresponding representations of $\alpha_j \pm g_j$. Then $g_1, \dots, g_m \in A^{\mathrm{bd}}(\mathcal{S})$.

Since $g_1, \dots, g_m$ generate the algebra A, $\mathcal{S}$ is Archimedean, so the Archimedean Positivstellensatz for semirings applies.

Supporting Polynomials of Convex Compact Sets

$$A_d := \{\alpha + \sum_{j=1}^{d} \beta_j x_j : \alpha, \beta_1, \dots, \beta_d \in \mathbb{R}\} \subseteq A := \mathbb{R}[x_1, \dots, x_d].$$

Let C is a convex set in $\mathbb{R}^d$. By a *supporting affine functional* at $t_0 \in C$ we mean a $h \in A_d$, $h \neq 0$, such that $h(t_0) = 0$ and $h(t) \geq 0$ for all $t \in C$. If such a functional exists, then t_0 is a boundary point of C.

Theorem 5:

Let C be a non-empty **compact convex** subset of $\mathbb{R}^d$. Suppose $H \subseteq A_d$ is a set of supporting affine functionals at points of C such that

$$C = \{t \in \mathbb{R}^d : h(t) \geq 0 \ \text{ for all } \ h \in H\}.$$

Then the semiring $\mathcal{S}(H)$ of $\mathbb{R}[x_1, \dots, x_d]$ generated by H is Archimedean. **If $f \in \mathbb{R}[x_1, \dots, x_d]$ satisfies $f(t) > 0$ for all $t \in C$, then $f \in \mathcal{S}(H)$.**

Theorem 6:

Let $A_1 := \mathbb{R}[x_1, \dots, x_n]$, $A_2 := \mathbb{R}[x_{n+1}, \dots, x_{n+k}]$ and $\mathcal{T}_1$, $\mathcal{T}_2$ finitely generated preorderings of A_1 and A_2. Suppose

$$\mathcal{K}(A_1; \mathcal{T}_1) := \{x \in \mathbb{R}^n : f(x) \geq 0 \text{ for } f \in \mathcal{T}_1\},$$
$$\mathcal{K}(A_2; \mathcal{T}_2) := \{x' \in \mathbb{R}^k : g(x') \geq 0 \text{ for } g \in \mathcal{T}_2\}.$$

are **compact**. Let $p \in A_1 \otimes A_2 \equiv \mathbb{R}[x_1, \dots, x_{n+k}]$.
If $p(x, x') > 0$ **for** $(x, x') \in \mathcal{K}(A_1, \mathcal{T}_1) \times \mathcal{K}(A_2, \mathcal{T}_2)$, **then** $f \in \mathcal{T}_1 \otimes \mathcal{T}_2$:

$$p(x, x') = \sum_{i=1}^{r} p_i(x) q_i(x'), \quad \text{with} \quad p_1, \dots, p_r \in \mathcal{T}_1,\ q_1, \dots, q_r \in \mathcal{T}_2.$$

Idea of proof:

$\mathcal{S} := \mathcal{T}_1 \otimes \mathcal{T}_2$ is a **semiring** of $A := A_1 \otimes A_2$. Since $\mathcal{K}(A_1; \mathcal{T}_1)$, $\mathcal{K}(A_2; \mathcal{T}_2)$ are compact, $\mathcal{T}_1$, $\mathcal{T}_2$ are Archimedean. Hence $f \otimes 1$, $1 \otimes g$ satisfy the Archimedean condition, so $\mathcal{S}$ is Archimedean. The Archimedean Positivstellensatz for semirings gives the assertion.

Semi-algebraic sets contained in compact polyhedra

Theorem 7:

Let $f_0 = 1, f_1, \dots, f_r \in \mathbb{R}_d[x]$. Suppose $g_1, \dots, g_m \in \mathbb{R}_d[x]$ have degree 1 s.t. the polyhedron $\mathcal{K}(g_1, \dots, g_m)$ is non-empty compact and contains the semi-algebraic set $\mathcal{K}(f_1, \dots, f_r)$.

Let $p \in \mathbb{R}_d[x]$. If $p(x) > 0$ for $x \in \mathcal{K}(f_1, \dots, f_r)$, then p is a finite sum of

$$\alpha\, f_j\, g_1^{n_1} \cdots g_m^{n_m}, \quad \textit{where} \ \ \alpha \geq 0,\ j = 0, \dots, r;\ n_1, \dots, n_m \in \mathbb{N}_0.$$

With f_j replaced by $f_{j_1} \cdots f_{j_r}$ this is contained in the book Prestel/Delzell.

Archimedean Positivstellensatz for semirings applied to the S-module $C = f_0 S + \cdots + f_r S$, where S is the semiring generated by $g_1, \dots, g_m$.

Cofinal elements and c-localizable semirings

Aim: Positivstellensätze for non-compact sets.

Suppose $\mathcal{S}$ is a semiring of A.

Definition:

An element $c \in 1 + \mathcal{S}$ is said to be **cofinal in** A if for all $a \in A$ there exist $\lambda \in (0, \infty)$ and $k \in \mathbb{N}_0$ such that $\lambda c^k - a \in \mathcal{S}$.

$\mathcal{S}$ is Archimedean if and only if the unit element 1 is cofinal in $\mathcal{S}$.

Proposition :

Suppose A has finitely many generators $y_1, \dots, y_n$ and there exist elements $a_1, \dots, a_n \in \mathcal{S}$ such that $a_j \pm y_j \in \mathcal{S}$ for $j = 1, \dots, n$. Then

$$c := 1 + \sum_{j=1}^{n} a_j$$

is cofinal in A.

Definition:

Let C be a cone of A and $c \in C$. C is c-**localizable** if the following holds:

Whenever $ca \in C$ for some $a \in A$, then $a \in C$.

Example:

Let $A = \mathbb{R}[x_1, \ldots, x_n]$ and $C := \{p \in A : p(\xi) \geq 0 \text{ for } \xi \in \mathbb{R}^n\}$.
Then C is p-localizable for each $p \in C$, $p \neq 0$.

Definition:

A cone C of A is called **filtered simplicial** if $C = \cup_{k\in\mathbb{N}_0} C_k$ for simplicial cones C_k such that $C_k \subseteq C_{k+1}$ for all $k \in \mathbb{N}_0$.

Proposition:

Suppose $C = \bigcup_{k\in\mathbb{N}_0} C_k$ be a filtered simplicial convex cone of A. Let $A[z]$ be the algebra of polynomials in one variable z with values in A, and

$$C[z]_k := \Big\{ \sum\nolimits_{j=1}^{\ell} c_j p_j : p_1, \dots, p_\ell \in \mathbb{R}[z]_+ \Big\},\ \ k \in \mathbb{N}_0, \tag{5}$$

where $c_1, \dots, c_\ell \in C_k$ are linearly independent generators of the cone C_k. Then $C[z] := \bigcup_{k\in\mathbb{N}_0} C[z]_k$ is a $(1+z^2)$-localizable convex cone of $A[z]$.

Cylindrical extensions of the Archimedean Positivstellensatz

Marshall's strip theorem (2009)

Let $f \in \mathbb{R}[x, y]$. If $f(x, y) \geq 0$ for all $x \in [-1, 1], y \in \mathbb{R}$, there exist $p, q \in \sum \mathbb{R}[x, y]^2$ such that

$$f(x, y) = p(x, y) + (1 - x^2)q(x, y).$$

Theorem 8:

Suppose A is a **finitely generated** commutative unital real algebra and $\mathcal{S}$ is an **Archimedean semiring** of A.

Let $a(z) = \sum_{i=0}^{2k} a_i z^i \in A[z] \cong A \otimes \mathbb{R}[z]$, where $a_0, \ldots, a_{2k} \in A$. The following are equivalent:

(i) $\varphi(a(\eta)) > 0$ for all $(\varphi, \eta) \in \mathcal{K}(A; \mathcal{S}) \times \mathbb{R}$ and $\psi(a_{2k}) > 0$ for all $\psi \in \mathcal{K}(A : \mathcal{S})$.

(ii) There exists $\varepsilon \in (0, \infty)$ such that

$$a \in \varepsilon(1 + z^2)^k + \mathcal{S} \otimes \sum \mathbb{R}[z]^2.$$

V. Powers (2004): Similar result for preorderings

Proof: M. Schweighofer's technique of using Polya's theorem

Example: Strip in $\mathbb{R}^2$

Let $A := \mathbb{R}[x]$ and S the semiring generated by $1 \pm x$. Then $\mathrm{char}(A, \mathcal{S}) \cong [-1, 1]$. Let $a(x, z) = \sum_{j=0}^{2k} a_j(x) z^j$.

If $a(x, z) > 0$ **for** $(x, z) \in [-1, 1] \times \mathbb{R}$ **and** $a_{2k}(x) > 0$ **for** $x \in \mathbb{R}$**, then**

$$a \in \varepsilon(1 + z^2)^k + S \otimes \sum \mathbb{R}[z]^2.$$

Relations to the moment property

Let $A = \mathbb{R}[x_1, \ldots, x_d]$ and let C be a cone in A.

Definition: Moment property

C has (MP) if each linear functional on A such that $L(c) \geq 0$ for all $c \in C$ is a moment functional, that is, there is a Radon measure μ on $\mathbb{R}^d$ s. t.

$$L(p) = \int_{\mathbb{R}^d} p(x)\, d\mu(x) \quad \text{for} \quad f \in A.$$

If a denominator-free Positivstellensatz for C holds, then C has (MP).

Example: Preordering for a semi-algebraic cylinder set with compact base.

Question: Is there a version of Theorem 8 for other sets satisfying (MP)?

For instance $\{(x, y) \in \mathbb{R}^2 : 1 \leq xy \leq 2, x \geq 0\}$?