Adams operations on hermitian forms

Nicolas Garrel

University of Antwerp

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Let K be a field of characteristic not 2.
$W(K)$: Witt ring of K
$GW(K)$: Grothendieck-Witt ring

Adams operations on $GW(K)$:

$$q = \langle a_1, \dots, a_n \rangle = \langle a_1 \rangle + \dots + \langle a_n \rangle.$$

Let $d \in \mathbb{N}^*$. The Adams operation ψ^d is:

$$\begin{aligned} \psi^d(q) &= (\langle a_1 \rangle)^d + \dots + (\langle a_n \rangle)^d \\ &= \langle a_1^d, \dots, a_n^d \rangle \\ &= \begin{cases} q & \text{if } d \text{ is odd} \\ n & \text{if } d \text{ is even.} \end{cases} \end{aligned}$$

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Adams operations on quadratic forms

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Let R be a commutative ring, and assume that $x \in R$ can be decomposed as a sum of "line elements":

$$x = \ell_1 + \cdots + \ell_n.$$

Then for any $d \in \mathbb{N}^*$:

$$\psi^d(x) = {\ell_1}^d + \cdots + {\ell_n}^d.$$

The idea of Adams operations

Let R be a commutative ring, and assume that $x \in R$ can be decomposed as a sum of "line elements":

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What is a line element?

Idea: from a decomposition $x = \ell_1 + \cdots + \ell_n$, one should be able to extract

$$f(x) = f(\ell_1, \ldots, \ell_n)$$

for every "symmetric function" f. The Adams operation ψ^d corresponds to $f(x_1, \ldots, x_n) = x_1^d + \cdots + x_n^d$ (Newton sum).

Elementary symmetric functions:

$$\lambda^d(x) = \sum_{i_1 < \cdots < i_d} \ell_{i_1} \cdots \ell_{i_d}.$$

Example: $\lambda^2(\ell_1 + \ell_2 + \ell_3) = \ell_1\ell_2 + \ell_1\ell_3 + \ell_2\ell_3$.

Then a line element satisfies $\lambda^d(x) = 0$ if $d > 1$.

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Pre-λ-rings

A pre-λ-ring structure on R is the data of

$$\lambda^d : R \to R$$

for $d \in \mathbb{N}$ such that $\lambda^0(x) = 1$, $\lambda^1(x) = x$, and

$$\lambda^d(x+y) = \sum_{p+q=d} \lambda^p(x)\lambda^q(y).$$

If $\lambda_t(x) = \sum_d \lambda^d(x)t^d \in GW(K)[[t]]$, then $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$.
An line element satisfies $\lambda^d(\ell) = 0$ for all $d > 1$, so $\lambda_t(\ell) = 1 + \ell t$.

$$\begin{aligned}
&\lambda^2(\ell_1+\ell_2+\ell_3+\ell_4)\\
&= \ell_1\ell_2 + \ell_1\ell_3 + \ell_1\ell_4 + \ell_2\ell_3 + \ell_2\ell_4 + \ell_3\ell_4\\
&= (\ell_1\ell_2) + (\ell_1+\ell_2)(\ell_3+\ell_4) + \ell_3\ell_4\\
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λ-rings

The previous axioms are not enough to ensure that the λ^d behave exactly like the elementary symmetric functions.

R is a λ-ring if certain additional axioms are satisfied, giving formulas for $\lambda^d(xy)$ and $\lambda^n(\lambda^m(x))$. If x and y are split (sums of line elements) they are satisfied.

$I \subset R$ is splittable if there is an embedding $R \to S$ such that the elements of I are split in I, and a product of line elements of S is a line element.

- If R is a λ-ring, every $I \subset R$ consisting of finite-dim elements is splittable.
- If R is additively generated by a set a splittable elements, it is a λ-ring.

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From λ to ψ

If R is a pre-λ-ring, one can define Adams operations ψ^d using the same equations that express Newton sums in terms of elementary symmetric polynomials. In practice:

$$\psi_{-t}(x) = -t\frac{(\lambda_t(x))'}{\lambda_t(x)}.$$

The ψ^d are always additive. If R is a λ-ring:

- ψ^d is a ring endomorphism.
- $\psi^m \circ \psi^m = \psi^{nm}$
- if p is prime, $\psi^p(x) \equiv x^p \mod p$.

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Endomorphism property

$$\begin{aligned}&\psi^d(\ell_1 + \cdots + \ell_n + \ell_{n+1} + \cdots + \ell_{n+m})\\ &= {\ell_1}^d + \cdots + {\ell_n}^d + {\ell_{n+1}}^d + \cdots + {\ell_{n+m}}^d\\ &= \psi^d(\ell_1 + \cdots + \ell_n) + \psi^d(\ell_{n+1} + \cdots + \ell_{n+m}).\end{aligned}$$

and

$$\begin{aligned}&\psi^d((\ell_1 + \cdots + \ell_n)(\ell'_1 + \cdots + \ell'_m))\\ &= \psi^d(\ell_1\ell'_1 + \cdots + \ell_n\ell'_m)\\ &= {\ell_1}^d{\ell'_1}^d + \cdots + {\ell_n}^d{\ell'_m}^d\\ &= ({\ell_1}^d + \cdots + {\ell_n}^d)({\ell'_1}^d + \cdots + {\ell'_m}^d)\\ &= \psi^d(\ell_1 + \cdots + \ell_n)\psi^d(\ell'_1 + \cdots + \ell'_m).\end{aligned}$$

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Composition property

$$
\begin{aligned}
&(\psi^n \circ \psi^m)(\ell_1 + \cdots + \ell_n) \\
&= \psi^n({\ell_1}^m + \cdots + {\ell_n}^m) \\
&= ({\ell_1}^m)^n + \cdots + ({\ell_n}^m)^n \\
&= {\ell_1}^{nm} + \cdots + {\ell_n}^{nm} \\
&= \psi^{nm}(\ell_1 + \cdots + \ell_n).
\end{aligned}
$$

Frobenius lifting property

$$\begin{aligned}
&\psi^{p}(\ell_1 + \cdots + \ell_n) \\
&= {\ell_1}^{p} + \cdots + {\ell_n}^{p} \\
&\equiv (\ell_1 + \cdots + \ell_n)^{p} \mod p.
\end{aligned}$$

Trace forms

Let B be a central simple algebra over K. The trace form of B is

$$\begin{array}{rccc} T_B : & B \times B & \longrightarrow & K \\ & (x,y) & \longmapsto & \mathrm{Trd}_B(xy) \end{array}$$

Example: if $B = M_n(K)$, $T_B = n \in W(K)$.

Let τ be an orthogonal involution of the first kind on B. The involution trace form is:

$$\begin{array}{rccc} T_\tau : & B \times B & \longrightarrow & K \\ & (x,y) & \longmapsto & \mathrm{Trd}_B(\tau(x)y) \end{array}$$

The restriction of T_τ to ε-symmetric elements is T_τ^ε.

$$T_\tau = T_\tau^+ + T_\tau^-$$

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Let (A, σ) be a central simple algebra with involution of the first kind over K, and $\varepsilon = \pm 1$. To simplify things, assume that σ is orthogonal and $\varepsilon = 1$, or that σ is sympectic and $\varepsilon = -1$.

One can define a pre-λ-ring

$$\widetilde{GW}^{\varepsilon}(A, \sigma) = GW(K) \oplus GW^{\varepsilon}(A, \sigma)$$

which is functorial with respect to hermitian Morita equivalences, and such that if h is an ε-hermitian form over (A, σ) with adjoint algebra with involution (B, τ) then

$$h^2 = T_\tau$$
$$\lambda^2(h) = \langle 2 \rangle T_\tau^- .$$

Mixed Witt ring of an algebra with involution

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Computation of ψ^d with d odd

If d is odd, $\psi^d : GW^\varepsilon(A,\sigma) \to GW^\varepsilon(A,\sigma)$.

By a theorem of Karpenko, the generic splitting map

$$GW^\varepsilon(A,\sigma) \to GW^\varepsilon(A_F,\sigma_F)$$

is injective, so one can compute ψ^d in a split algebra, where it comes down to the case of quadratic forms.

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Computation of ψ^2

In general, in a pre-λ-ring,

$$x^2 = \psi^2(x) + 2\lambda^2(x)$$
$$(\ell_1 + \cdots + \ell_n)^2 = {\ell_1}^2 + \cdots + {\ell_n}^2 + 2(\ell_1\ell_2 + \cdots + \ell_{n-1}\ell_n)$$

If h is an ε-hermitian form, in $W(K)$ we have:

$$\begin{aligned} h^2 = T_\tau &= T_\tau^+ - T_\tau^- \\ &= (T_\tau^+ + T_\tau^-) - 2T_\tau^- \\ &= T_B - 2\lambda^2(h) \end{aligned}$$

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Trace forms as twisted integers

Given a central simple algebra B, there is a unique element $\theta(B) \in GW(K)$ with

$$[\theta(B)] = [T_B] \in W(K)$$
$$\dim(\theta(B)) = \deg(B).$$

Given a Brauer class $\alpha \in \mathrm{Br}(K)$ and an integer $n \in \mathbb{N}^*$ divisible by $\mathrm{ind}(\alpha)$, there is a unique central simple algebra B, with $[B] = \alpha$ and $\deg(B) = n$, and we can define

$$[n]_\alpha = \theta(B) \in GW(K).$$

We can see $[n]_\alpha$ as a version of n, twisted by the Brauer class α. When $\alpha = 0$, $[n]_\alpha = n$. Then

$$\psi^2(h) = [\mathrm{rdim}(h)]_{[A]}.$$

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Brauer-Witt integers combinatorics

Given some CSA B of degree n and Brauer class α, and some $k \in \mathbb{N}$, we can define

$$\binom{B}{k} = \theta(\lambda^k(B)) = \left[\binom{n}{k}\right]_{k\alpha}$$

so in particular $\binom{B}{1} = \theta(B) = [n]_\alpha$, and

$$(1+t)^B = \sum_{k=0}^{n} \binom{B}{k} t^k \in GW(K)[t].$$

When B has exponent 2, then

$$\binom{B}{k} = \begin{cases} \binom{n}{k} & \text{if } k \text{ is even} \\ [\binom{n}{k}]_\alpha & \text{if } k \text{ is odd.} \end{cases}$$

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Other Adams operations

If $\widetilde{GW}^{\varepsilon}(A,\sigma)$ is a λ-ring, then

$$\psi^d(h) = \begin{cases} h & \text{if } d \text{ is odd} \\ \mathrm{rdim}(h) & \text{if } d \equiv 0 \mod 4 \\ [\mathrm{rdim}(h)]_{[A]} & \text{if } d \equiv 2 \mod 4 \end{cases}$$

Theorem

If $\mathrm{ind}(A) \leqslant 2$, *then* $\widetilde{GW}^{\varepsilon}(A,\sigma)$ *is a* λ*-ring.*

Products of λ-powers of hermitian forms

For a quadratic form q, the products $\lambda^i(q)\lambda^j(q)$ are $\mathbb{Z}$-linear combinations of the $\lambda^k(q)$. Precisely, if $\dim(q) = n = 2m$:

$$\lambda_u(q)\lambda_v(q) = \sum_k (1+uv)^{n-k}(u+v)^k\lambda^k(q)$$

$$= \sum_k (1+uv)^{2(m-k)}(u+v)^{2k}\lambda^{2k}(q)$$

$$+ \sum_{k \text{odd}} (1+uv)^{n-k}(u+v)^k\lambda^k(q).$$

In general, we can hope for a similar formula with coefficients in twisted integers. If $A = Q$ is a quaternion algebra and $\mathrm{rdim}(h) = n = 2m$, we have

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Products of λ-powers of hermitian forms

For a quadratic form q, the products $\lambda^i(q)\lambda^j(q)$ are $\mathbb{Z}$-linear combinations of the $\lambda^k(q)$. Precisely, if $\dim(q) = n = 2m$:

$$\begin{aligned}\lambda_u(q)\lambda_v(q) &= \sum_k (1+uv)^{n-k}(u+v)^k\lambda^k(q) \\ &= \sum_k (1+uv)^{2(m-k)}(u+v)^{2k}\lambda^{2k}(q) \\ &+ \sum_{k \text{odd}} (1+uv)^{n-k}(u+v)^k\lambda^k(q).\end{aligned}$$

In general, we can hope for a similar formula with coefficients in twisted integers. If $A = Q$ is a quaternion algebra and $\operatorname{rdim}(h) = n = 2m$, we have

$$\begin{aligned}\lambda_u(h)\lambda_v(h) &= \sum_k (1+uv)^{Q(m-k)}(u+v)^{2k}\lambda^{2k}(q) \\ &+ \sum_{k \text{odd}} (1+uv)^{n-k}(u+v)^k\lambda^k(q).\end{aligned}$$

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