Sums of squares of analytic functions

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I. Cohomological dimension

Def: A field F has cohomological dimension $\leq n$ ($cd(F) \leq n$) if

$$H^k(\Gamma, M) = 0 \quad \text{for} \quad \begin{cases} \Gamma = \mathrm{Gal}(\bar{F}/F) \\ M \text{ finite } \Gamma\text{-module} \\ k > n \end{cases}$$

Reflects the arithmetic complexity of F (Galois cohomology controls
controls quadratic forms, Brauer classes ...).

ex: (*) X connected smooth alg. variety of dim n $/\mathbb{C}$
Then $cd(\mathbb{C}(X)) = n$.

Today: complex-analytic analogue + application to sums of squares

analytic analogue of affine varieties

Def: A complex-analytic manifold S is Stein if there exists $S \hookrightarrow \mathbb{C}^N$ holomorphic embedding.

[to simplify, could allow singularities.

ex: $\mathbb{C}^n$, closed subspaces of $\mathbb{C}^n$, open ball in $\mathbb{C}^n$

Def: A compact subset $K \subseteq S$ is Stein if it admits a basis of Stein open neighborhoods

ex closed ball in $\mathbb{C}^n$.

$\mathcal{O}(S)$, $\mathcal{O}(K)$: rings of holomorphic functions on S, or in some neighb of K
$\mathcal{M}(S)$, $\mathcal{M}(K)$: fields meromorphic
$\mathcal{M}(S) = \mathrm{Frac}\,\mathcal{O}(S)$, $\mathcal{M}(K) = \mathrm{Frac}\,\mathcal{O}(K)$

ex: $\mathcal{O}(\mathbb{C}^n) = \{ f = \sum a_I z^I \mid f \text{ converges on } \mathbb{C}^n \}$

Th 1 (B. 2023) S Stein manifold of dim n.

$K \subset S$ connected Stein compact subset.

Then $\mathrm{cd}(\mathcal{M}(K)) = n$.

Th 2 (B. 2024) S connected Stein manifold of dim. 2.

Then $\mathrm{cd}(\mathcal{M}(S)) = 2$.

Remarks:

- dim 1 case (= noncompact Riemann surfaces) due to M. Artin.
- $\mathrm{cd}(\mathcal{M}(S)) = n$ in general?
- starting from Th 2, one shows: for $\alpha \in \mathrm{Br}\,\mathcal{M}(S)$, $\underbrace{\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)}_{\text{period-index result.}}$.

II Application to Hilbert's 17^{th} problem

Th (Artin 1927): $f \in \mathbb{R}[x_1, \dots, x_n] \geq 0$ ($f(x) \geq 0 \ \forall x \in \mathbb{R}^n$)

Then f is a sum of 2^n squares in $\mathbb{R}(x_1, \dots, x_n)$.

Pfister 1967.

Open question: is that optimal? [yes if $n \leq 2$, Cassels-Ellison-Pfister 1971]

Real-analytic variant:

Th (Jaworski 1986): Let M be a compact real-analytic manifold of dim n (e.g. S^n). Fix $f \in A(M) \geq 0$. real-analytic function

Then f is a sum of 2^n squares in $F(M) := \mathrm{Frac}\, A(M)$.

B 2023. real-analytic meromorphic functions

Open question: Are all $f \in A(\mathbb{R}^n) \geq 0$ sums of squares in $F(\mathbb{R}^n)$?

[yes if $n \leq 2$, Bochnak-Risler 1975].

__Sketch__ : • Milnor conjectures proved by Voevodsky relate quadratic forms and Galois cohomology :

$\left[\begin{array}{l} F \text{ field, } \mathrm{car} \neq 2 \quad F^*/F^{*2} \xrightarrow{\sim} H^1(F, \mathbb{Z}/2) \\ \qquad\qquad\qquad\qquad\qquad f \longmapsto \{f\} \\ f \text{ is a sum of } 2^n \text{ squares} \iff \{f\}\{-1\}^n = 0 \text{ in } H^{n+1}(F, \mathbb{Z}/2) \end{array}\right.$

• By Cartan and Grauert, one can realize M as a Stein compact subset in a Stein manifold S ("its complexification", "Grauert tube")

Th 1 $\Rightarrow$ $\mathrm{cd}\ \mathcal{M}(M) \leq n$.

$\quad F(M)[\sqrt{-1}]$

• Unfortunately, cannot hope that $\mathrm{cd}(F(M)) \leq n$ (because of orderings).

However, Arason shows that

$$\left\{\begin{array}{l} f \in F \text{ sum of squares} \\ \mathrm{cd}(F[\sqrt{-1}]) \leq n \end{array}\right. \rightarrow \{f\}\{-1\}^n = 0.$$

[Why Stein?

• many meromorphic functions, so $\mathcal{M}(S)$ interesting field.

• applications to real-analytic geometry.]

III Weak Lefschetz

Th (Andreotti–Frankel) : Any Stein manifold S of dim n has the homotopy type of a CW-complex of dim. n. ($\Rightarrow H^k(S, A) = 0$, $k > n$)

How to go from singular to Galois cohomology?
$\rightsquigarrow$ étale cohomology!

Warm-up: proof of $(*)$

$\mathbb{C}(X) = \bigcup_{\substack{\emptyset \subsetneq U \subseteq X \\ \text{open affine}}} G(U)$

$$H^k(\mathbb{C}(X), \mathbb{Z}/n) = \varinjlim_{\emptyset \subsetneq U \subseteq X} H^k_{ét}(U, \mathbb{Z}/n)$$

Artin comparison theorem
$$= \varinjlim_{\emptyset \subsetneq U \subseteq X} H^k(U(\mathbb{C}), \mathbb{Z}/n)$$

weak Lefschetz
$$= 0 \quad \text{si } k > n.$$

(Galois modules $\Leftrightarrow$ local systems)

Proof of Th 1 :

$$\mathcal{M}(K) = \varinjlim_{\substack{U \text{ Stein neigh of } K \\ 0 \neq h \in G(U)}} G(U)\left[\tfrac{1}{h}\right]$$

$$H^k(\mathcal{M}(K), \mathbb{Z}/n) = \varinjlim_{U, h} H^k_{ét}\left(\mathrm{Spec}\left(G(U)\left[\tfrac{1}{h}\right]\right), \mathbb{Z}/n\right)$$

un th de comparaison d'Artin en géométrie Stein
$$= \varinjlim_{U, h} H^k\Big(\overbrace{U \setminus \{h = 0\}}^{\text{Stein}}, \mathbb{Z}/n\Big)$$

$$= 0 \leftarrow \text{weak Lefschetz}$$

Proof of Th2:

$$\mathcal{M}(S) = \varinjlim_{0 \neq h \in \mathcal{O}(S)} \mathcal{O}(S)\left[\tfrac{1}{h}\right]$$

$$H^k(\mathcal{M}(S), \mathbb{Z}/m) = \varinjlim_{h} H^k_{\text{ét}}\left(\operatorname{Spec} \mathcal{O}(S)\left[\tfrac{1}{h}\right], \mathbb{Z}/m\right)$$

An Artin comparison theorem in Stein geometry

$$= \varinjlim_{h} H^k(\underbrace{S \setminus \{h=0\}}_{\text{Stein}}, \mathbb{Z}/m)$$

$$= 0 \quad \leftarrow \text{weak Lefschetz}$$

IV Comparison theorems.

Th: S Stein manifold of dim. 2

$$\varinjlim_{h} H^k_{\text{ét}}\left(\operatorname{Spec} \mathcal{O}(S)\left[\tfrac{1}{h}\right], \mathbb{Z}/m\right) \xrightarrow[\sim]{\varepsilon^*} \varinjlim_{h} H^k(S \setminus \{h=0\}, \mathbb{Z}/m)$$

⚠ these rings are not noetherian!

false without $\varinjlim_{h}$ in general!

Remarks: I do not know if it holds in any dimension. [would solve the real-analytic Hilbert's 17th problem]

The variant for Stein compact sets does!

There is a morphism

$$\underset{\text{classical topology}}{S} \xrightarrow{\varepsilon} \underset{\text{étale topology}}{\operatorname{Spec} \mathcal{O}(S)}$$

$$x \longmapsto \mathfrak{m}_x = \{ f \in \mathcal{O}(S) \mid f(x) = 0 \}$$

V Proof of the comparison theorem.

Both in the surface and compact cases, consider the Leray spectral sequence of ε:

$$H^p_{\text{ét}}\left(\operatorname{Spec} \mathcal{O}(S)\left[\tfrac{1}{h}\right], R^q\varepsilon_*\mathbb{Z}/m\right) \Rightarrow H^{p+q}\left(S\setminus\{h=0\}, \mathbb{Z}/m\right)$$

and show that (a) $\mathbb{Z}/m \xrightarrow{\sim} \varepsilon_*\mathbb{Z}/m$

(b) $R^q\varepsilon_*\mathbb{Z}/m = 0$ $(q\geq 1)$ [at least in the limit over h].

Concretely, (a) means $S\setminus\{h=0\}$ connected $\Leftrightarrow$ $\operatorname{Spec}\mathcal{O}(S)\left[\tfrac{1}{h}\right]$ connected.

[consequence of GAGA-type theorems, due to Bingener in the compact case].

(b) means more or less:

"for $\alpha \in H^q(S\setminus\{h=0\}, \mathbb{Z}/m)$, there is an étale covering $X \to \operatorname{Spec}\mathcal{O}(S)\left[\tfrac{1}{h}\right]$ such that $\alpha|_{X^{an}} = 0$ in $H^q(X^{an}, \mathbb{Z}/m)$".

[compact case: complex analysis, Grauert's bump method.

surface case: commutative algebra of the ring $\mathcal{O}(S)$, argue degree by degree $(q\in\{0,1,2\})$]