Squares in Dortmund
31 March – 4 April 2025
Programme

All talks take place in **lecture hall E29.**

Monday, 31 March

9:00 – 9:15	Welcome	
9:15 – 10:15	O. Benoist	*Sums of squares of analytic functions*
	— Coffee break —	
10:45 – 11:15	M. Zaninelli	*Sums of squares in hyperelliptic function fields*
11:30 – 12:30	G. Nebe	*Orthogonal representations of finite groups*
	— Lunch —	
14:30 – 15:00	J. Krásenský	*Sums of integral squares in totally real number fields*
15:10 – 15:40	M. Tinková	*Bounds on the Pythagoras number and indecomposables in biquadratic fields*
	— Coffee break —	
16:10–16:40	J. Yoon	*Minimal rank of primitively n-universal quadratic forms over local fields*
16:50–17:20	O. Prakash	*Universality beyond quadratic forms*

Tuesday, 1 April

9:15 – 10:15	A. Vishik	*Varieties and pure symbols*
	— Coffee break —	
10:45 – 11:15	P. Sechin	*Morava motives of quadrics*
11:30 – 12:30	S. Scully	*On the holes in I^n for symmetric bilinear forms in characteristic 2*
	— Lunch —	
14:30 – 15:00	N. Lorenz	*Isotropy indices of Pfister multiples in characteristic 2*
15:10 – 15:40	A. Laghribi	*Around the 3-Pfister number in characteristic 2*
	— Coffee break —	
16:10–16:40	A. Schönert	*Witt rings of fields with 64 square classes*
16:50–17:20	N. Garrel	*Adams operations on hermitian forms*

Wednesday, 2 April

9:15 – 10:15	F. Scavia	*The lifting problem for Galois representations*
	— Coffee break —	
10:45 – 11:15	R. Sinn	*Graded Pythagoras numbers*
11:30 – 12:30	E. Becker	*A Waring problem for real function fields over* $\mathbb{R}$
	— Lunch —	
13:45	Excursion	Zeche Zollern (Taxi-transfer from university)
19:00	Conference dinner	Restaurant *Schönes Leben*

Thursday, 3 April

9:15 – 10:15	A. Quéguiner-Mathieu	*Higher Tate trace and classification of Chow motives*
	— Coffee break —	
10:45 – 11:15	M. Zhykhovich	*The J-invariant of algebraic groups of outer type and symplectic involutions*
11:30 – 12:30	R. Parimala	*Projective homogeneous spaces and Hasse principle*
	— Lunch —	
14:30 – 15:00	A. Mondal	*R-triviality of groups of projective similitudes over special fields*
15:10 – 15:40	A. Chapman	*Zeros of multivariate quaternionic polynomials*
	— Coffee break —	
16:10–16:40	U. First	*The number of generators of Azumaya algebras with involution*
16:50–17:20	A. Soman	*Totally positive field extensions and Pythagorean index*

Friday, 4 April

9:15 – 10:15	V. Mehmeti	*Local-global principles and the u-invariant*
	— Coffee break —	
10:45 – 11:15	N. Daans	*Linkage of Pfister forms over local and semi-global fields*
11:30 – 12:30	D. Leep	*Systems of quadratic forms and u-invariants*

Abstracts

Eberhard Becker: *A Waring problem for real function fields over* $\mathbb{R}$
Wednesday, 11:30–12:30

Abstract. The classical Waring Problem deals with the representation of natural numbers as sums of a fixed number $g(n)$ of nth powers of natural numbers, for each exponent n. That this is possible was first proven by Hilbert in 1909.

In this talk I deal with a variant of the classical Waring Problem in the context of a real function field $F/\mathbb{R}$: Find the largest subset $W \subseteq F^*$ such that for each exponent n there exists a number w_n such that every element of W can be written as the sum of at most w_n n-th powers of elements of W.

In fact, there is such a distinguished subset. It is identified as the group of totally positive units

$$\mathbb{E}_+ := H^* \cap \sum F^2$$

where H denotes the real holomorphy ring of F, defined as the intersection of all real valuation rings of F. Let F be a function field of d variables and p its Pythagoras number. It can be proven:

$$w_2 \leq p \leq (d+1) * w_2, \quad w_n \leq \binom{2n + w_2}{2n}.$$

The proof makes use of the descriptions of the real holomorphy ring and the space M of real places of F by means of all smooth affine models of F with compact real locus, the representation

$$S^k(H) \longrightarrow C(M, S^k)$$

and a recent result of W. Kucharz on rings of continuous rational functions as well as the socalled Hilbert's identities.

Olivier Benoist: *Sums of squares of analytic functions*
Monday, 9:15–10:15

Abstract. Artin solved Hilbert's 17th problem by showing that any real polynomial in n variables that is nonnegative is a sum of squares of rational functions. Pfister improved quantitatively Artin's theorem by showing that 2^n squares suffice. In this talk, we will present new quantitative results à la Pfister in the real-analytic setting (where polynomials are replaced with real-analytic functions).

Adam Chapman: *Zeros of multivariate quaternionic polynomials*
Thursday, 15:10–15:40

Abstract. Alon and Paran showed that over Hamilton's quaternion algebra, if one multivariate polynomial vanishes over all the zeros of another polynomial where all the slots commute, then the first polynomial vanishes over all the zeros of the second polynomial. They raised the question of whether this phenomenon holds true over any quaternion division algebra. I joined Alon and Paran to answer this question in the negative, by producing an example of a quaternion division algebra over the function field in three algebraically independent variables over the field of real numbers over which there exist two polynomials that disprove the hypothesis. The argument involves quadratic form theory and valuation theory.

Nicolas Daans: *Linkage of Pfister forms over local and semi-global fields*
Friday, 10:45–11:15

Abstract. Two quadratic d-fold Pfister forms over a field are called linked if they share a common $(d-1)$-fold Pfister form as a subform. When K is a field and d is minimal such that all $(d+1)$-fold Pfister forms over K are isotropic, then it is often the case that any two d-fold Pfister forms over K are linked - but not always! In this talk, we discuss some examples. We will see that properties on linkage of Pfister forms can sometimes be transferred from a henselian valued field to its residue field and vice versa. More surprisingly, in view of some recent local-global principles for isotropy of quadratic forms, it may be possible to transfer properties on linkage of Pfister forms over function fields between a henselian valued field and its residue field, even in characteristic 2. Research paper: Daans, N. *Linkage of Pfister forms over semi-global fields.* Math. Z. 308, 41 (2024). https://doi.org/10.1007/s00209-024-03598-2

Uriya First: *The number of generators of Azumaya algebras with involution*
Thursday, 16:10–16:40

Abstract. It is well-known that every central simple algebra can be generated by 2 elements. A slightly less known fact is that, with one exception in degree 2, every CSA with involution can be generated by a single element (as an algebra with involution). Fix an infinite field K and let R be a finitely generated commutative K-algebra of Krull dimension d. I will discuss three recent works of First-Reichstein-Williams, Nam-Tan-Williams and Cantor-First which globalize the previous results by giving upper bounds on the number of generators of Azumaya algebras with and without involution over R. For example, every Azumaya algebra of degree n over R can be generated using at most $\lfloor d/(n-1)) + 2\rfloor$ elements, and there are examples requiring about half that many generators. All the results stem from one general result in F-R-W which reduces the problem to a geometric question: determining the dimension of the variety of r-tuples which fail to generate a certain finite-dimensional K-algebra — in our case it would be a split CSA with or without an involution. After some case-specific computations in F-R-W and N-T-W, a general recipe for computing this dimension of this variety was given and applied in C-F.

Nicolas Garrel: *Adams operations on hermitian forms*
Tuesday, 16:50–17:20

Abstract. The so-called Adams operations which are familiar in K-theory can be defined in any lambda-ring, and operate as power sums on sums of 1-dimensional elements. In particular, in the Grothendieck-Witt ring, the second Adams operation sends a quadratic form $\langle a_1, ..., a_n\rangle$ to the "sum of squares" $\langle a_1^2, ..., a_n^2\rangle$, which is just $\langle 1, ..., 1\rangle = n$. For hermitian forms over an algebra with involution, there is also a structure of lambda-ring on the so-called "mixed Grothendieck-Witt ring", so the Adams operations are defined. When the algebra is not split, we cannot write a hermitian form as a sum of 1-dimensional elements, but we can still interpret the second Adams operation as the sum of the squares of a virtual decomposition. It turns out that the value of this Adams operation on a hermitian form of reduced dimension n is not just $\langle 1, ..., 1\rangle = n$, but rather the trace form of the underlying central simple algebra, which we interpret as a twisted version of the integer n. We will explain how that leads to a twisted version of the usual combinatorics involved when mutiplying quadratic forms.

Jakub Krásenský: *Sums of integral squares in totally real number fields*
Monday, 14:30–15:00

Abstract. Every totally positive element of a number field can be written as a sum of at most four squares. When we restrict ourselves to integral squares, then not every totally positive integral element is a sum of squares anymore. However, if the field is not totally real, then every sum of squares can still be rewritten

as a sum of at most four squares. In other words, the Pythagoras number $\mathrm{P}(O_K)$ of the ring of integers O_K in a not-totally-real number field K is at most 4. In contrast, Scharlau proved that for totally real fields K, $\mathrm{P}(O_K)$ can be arbitrarily large. Among those who contributed to the study of $\mathrm{P}(O_K)$ in totally real number fields we can name Siegel, Maaß, Peters, and in recent years Tinková; Kala and Yatsyna; He and Hu; K., Raška and Sgallová. I will touch on their results, and in particular I will present a recent work with R. Scharlau which exhibits two fields of degree 4 with $\mathrm{P}(O_K) = 3$, which is the minimal possible value.

Ahmed Laghribi: *Around the 3-Pfister number in characteristic 2*
Tuesday, 15:10–15:40

Abstract. Let F be a field of characteristic 2, $W_q(F)$ the Witt group of nonsingular quadratic forms over F, and $W(F)$ the Witt ring of symmetric bilinear forms over F. For any integer $m \geq 2$, let $I_q^m(F)$ denote the subgroup $I^{m-1}(F) \otimes W_q(F)$ of $W_q(F)$, where $I^n(F)$ is the n-th power of the fundamental ideal IF of $W(F)$ (we take $I_q^1(F) = W_q(F)$). Any quadratic form $\varphi \in I_q^m(F)$ is Witt equivalent to a sum of forms similar to m-fold Pfister forms. The m-Pfister number of φ, denoted by $\mathrm{Pf}_m(\varphi)$, is the least number of forms similar to m-fold Pfister forms needed to express φ up to Witt equivalence. Our aim in this talk is to discuss the case $m = 3$ by giving a formula that bounds $\mathrm{Pf}_3(\varphi)$ for any φ. The case where φ is of dimension 14 will be detailed. (This talk is based on joint works with Chapman, Maïti and Mukhija).

David Leep: *Systems of quadratic forms and u-invariants*
Friday, 11:30–12:30

Abstract. This lecture will survey the topic of finding zeros of systems of quadratic forms and applications of these results to the question of computing the u-invariant of a field.
The u-invariant of a field F, $u(F)$, is the supremum of the dimensions of anisotropic quadratic forms defined over F.
Question: Can the u-invariant of a finite extension of F or the u-invariant of the rational function field $F(x)$ be expressed in terms of $u(F)$?
One method to study these questions is to relate the computations of these u-invariants to finding non-trivial zeros of certain systems of quadratic forms.
Another method is to relate the computations of these u-invariants to studying conditions guaranteeing that a pair of quadratic forms defined over F vanishes on a linear subspace of dimension m for values of $m \geq 1$.
These two approaches will be surveyed. Various theorems and conjectures will be discussed. Fields with $u(F) = 1$ and $u(F) = 2$ will receive special attention in this talk.

Nico Lorenz: *Isotropy indices of Pfister multiples in characteristic 2*
Tuesday, 14:30–15:00

Abstract. Let φ be a quadratic form and π be a bilinear Pfister form over some field F of characteristic 2. Given a further quadratic form ψ, we compare the isotropy behaviour of φ over the function field $F(\psi)$ with the isotropy behaviour of $\pi \otimes \varphi$ over $F(\pi \otimes \psi)$. In particular, when $\psi = \varphi$, we investigate the first isotropy index of these forms and show the inequality $i_1(\pi \otimes \varphi) \geq i_1(\varphi) \cdot \dim(\pi)$. We finally discuss situations in which equality and strict inequality hold.

Vlerë Mehmeti: *Local-global principles and the u-invariant*
Friday, 9:15–10:15

Abstract. I will speak of local-global principles obtained by working with non-Archimedean analytic spaces. By local considerations in the case of quadratic forms, one can then obtain upper bounds on the u-invariant of function fields of curves. I will also speak of some recent generalizations obtained through this approach together with K.J. Becher and N. Daans.

Archita Mondal: *R-triviality of groups of projective similitudes over special fields*
Thursday, 14:30–15:00

Abstract. Around 1996, A.S. Merkurjev and P. Gille gave the first examples of quadratic forms and orthogonal involutions whose groups of proper similitudes are not R-trivial, and hence not stably rational. The method of proof relies on a comparison of the groups of similarity factors with a subgroup of hyperbolic norms. It has been shown later that, nevertheless, in many cases, depending on special cohomological conditions on the base field, these two groups are nevertheless equal. In my talk, I will survey some of these results and related questions over various fields with some cohomological properties.

Gabriele Nebe: *Orthogonal representations of finite groups*
Monday, 11:30–12:30

Abstract. A complex irreducible representation of a finite group fixes a quadratic form, if and only if it can be realised over a totally real number field. In this case the associated character has Frobenius-Schur indicator +. For even degree absolutely irreducible indicator + characters, there is a unique square class of the character field that contains the discriminants of the invariant quadratic forms, the orthogonal discriminant of the character. Together with Richard Parker we developed methods to compute these orthogonal discriminants and determined them for all even degree indicator + characters of the ATLAS groups up to the Harada-Norton group.
There are certain surprising facts: For some characters of sporadic groups the quadratic extension of the character field determined by the orthogonal discriminant is not Galois over the rationals.
Also we never observed square classes having an odd dyadic valuation, motivating Richard Parker to conjecture that orthogonal discriminants are always odd. This conjecture is a theorem for solvable groups and for some infinte series of finite groups of Lie type, however we do not have a structural explanation for this phenomenon.

R. Parimala: *Projective homogeneous spaces and Hasse principle*
Thursday, 11:30–12:30

Abstract. In this talk, we discuss Hasse principle for function fields of curves over complete discrete valued fields. We shall review progress towards Hasse principle for certain classes of homogeneous spaces under linear algebraic groups over such fields.

Om Prakash: *Universality beyond quadratic forms*
Monday, 16:50–17:20

Abstract. A universal quadratic form is a positive definite quadratic form with integral coefficients that represents all positive integers – a classical example being the sum of four squares $x^2+y^2+z^2+w^2$. The 290-Theorem of Bhargava and Hanke characterizes positive definite quadratic forms over rational integers are universal as exactly the forms that represent $1, 2, 3, \dots, 290$. In this talk, I will discuss universality of higher degree forms (i.e. homogeneous polynomials of degree $m > 2$) and I will prove that no statement like the 290-Theorem can hold for them. If time permits, I will conclude with the more general case of forms over

totally real number fields. This is a joint work with Vitezslav Kala.

Anne Quéguiner-Mathieu: *Higher Tate trace and classification of Chow motives*
Thursday, 9:15–10:15

Abstract. The talk is based on a joint work with de Clercq, and a joint work wit de Clercq and Karpenko. The classification of projective homogeneous varieties up to motivic isomorphism is a long-standing problem, opened more than 30 years ago with the classification over a separably closed field, involving Poincaré polynomials. Over an arbitrary field, Vishik provided a criterion for quadrics, in terms of the higher Witt index of the underlying quadratic form. Working with Chow motives with finite coefficients, we will introduce new invariants, the Tate trace and the Artin-Tate trace, and we will prove they provide a criterion for a large class projective homogeneous varieties.

Federico Scavia: *The lifting problem for Galois representations*
Wednesday, 9:15–10:15

Abstract. For every finite group H and every finite H-module A, we determine the subgroup of negligible classes in $H^2(H, A)$, in the sense of Serre, over fields with enough roots of unity. As a consequence, we show that for every odd prime p and every field F containing a primitive p-th root of unity, there exists a continuous 3-dimensional mod p representation of the absolute Galois group of $F(x_1, ..., x_p)$ which does not lift modulo p^2. We also construct continuous 5-dimensional Galois representations mod 2 which do not lift modulo 4. This answers a question of Khare and Serre, and disproves a conjecture of Florence. This is joint work with Alexander Merkurjev.

Alexander Schönert: *Witt rings of fields with* 64 *square classes*
Tuesday, 16:10–16:40

Abstract. Every Witt ring is generated by the square class group of the underlying field which has - if it is finite - 2^n elements for some n which we call the order of the Witt ring. In 1982, Carson and Marshall showed that all (abstract) Witt rings of order up to 5 are of so called 'elementary type'. We present a new approach to classifying (abstract) Witt rings of small order using an abstract version of the 2-torsion part of the Brauer group. Applying this, we outline a joint effort with Nico Lorenz, giving a computational verification that the 'elementary type conjecture' also holds for Witt rings of order 6.

Stephen Scully: *On the holes in I^n for symmetric bilinear forms in characteristic* 2
Tuesday, 11:30–12:30

Abstract. Following the proof of Milnor's conjecture relating the graded Witt ring of a field to its mod $-$ 2 Milnor K-theory, a major problem in the theory of symmetric bilinear forms (or quadratic forms in characteristic not 2) is to understand, for each n, the low-dimensional part of the nth power of the fundamental ideal in the Witt ring of a field. Using methods from the theory of algebraic cycles, Karpenko showed that a non-zero anisotropic form that represents an element of I^n has dimension $2^{n+1} - 2^i$ for some $1 \leq i \leq n$. When $i = n$, a classical result of Arason-Pfister says that the form is then similar to a Pfister form. When $i = n - 1$, a conjecture of Hoffmann predicts that the form should be isometric to the tensor product of an $(n - 2)$-fold Pfister form and an Albert form of dimension 6. Over fields of characteristic not 2, this is open for all $n \geq 4$. In this talk, I will give an overview of this general picture, and then discuss the (simpler) case where the characteristic is 2. Here, we can give a direct and elementary proof of Karpenko's theorem, and, more interestingly, a proof of the conjecture of Hoffmann for all values of n.

Pavel Sechin: *Morava motives of quadrics*
Tuesday, 10:45–11:15

Abstract. By using Morava K-theory $K(n)$ instead of Chow groups in the definition of motives we give a new perspective on motivic decomposition of quadrics. Also, we construct an injective group homomorphism from I^{n+1}/I^{n+2} to the group of invertible $K(n)$-motives. These invertible motives appear in quadrics with Kahn dimension $\dim_n q < 2^n$, and conjecturally only in those. Based on the joint work with A. Lavrenov.

Rainer Sinn: *Graded Pythagoras numbers*
Wednesday, 10:45–11:15

Abstract. I will present joint work with Greg Blekherman, Greg Smith, and Mauricio Velasco about bounds on the smallest number k such that every sum of squares of degree $2d$ in n variables can be written as a sum of k squares. I will look at this question from the point of view of projective real algebraic geometry. I might also mention improvements in joint work with Greg Blekherman and Alex Dunbar for the case of three variables.

Abhay Soman: *Totally positive field extensions and Pythagorean index*
Thursday, 16:50–17:20

Abstract. We show that if K/F is a Galois totally positive field extension then so is the corresponding extensions of their Pythagorean closures $K_{\mathrm{py}}/F_{\mathrm{py}}$. We also discuss weak isotropy of orthogonal involutions under totally positive field extensions. This is a joint work with Priyabrata Mandal and Preeti Raman. The talk will be based on the paper at this link: https://doi.org/10.1142/S0219498824500348

Magdaléna Tinková: *Bounds on the Pythagoras number and indecomposables in biquadratic fields*
Monday, 15:10–15:40

Abstract. In this talk, we will show that, except for a few cases, the Pythagoras number of the ring of algebraic integers of real biquadratic fields is at least 6. Moreover, we will discuss a family of such fields for which the Pythagoras number is exactly 7. Part of the talk will also focus on additively indecomposable integers in some families of these fields.

Alexander Vishik: *Varieties and pure symbols*
Tuesday, 9:15–10:15

Abstract. The norm-varieties for pure symbols in Milnor's K-theory mod p were indispensable in the proof of all cases of Milnor and Bloch-Kato conjectures since the work of Merkurjev from 1981. For $p = 2$, these varieties are Pfister quadrics. I will discuss the result claiming that an arbitrary (!) projective variety is a norm-variety for an appropriate pure symbol mod 2. The only thing, this symbol is defined not over the base field, but over the flexible closure of it. This allows to distinguish the 2-equivalence class of a field extension by the kernel on the pure part of the Milnor's K-theory mod 2 it produces. This has consequences for the topology of the Balmer spectrum of Voevodsky category of (geometric) motives. The proof of the mentioned result also provides a new point of view on the Cassels-Pfister theorem.

Jongheun Yoon: *Minimal rank of primitively n-universal quadratic forms over local fields*
Monday, 16:10–16:40

Abstract. For a positive integer n, a (integral) quadratic form is called primitively n-universal if it primi-

tively represents all quadratic forms of rank n. In 2021, Earnest and Gunawardana proved that the minimal rank of primitively 1-universal quadratic forms over $\mathbb{Z}_p$ is 2 or 3, depending on the prime p and the concept of integrality in use. In this talk, we prove that the minimal rank of primitively n-universal quadratic forms over the ring of integers R in a local field is $2n$ in general, and it is $2n + 1$ for classically integral quadratic forms when $n \leq 2$ if $R \neq \mathbb{Z}_2$ is unramified dyadic, and when $n \leq 4$ if $R = \mathbb{Z}_2$. This is a joint work with Prof. Byeong-Kweon Oh.

Marco Zaninelli: *Sums of squares in hyperelliptic function fields*
Monday, 10:45–11:15

Abstract. Building on work by Becher and Van Geel, we study sums of squares in function fields of hyperelliptic curves, mostly over hereditarily pythagorean fields. A refinement of their techniques allows us to obtain upper bound 2 for the Pythagoras number of certain such fields. As a consequence, we give a negative answer to Becher and Van Geel's question whether Pythagoras number 2 for the function field of the elliptic curve $X(X^2 + 1)$ implies that the base field K is ∞-hereditarily pythagorean or $-1 \in K^2$.

Maxim Zhykhovich: *The J-invariant of algebraic groups of outer type and symplectic involutions*
Thursday, 10:45–11:15

Abstract. The J-invariant was originally introduced by Vishik for quadratic forms and then generalized for semi-simple algebraic groups of inner type by Geldhauser, Petrov and Zaynullin. The J-invariant of a group G is a discrete invariant which encodes the Chow-motivic decomposition of the variety of Borel subgroups in G. In this talk I will discuss the extension of the theory to certain groups of outer type and symplectic involutions. The talk is based on the joint works with Geldhauser and Hendrichs.