Isotropy Indices of Pfister multiples in Characteristic 2 (joint with K. Zemková)

Squares in Dortmund

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Witt decomposition

- in characteristic 2, Witt decomposition is slightly more involved:

$$\varphi \cong \varphi_1 \perp \varphi_2 \perp i \times \mathbb{H} \perp j \times \langle 0 \rangle$$

  with
  - $\varphi_1 \cong [a_1, b_1] \perp \ldots \perp [a_r, b_r]$ *nonsingular*
    recall the notation $[a, b] : F^2 \to F, (x, y) \mapsto ax^2 + xy + by^2$.
  - $\varphi_2 \cong \langle c_1, \ldots c_s \rangle$ *totally singular* or *quasilinear* – unique up to isometry
  - $\varphi_1 \perp \varphi_2$ anisotropic
- *type*$(r, s) \in \mathbb{N}^2$ is unique;
- *Witt index* $i = \mathrm{i_W}(\varphi)$ and *defect* $j = \mathrm{i_d}(\varphi)$ are unique;
  the *anisotropic part* $\varphi_1 \perp \varphi_2$ is unique up to isometry
- *total isotropy index* $\mathrm{i_t}(\varphi) = \mathrm{i_W}(\varphi) + \mathrm{i_d}(\varphi)$

We have multiple ways to measure isotropy!

Main Question

- We consider the tensor product of a bilinear form $\mathfrak{b} = \langle a_1, \dots, a_k \rangle_b$ and a quadratic form φ
$$\mathfrak{b} \otimes \varphi = a_1\varphi \perp \dots \perp a_k\varphi$$
- Questions:
  - How are the different isotropy indices of $\mathfrak{b} \otimes \varphi$ and φ related to each other?
  - What about isotropy indices over field extensions?
    Of most interest: connections between $\mathfrak{i}_*(\varphi_{F(\psi)})$ and $\mathfrak{i}_*(\mathfrak{b} \otimes \varphi_{F(\mathfrak{b} \otimes \psi)})$ for another quadratic form ψ with $\mathfrak{i}_* \in \{\mathfrak{i}_W, \mathfrak{i}_d, \mathfrak{i}_t\}$.
    In particular, for $\psi = \varphi$ and $\mathfrak{i}_* = \mathfrak{i}_t$, we obtain the *first isotropy indices*
$$\mathfrak{i}_1(\varphi) = \mathfrak{i}_t(\varphi_{F(\varphi)}) \quad \textit{and} \quad \mathfrak{i}_1(\mathfrak{b} \otimes \varphi) = \mathfrak{i}_t(\mathfrak{b} \otimes \varphi_{F(\mathfrak{b} \otimes \varphi)})$$

    (Theorem by A. Laghribi:
    if φ is not totally singular, $\mathfrak{i}_1(\varphi) = \mathfrak{i}_W(\varphi_{F(\varphi)})$ and if φ is totally singular, $\mathfrak{i}_1(\varphi) = \mathfrak{i}_d(\varphi_{F(\varphi)})$)

Conventions

- We cannot expect deep results without constrains on our forms
- Motivated by J. O'Shea who asked the same question in characteristic $\neq 2$:
  We can expect results for n-fold bilinear Pfister form: defined as in characteristic $\neq 2$ as tensor product

  $$\langle\langle a_1, \ldots, a_n\rangle\rangle_b = \langle\langle a_1\rangle\rangle_b \otimes \ldots \otimes \langle\langle a_n\rangle\rangle_b$$

  where $\langle\langle a\rangle\rangle_b = \langle 1, a\rangle_b$
  (Don't worry about sign conventions here!)

Notation for the rest of the talk:

- φ a quadratic form of dimension $\dim(\varphi) \geqslant 2$ and of type (r, s)
- π an anisotropic n-fold bilinear Pfister form for some $n \in \mathbb{N}_+$

The base field

Over F, the situation is covered by the following well known result:

Proposition

If $\pi \otimes \varphi$ is isotropic, there are $k, \ell \in \mathbb{N}$ and a quadratic form φ_1 of type $(r-k, s-\ell)$ such that

$$\pi \otimes \varphi \cong \pi \otimes (\varphi_1 \perp k \times \mathbb{H} \perp \ell \times \langle 0 \rangle)$$

and $\pi \otimes \varphi_1$ is anisotropic.
In particular, we have

$$\mathrm{i_W}(\pi \otimes \varphi) = k \cdot \dim(\pi) \textit{ and } \mathrm{i_d}(\pi \otimes \varphi) = \ell \cdot \dim(\pi).$$

Proof: The proof is just a modification of the usual proof in characteristic $\neq 2$. □

The First Isotropy Index

If $\pi \otimes \varphi$ is anisotropic, we have

$$\mathfrak{i}_1(\pi \otimes \varphi) = \mathfrak{i}_t((\pi \otimes \varphi)_{F(\pi\otimes\varphi)}) = \mathfrak{i}_t(\pi_{F(\pi\otimes\varphi)} \otimes \varphi_{F(\pi\otimes\varphi)}),$$

and thus readily obtain $\mathfrak{i}_1(\pi \otimes \varphi) \geqslant \dim(\pi)$ by the above proposition.
This is the base case for the proof of the following:

Theorem

If $\pi \otimes \varphi$ is anisotropic, we have

$$\mathfrak{i}_1(\pi \otimes \varphi) \geqslant (\dim\pi)\, \mathfrak{i}_1(\varphi).$$

Proof: Because of the above, we may assume $\mathfrak{i}_1(\varphi) > 1$. We choose a dominated form $\varphi' \preccurlyeq \varphi$ (i.e. $\varphi' \cong \varphi|_U$ for some subspace U) of dimension $\dim(\varphi) - \mathfrak{i}_1(\varphi) + 1$. The dimension is big enough to assure that $\varphi'_{F(\varphi)}$ is isotropic and thus, $(\pi \otimes \varphi')_{F(\pi\otimes\varphi)}$ is isotropic by a Theorem of K. Zemková. Clearly, $(\pi \otimes \varphi)_{F(\pi\otimes\varphi')}$ is isotropic as well. It follows

$$\dim(\pi \otimes \varphi) - \mathfrak{i}_1(\pi \otimes \varphi) = \dim(\pi \otimes \varphi') - \underbrace{\mathfrak{i}_1(\pi \otimes \varphi')}_{\geqslant \dim \pi} \qquad \text{(key equation, B. Totaro)}$$

hence the assertion. □

Examples for (strict in-) equality?

Strict Inequality

Example

(a) Let $F = \mathbb{F}_2(a, b, t)$ with independent indeterminants a, b, t, $\varphi = [1, a] \perp t[1, b]$ and $\pi = \langle\langle a + b\rangle\rangle_b$.
Then $\mathfrak{i}_1(\varphi) = 1$, but one can show that

$$\pi \otimes \varphi \cong \langle\langle t, a + b\rangle\rangle_b \otimes [1, a]$$

is a quadratic Pfister form and thus has

$$\mathfrak{i}_1(\pi \otimes \varphi) = 4 > 2 \cdot 1 = \dim(\pi) \cdot \mathfrak{i}_1(\varphi).$$

(b) Similarly, for $\pi = \langle\langle b\rangle\rangle_b$ and $\varphi = \langle 1, a, t, abt\rangle$, we obtain an example for strict inequality with totally singular forms
(using that $\pi \otimes \varphi$ is a *quasi Pfister form* so that similar arguments apply).

Explicit example for equality will follow at the end of the talk, because this will be an example showing strictness for multiple inequalities.

Function fields of Quadrics

- We now would like to compare $\mathfrak{i}_*((\pi \otimes \varphi)_{F(\pi\otimes\psi)})$ with $\mathfrak{i}_*(\varphi_{F(\psi)})$ and hope for the analogous inequality

$$\mathfrak{i}_*((\pi \otimes \varphi)_{F(\pi\otimes\psi)}) \geqslant (\dim \pi)\mathfrak{i}_*(\varphi_{F(\psi)})$$

- While this is a natural generalization, the above techniques do not work anymore since there is no analogue for the *key equation* from above
- We do not show this in complete generality, but have to make assumptions on the types of φ, ψ.
- For our result, there are key reduction steps:
  - Reduction to anisotropic forms – technical, won't give further details on this part
  - Study isotropy behaviour over $F(\mu)$ where $\mu \preccurlyeq \psi$ – we will show that we find a suitable form of μ of dimension 2.

Lemma

For a form μ *with* $\dim\mu \geqslant 2$ *and* $\mu \preccurlyeq \psi$ *with* ψ *not totally singular, we have*

$$\mathfrak{i}_W(\varphi_{F(\mu)}) \geqslant \mathfrak{i}_W(\varphi_{F(\psi)}),$$
$$\mathfrak{i}_d(\varphi_{F(\mu)}) \geqslant \mathfrak{i}_d(\varphi_{F(\psi)}),$$
$$\mathfrak{i}_t(\varphi_{F(\mu)}) \geqslant \mathfrak{i}_t(\varphi_{F(\psi)}).$$

Proof:

- For the defect, this follows from results of Stephan Scully (which apply more generally to *quasilinear p-forms*)
- For the Witt index, since ψ is not totally singular, we have

$$\mathfrak{i}_W(\psi_{F(\mu)}) \geqslant \mathfrak{i}_W(\psi_{F(\psi)}) = \mathfrak{i}_1(\psi) > 0$$

  and hence, $F(\mu)(\psi)/F(\mu)$ is purely transcendental. We thus have

$$\mathfrak{i}_W(\varphi_{F(\mu)}) = \mathfrak{i}_W(\varphi_{F(\mu)(\psi)}) \geqslant \mathfrak{i}_W(\varphi_{F(\psi)}).$$

$\square$

Main Theorem

Theorem

Let φ and ψ be of dimension at least 2, and let π be an n-fold bilinear Pfister form.

(a) We have

$$\mathrm{i}_{\mathrm{d}}((\pi \otimes \varphi)_{F(\pi\otimes\psi)}) \geqslant (\dim \pi)\, \mathrm{i}_{\mathrm{d}}(\varphi_{F(\psi)}).$$

(b) Moreover, if φ is nonsingular and ψ is not totally singular, then we also have

$$\mathrm{i}_{\mathrm{W}}((\pi \otimes \varphi)_{F(\pi\otimes\psi)}) \geqslant (\dim \pi)\, \mathrm{i}_{\mathrm{W}}(\varphi_{F(\psi)}).$$

Proof (1/2):

- Both proofs work similarly and make use of similar lemmas for the respective isotropy indices.
- We use induction on n, the induction step being the easier part.
- For $n \geqslant 2$, write $\pi = \langle\langle a \rangle\rangle_b \otimes \pi'$. By using the case $n = 1$, we reduce to the calculation of $\mathrm{i}_*((\pi' \otimes \varphi)_{F(\pi'\otimes\psi)})$, which is covered by induction.

Proof (2/2):

- *For $n = 1$, i.e. $\pi = \langle 1, a\rangle_b$, we find linear independent vectors u, v over $E = F(\pi \otimes \psi)$ with $\psi(u) + a\psi(v) = 0$. Thus, for $\mu = \psi|_{\mathrm{span}(u,v)}$, we have $\mathrm{i}_*(\pi_E \otimes \mu) = 2$.*

  If μ is nonsingular or if we consider the defect, we can choose a bilinear form τ over E of dimension $\mathrm{i}_*(\varphi_{E(\mu)})$ such that $\tau \otimes \mu \preccurlyeq \varphi_E$. We obtain

$$\begin{aligned}\mathrm{i}_*((\pi \otimes \varphi)_E) &\geqslant \mathrm{i}_*(\pi_E \otimes \tau \otimes \mu)\\ &\geqslant \mathrm{i}_*(\pi_E \otimes \mu)\mathrm{dim}\tau = 2\mathrm{dim}\tau = 2\mathrm{i}_*(\varphi_{E(\mu)}) \geqslant 2\mathrm{i}_*(\varphi_{F(\psi)}).\end{aligned}$$

  If $\mu \cong \langle 1, d\rangle$ is totally singular and we consider the Witt index, we have

$$\varphi_E \cong \mathop{\perp}_{i=1}^{k} \left([a_i, b_i] \perp d[a_i, c_i]\right) \perp \varphi'$$

  with $k = \mathrm{i_W}(\varphi_{E(\mu)})$ and $\varphi'_{F(\sqrt{d})}$ anisotropic and use

$$\pi_E \otimes ([a_i, b_i] \perp d[a_i, c_i]) \cong \langle 1, d\rangle \otimes [a_i, b_i + c_i] \perp 2 \times \mathbb{H}\,.$$

$\square$

Some remarks about the assumptions

We want to clarify why we needed the restrictions on φ and ψ in part (b) of the above Theorem in contrast to part (a).

- We needed ψ to be not totally singular so that we can apply the above lemma – recall that we used a chain of inequalities

$$\mathrm{i_W}(\psi_{F(\mu)}) \geqslant \mathrm{i_W}(\psi_{F(\psi)}) = \mathrm{i}_1(\psi) > 0$$

  where the first inequality only holds for ψ not totally singular
- Further, we needed φ to be nonsingular since only for nonsingular forms, we find a decomposition related to the Witt index resp. defect r over a quadratic extension E/F where $E = F(\wp^{-1}(d))$ resp. $E = F(\sqrt{d})$:
  Recall that we have

$$\varphi \cong \varphi' \perp \tau \otimes [1, d] \quad \textit{resp.} \quad \varphi \cong \mathop{\perp}_{i=0}^{r} \big([a_i, b_i] \perp d[a_i, c_i]\big) \perp \varphi'$$

  with $\dim(\tau) = r$ and φ'_E anisotropic.
- We did not need similar assumptions for our results about the defect, since the defect only depends on the quasilinear part of a quadratic form and we were thus able to reduce to the case of totally singular forms

Theorem

Let φ, ψ be quadratic forms over F as above, let $K = F((X_1))\dots((X_n))$ be the field of iterated Laurent series and $\pi = \langle\langle X_1, \dots, X_n\rangle\rangle_b$.

(a) If φ is nonsingular and ψ is not totally singular, then we have

$$\mathrm{i_W}((\pi \otimes \varphi)_{K(\pi\otimes\psi)}) = (\dim \pi)\, \mathrm{i_W}(\varphi_{F(\psi)}).$$

(b) For arbitrary quadratic forms φ and ψ, we have

$$\mathrm{i_d}((\pi \otimes \varphi)_{K(\pi\otimes\psi)}) = (\dim \pi)\, \mathrm{i_d}(\varphi_{F(\psi)}).$$

(c) For an arbitrary quadratic form φ, we have

$$\mathrm{i_1}(\pi \otimes \varphi) = (\dim \pi)\, \mathrm{i_1}(\varphi).$$

Proof: Subtle calculations with our lemmas and using that for $\varphi \cong \varphi_0 \perp X\varphi_1$ with φ_0, φ_1 defined over F, we have

$$\mathrm{i_W}(\varphi) = \mathrm{i_W}(\varphi_0) + \mathrm{i_W}(\varphi_1) \quad \text{and} \quad \mathrm{i_d}(\varphi) = \mathrm{i_d}(\varphi_0) + \mathrm{i_d}(\varphi_1).$$

□

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