Orthogonal Representations of Finite Groups

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Group algebras

- G finite group, K field
- $KG = \bigoplus_{g \in G} Kg$ group algebra
- $\mathbb{Q}G \cong \bigoplus_{i=1}^{h} A_i$ with $A_i \cong D_i^{n_i \times n_i}$ semisimple algebra
- $K_i := Z(D_i)$ abelian number fields, conductor divides $|G|$
- D_i division algebra $\dim_{K_i}(D_i) = m_i^2$
- D_i has uniformly distributed invariants
- $m_i n_i$ divides $|G| = \sum_{i=1}^{h} [K_i : \mathbb{Q}](m_i n_i)^2$

Natural involution

- $\iota : \mathbb{Q}G \to \mathbb{Q}G, \sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g g^{-1}$.
- $\iota(A_i) = A_i$, $\iota_i := \iota_{|A_i}$
- K_i real $\Rightarrow \iota_i$ involution of first kind,

  then $m_i \in \{1, 2\}$ and $m_i = 2$ if ι_i symplectic.
- K_i not real ι_i involution of second kind.

Determine invariants of ι_i

Invariant forms

- $\rho_i : G \to A_i^\times$ group homomorphism
- $\chi_i : G \to K_i, g \mapsto \mathrm{trace}(\rho_i(g))$ character,
  $K_i = \mathbb{Q}(\chi_i)$ character field
  χ_i constant on conjugacy classes, $\chi_i(1) = n_i m_i$
- Frobenius Schur indicator $\mathrm{ind}(\chi_i) \in \{+, o, -\}$

\+ if K_i is real and $\rho_i(G)$ stabilises a quadratic form Q_i

\- if K_i is real and $\rho_i(G)$ stabilises a symplectic form S_i

o if K_i is complex, then $\rho_i(G)$ stabilises a Hermitian form H_i
$F_i := \mathrm{Fix}_{K_i}(\iota_i)$

$\mathcal{F}(\rho_i) = \{aQ_i \mid a \in K_i\}$ resp. $= \{aH_i \mid a \in F_i\}$
space of $\rho_i(G)$-invariant forms.
Invariants of ι_i are the invariants of Q_i resp. H_i that are independent of scaling.

Discriminants for even character degree

ι_i orthogonal

$\mathcal{F}(\rho_i) = \{aQ_i \mid a \in K_i\}$, $\rho_i : G \to O(Q_i)$ orthogonal
$\mathrm{disc}(aQ_i) = a^{\chi_i(1)} \mathrm{disc}(Q_i)$
so $\mathrm{disc}(\iota_i) \in K_i^\times/(K_i^\times)^2$ well defined, if and only if $\chi_i(1)$ even.

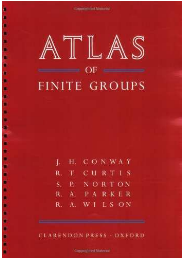
$$\mathrm{Irr}^+(G) := \{\chi \in \mathrm{Irr}(G) \mid \mathrm{ind}(\chi) = + \text{ and } \chi(1) \text{ even }\}$$

ι_i of second kind

$\mathcal{F}(\rho_i) = \{aH_i \mid a \in F_i\}$, $\rho_i : G \to U(H_i)$ unitary
$\mathrm{disc}(\iota_i) \in F_i^\times/N_{K_i/F_i}(K_i^\times)$ well defined, if and only if $\chi_i(1)$ even.

$$\mathrm{Irr}^o(G) := \{\chi \in \mathrm{Irr}(G) \mid \mathrm{ind}(\chi) = o \text{ and } \chi(1) \text{ even }\}$$

Determine discriminants for the characters in $\mathrm{Irr}^+(G)$ and $\mathrm{Irr}^o(G)$ for all but the largest few ATLAS groups.



- Building blocks of finite groups:
  finite simple groups
- alternating groups
- classical groups
  linear, symplectic, unitary, orthogonal groups over finite fields
- 26 sporadic simple groups: Matthieu groups ... Monster
- ATLAS of finite groups
  ordinary character tables of finite simple groups
  classifying simple $\mathbb{Q}G$-modules

The character table of A_7

		1a	2a	3a	3b	4a	5a	6a	7a	7b
X.1	+	1	1	1	1	1	1	1	1	1
X.2	+	6	2	3	.	.	1	-1	-1	-1
X.3	o	10	-2	1	1	.	.	1	A	B
X.4	o	10	-2	1	1	.	.	1	B	A
X.5	+	14	2	2	-1	.	-1	2	.	.
X.6	+	14	2	-1	2	.	-1	-1	.	.
X.7	+	15	-1	3	.	-1	.	-1	1	1
X.8	+	21	1	-3	.	-1	1	1	.	.
X.9	+	35	-1	-1	-1	1	.	-1	.	.

$$A = (-1+\sqrt{-7})/2, B = (-1-\sqrt{-7})/2$$

The character table of A_7

	OD		1a	2a	3a	3b	4a	5a	6a	7a	7b
X.1		+	1	1	1	1	1	1	1	1	1
X.2	-7	+	6	2	3	.	.	1	-1	-1	-1
X.3	-1	o	10	-2	1	1	.	.	1	A	B
X.4	-1	o	10	-2	1	1	.	.	1	B	A
X.5	-3	+	14	2	2	-1	.	-1	2	.	.
X.6	-15	+	14	2	-1	2	.	-1	-1	.	.
X.7		+	15	-1	3	.	-1	.	-1	1	1
X.8		+	21	1	-3	.	-1	1	1	.	.
X.9		+	35	-1	-1	-1	1	.	-1	.	.

$$A = (-1 + \sqrt{-7})/2, B = (-1 - \sqrt{-7})/2$$

Orthogonal stability

A character χ is called orthogonal if there is a representation ρ with character χ admitting a non-degenerate invariant quadratic form Q. Then $\rho : G \to O(Q)$.
An orthogonal character χ is called orthogonally stable if there is a square class $d(\mathbb{Q}(\chi)^\times)^2$ such that for all representations $\rho : G \to \mathrm{GL}_n(L)$ with character χ and all non-degenerate quadratic forms $Q \in \mathcal{F}(\rho)$

$$\mathrm{disc}(Q) = d(L^\times)^2.$$

χ orthogonally stable, then

$$\mathrm{disc}(\chi) := d(\mathbb{Q}(\chi)^\times)^2$$

is called the orthogonal discriminant of χ.

The discriminant of a quadratic form

- B non-degenerate symmetric bilinear form on V
- adjoint involution ι_B on $\mathrm{End}(V)$

$$B(\alpha(v), w) = B(v, \iota_B(\alpha)(w)) \text{ for all } v, w \in V.$$

$$E_-(B) := \{\alpha \in \mathrm{End}_K(V) \mid \iota_B(\alpha) = -\alpha\}$$

- basis $(v_1, \dots, v_n)$, $\mathrm{End}(V) \cong K^{n\times n}$, $B := (B(v_i, v_j)) \in K^{n\times n}$
- $\iota_B(A) = BA^{tr}B^{-1}$ and $E_-(B) = \{BX \mid X = -X^{tr}\}$ as
- $\iota_B(BX) = B(BX)^{tr}B^{-1} = BX^{tr}$.
- $X = -X^{tr}$ then $\det(X)$ is a square.

Proposition (Knus, Parimala, Sridharan 1991)

$\dim(V)$ even $\Leftrightarrow E_-(B) \cap \mathrm{GL}(V) \neq \{\}$.
Then $\det(B) = \det(\alpha)(K^\times)^2$ for any invertible $\alpha \in E_-(B)$.

Computing the orthogonal discriminant

Proposition (Knus, Parimala, Sridharan 1991)

$\dim(V)$ even $\Rightarrow E_-(B) \cap \mathrm{GL}(V) \neq \{\}$.
Then $\det(B) = \det(\alpha)(K^\times)^2$ for any invertible $\alpha \in E_-(B)$.

Theorem (GN 22)

χ is orthogonally stable, if and only if all its absolutely irreducible indicator + constituents have even degree.

- $Q \in \mathcal{F}(\rho)$ non-degenerate, $\rho(G) \leq O(Q)$, $n := \dim(\rho)$ even
- $\iota_Q(g) = g^{-1}$ for all $g \in \rho(G)$
- Take three random elements g, h, k in $\rho(G)$
- compute $X = X(g, h, k) = g - g^{-1} + h - h^{-1} + k - k^{-1}$
- If $\det(X) \neq 0$ then
  (a) χ_ρ is orthogonally stable and
  (b) $\mathrm{disc}(\chi_\rho) = (-1)^{n/2} \det(X)(\mathbb{Q}(\chi)^\times)^2$.

The character table of J_2

;		@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	;	;	@	@	@	@	@	@	@	@	@	@	@
		604800	1 920	240	1 080	36	96	300	300	50	50	24	12	7	8	20	20	10	10	12	15	15			336	48	12	6	48	16	6	6	7	12	12	
p power			A	A	A	A	A	A	A	A	A	AA	BB	A	A	BB	AB	DA	CA	AA	BA	AA			A	A	B	BC	A	A	AB	BC	AC	AB	AB	
p' part			A	A	A	A	A	A	A	A	A	AA	BB	A	A	AB	BB	CA	DA	AA	AA	BA			A	A	A	BC	A	A	AB	BC	AC	AB	AB	
ind		1A	2A	2B	3A	3B	4A	5A	B*	5C	D*	6A	6B	7A	8A	10A	B*	10C	D*	12A	15A	B*	fus	ind	2C	4B	4C	6C	8B	8C	12B	12C	14A	24A	B*	
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	:	++	1	1	1	1	1	1	1	1	1	1	1	χ_1
χ_2	+	14	-2	2	5	-1	2	-3b5	*	b5+2	*	1	-1	0	0	b5	*	-b5	*	-1	0	0	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_2
χ_3	+	14	-2	2	5	-1	2	*	-3b5	*	b5+2	1	-1	0	0	*	b5	*	-b5	-1	0	0														χ_3
χ_4	+	21	5	-3	3	0	1	b5+4	*	-2b5	*	-1	0	0	-1	b5	*	0	0	1	*	-b5	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_4
χ_5	+	21	5	-3	3	0	1	*	b5+4	*	-2b5	-1	0	0	-1	*	b5	0	0	1	-b5	*														χ_5
χ_6	+	36	4	0	9	0	4	-4	-4	1	1	1	0	1	0	0	0	-1	-1	1	-1	-1	:	++	6	-2	0	0	2	2	1	0	-1	-1	-1	χ_6
χ_7	+	63	15	-1	0	3	3	3	3	-2	-2	0	-1	0	1	-1	-1	0	0	0	0	0	:	++	7	3	-1	1	-3	1	0	-1	0	0	0	χ_7
χ_8	+	70	-10	-2	7	1	2	-5b5	*	0	0	-1	1	0	0	-b5	*	0	0	-1	b5	*	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_8
χ_9	+	70	-10	-2	7	1	2	*	-5b5	0	0	-1	1	0	0	*	-b5	0	0	-1	*	b5														χ_9
χ_{10}	+	90	10	6	9	0	-2	5	5	0	0	1	0	-1	0	1	1	0	0	1	-1	-1	:	++	6	2	0	0	4	0	-1	0	-1	1	1	χ_{10}
χ_{11}	+	126	14	6	-9	0	2	1	1	1	1	-1	0	0	0	1	1	-1	-1	-1	1	1	:	++	0	4	0	0	2	-2	1	0	0	-1	-1	χ_{11}
χ_{12}	+	160	0	4	16	1	0	-5	-5	0	0	0	1	-1	0	-1	-1	0	0	0	1	1	:	++	8	0	2	-1	0	0	0	-1	1	0	0	χ_{12}
χ_{13}	+	175	15	-5	-5	1	-1	0	0	0	0	3	1	0	-1	0	0	0	0	-1	0	0	:	++	7	-1	1	1	-1	-1	-1	1	0	-1	-1	χ_{13}
χ_{14}	+	189	-3	-3	0	0	-3	-3b5	*	b5+2	*	0	0	0	1	b5	*	b5	*	0	0	0	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_{14}
χ_{15}	+	189	-3	-3	0	0	-3	*	-3b5	*	b5+2	0	0	0	1	*	b5	*	b5	0	0	0														χ_{15}
χ_{16}	+	224	0	-4	8	-1	0	2r5-1	-2r5-1	2b5	*	0	-1	0	0	1	1	0	0	0	*	-b5	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_{16}
χ_{17}	+	224	0	-4	8	-1	0	-2r5-1	2r5-1	*	2b5	0	-1	0	0	1	1	0	0	0	-b5	*														χ_{17}
χ_{18}	+	225	-15	5	0	3	-3	0	0	0	0	0	-1	1	-1	0	0	0	0	0	0	0	:	++	1	-3	-1	1	3	-1	0	-1	1	0	0	χ_{18}
χ_{19}	+	288	0	4	0	-3	0	3	3	-2	-2	0	1	1	0	-1	-1	0	0	0	0	0	:	++	8	0	-2	-1	0	0	0	1	1	0	0	χ_{19}
χ_{20}	+	300	-20	0	-15	0	4	0	0	0	0	1	0	-1	0	0	0	0	0	1	0	0	:	++	6	-2	0	0	-2	-2	1	0	-1	1	1	χ_{20}
χ_{21} ($Q_{2,3}$)	+	336	16	0	-6	0	0	-4	-4	1	1	-2	0	0	0	0	0	1	1	0	-1	-1	:	++	0	0	0	0	0	0	0	0	0	r6	-r6	χ_{21}
ind		1 2	2 2	4	3 6	3 6	4 4	5 10	5 10	5 10	5 10	6 6	12	7 14	8 8	20	20	10 10	10 10	12 12	15 30	15 30	fus	ind	4	4	8 8	12	8 8	8	12 12	24 24	28 28	24 24	24 24	
χ_{22}		6	2	0	-3	0	2	-2b5	*	*	b5-1	1	0	-1	0	0	0	-b5	*	-1	b5	*	|	-	0	0	0	0	0	0	0	0	0	0	0	χ_{22}
																																			χ_{23}

The character table of J_2

- Rational Schur index of χ_{21} is 2, $A_{21} = \left(\frac{2,3}{\mathbb{Q}}\right)^{168 \times 168}$
- No rational representation with character χ_{21}
- But over $\mathbb{F}_p$, $p \geq 7$, there is an orthogonal representation of degree 336 with character χ_{21}.

- Compute $\mathrm{disc}(\chi_{21})$:
- ρ a 672 dimensional rational representation affording $2\chi_{21}$

- Choose $g, h, k \in \rho(J_2)$, compute $X = X(g, h, k) = g - g^{-1} + h - h^{-1} + k - k^{-1} \in \rho(E_{-}(\mathbb{Q}J_2))$.
- $\mathrm{disc}(\chi_{21}) = N_{red}(X)(\mathbb{Q}^{\times})^2$.

The discriminant field extension

- $d(K^\times)^2$ determines $K[\sqrt{d}]/K$.
- $\chi \in \mathrm{Irr}^+(G)$, $K := \mathbb{Q}(\chi)$
- $\mathrm{disc}(\chi) = d(K^\times)^2 \Rightarrow \Delta(\chi) := K[\sqrt{d}]$ discriminant field
- $\chi \pmod{\wp}$ orthogonally stable $\Rightarrow \wp$ unramified in $\Delta(\chi)$
  Then $\wp$ inert $\Leftrightarrow \mathrm{disc}(\chi \pmod{\wp})$ not a square.
- primes that ramify in $\Delta(\chi)/\mathbb{Q}(\chi)$ divide the group order.
- a priori finite list of possibilities for $\mathrm{disc}(\chi)$
- Determine $\mathrm{disc}(\chi)$ by reducing it modulo enough primes (not dividing the group order)

$\Delta(\chi)/\mathbb{Q}$ is not always Galois.

E.g. $G = J_1$, $\chi(1) = 56$, $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{5})$,

$$\mathrm{disc}(\chi) = (31 + 5\sqrt{5})/2,\ \mathrm{Gal}(\Delta(\chi))/\mathbb{Q} \cong D_8$$

2

ordinaries 20 Dec 2021

	11	19	29	31	41	59		13	17
14	0−	0+	0−	0+	0−	0−	−3	x	x
14	0−	0+	0−	0+	0−	0−	−3	x	x
36	0+	0+	0+	0+	0+	0+	~~✓~~ 5	0−	0−
70	0−	0+	0−	0+	0−	0−	−3	x	x
70	0−	0+	0−	0+	0−	0−	−3	x	x
90	0+	0−	0+	0−	0−	0−	−7	0−	0−
126	0−	0−	0+	0−	0+	0−	~~−1~~ −5	0−	0−
160	0+	0+	0+	0+	0+	0+	1	0+	0+
224	0+	0+	0+	0+	0+	0+	1	x	x
224	0+	0+	0+	0+	0+	0+	1	x	x
288	0−	0−	0−	0−	0+	0+	~~21~~ 105	0+	0−
300	0−	0−	0−	0−	0+	0+	21	0−	0+
336	0+	0+	0+	0+	0+	0+	1	0+	0+

lab 63 175 189_{ab} 225

(1 12) (2 11) (4 9) (8 5) (3 10) (6 7) 13

<Prime> = 1 + k63 + k36 | e2 $k36 = 224_{ab} + 70_{ab} + 21_{ab}$

$\Sigma^-(G) := \langle g - g^{-1} \mid g \in G \rangle \leq \mathbb{Z}G$.

χ orthogonally stable $\Leftrightarrow$ there is $X \in \Sigma^-(G)$ such that $\det(\rho(X)) \neq 0$ for any representation ρ affording χ. Then

$$\mathrm{disc}(\chi) = (-1)^{\chi(1)/2} \det(\rho(X))(\mathbb{Q}(\chi)^\times)^2.$$

$H \leq G$ such that $\chi_{|H}$ orthogonally stable $\Leftrightarrow$ $\exists$ such $X \in \Sigma^-(H)$.

A simple algebra with orthogonal involution ι.
$\Sigma^-(A) := \{a \in A \mid a = -\iota(a)\}$.
Subalgebra $B \leq A$ orthogonally stable if and only if
(a) $\iota(B) = B$ and (b) $\Sigma^-(B) \cap A^\times \neq \emptyset$.
Then

$$\mathrm{disc}(\iota) = \mathrm{disc}(\iota_{|B}).$$

Parker's conjecture

Discriminants of rational orthogonally stable characters are odd.
If $\mathrm{disc}(\chi) = d(K^\times)^2$ then $\nu(d)$ is even for all dyadic valuations ν of K.

Parker's conjecture holds

- for the ATLAS groups up to HN
  order $|HN| = 2^{14}3^6 5^6 7 \cdot 11 \cdot 19 = 273,030,912,000,000$
  largest $\chi(1) = 5,103,000$, $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{5})$
- for characters of the form $\psi + \overline{\psi}$ where ind$(\psi) =$o (Navarro, Tiep, Isaacs, Liebeck)
- for solvable groups (GN)
- $\mathrm{SL}_2(q)$, $\mathrm{SL}_3(q)$, $\mathrm{SU}_3(q)$ (all q, OB,LH,GN),
- for all Coxeter groups (Linda Hoyer)
- for all groups $\mathrm{GL}_n(q)$, $\mathrm{G}_2(q)$ with q odd (Linda Hoyer)

No counterexamples to Parker's conjecture so far.

Hermitian forms

Discriminant of quaternion algebra

- K field, $\sigma \in \mathrm{Aut}(K)$ of order 2, $F := \mathrm{Fix}_K(\sigma)$, $K = F[\sqrt{-\delta}]$
- $d \in F^{\times}$, quaternion algebra

$$(K,d)_F := \langle 1, i, j, k \mid i^2 = -\delta, j^2 = d, ij = -ji = k \rangle_F$$

- $\mathrm{disc}_K([(K,d)_F]) =: dN_{K/F}(K^{\times})$ K-discriminant of $[(K,d)_F]$.

Discriminant of Hermitian form

- $H : V \times V \to K$ Hermitian form
- $H_B := (H(b_i, b_j))_{i,j=1}^{n} \in K^{n \times n}$, $B = (b_1, \ldots, b_n)$ an K-basis of V
- $\mathrm{disc}(H) := (-1)^{\binom{n}{2}} \det(H_B) N_{K/F}(K^{\times})$ discriminant of H.
- $\Delta(H) := [(K,d)_F] \in \mathrm{Br}_2(K,F)$ discriminant algebra of H.
- $\chi \in \mathrm{Irr}^o(G)$, well defined $\mathrm{disc}(\chi)$ and $\Delta(\chi)$
- Primes that ramify in $\Delta(\chi)$ do divide the group order.

Orthogonal subalgebras

$\mathfrak{A}$ central simple K-algebra of even degree $2m$
ι involution of second kind, $F := \mathrm{Fix}_K(\iota)$.

A F-subalgebra A of $\mathfrak{A}$ is called an orthogonal subalgebra of $(\mathfrak{A}, \iota)$ if

(a) A is a central simple F-algebra with $KA = \mathfrak{A}$.

(b) A is invariant under ι, i.e. $\iota(A) = A$.

(c) The restriction of ι to A is an orthogonal involution of A.

(V, H) Hermitian space of dimension $2m$

B an orthogonal basis, so H_B diagonal matrix.

$\mathfrak{A} = \mathrm{End}(V) = K^{2m\times 2m}$, $\iota = \iota_H$.

Then $A := F^{2m\times 2m}$ is an orthogonal subalgebra.

$\mathrm{disc}(H) = \mathrm{disc}(\iota_A) N_{K/F}(K^{\times})$.

Orthogonal subalgebras

- $K = F[\sqrt{-\delta}]$,
- $V = Kb_1 \oplus Kb_2$, $d \in F^\times$, $H_B := \mathrm{diag}(1, d) \in K^{2\times 2}$.
- $\mathrm{disc}(H) = \mathrm{disc}(H_B) = -dN_{K/F}(K^\times)$
- $$\mathfrak{Q} = \langle i := \mathrm{diag}(\sqrt{-\delta}, -\sqrt{-\delta}), j := \begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix} \rangle_F \subseteq K^{2\times 2}$$
- adjoint involution ι_H restricts to orthogonal involution $\iota_{\mathfrak{Q}}$ of $\mathfrak{Q}$.
- $\iota(i) = -i$, so $\mathrm{disc}(\iota_{\mathfrak{Q}}) = -\det(i)(F^\times)^2 = -\delta(F^\times)^2$.
- $\mathrm{disc}(\iota_H) = -dN_{K/F}(K^\times)$
- $\delta \in N_{K/F}(K^\times)$

$$\mathrm{disc}(H_B) = \mathrm{disc}(\iota_H) = \mathrm{disc}(\iota_{\mathfrak{Q}})\,\mathrm{disc}_K([\mathfrak{Q}])$$

Unitary discriminant from orthogonal subalgebra

Theorem (GN 24)

Let (V, H) be a Hermitian K-space of even dimension $2m$, $\mathfrak{A} := \mathrm{End}_K(V)$, and $\iota = \iota_H$ the adjoint involution of the non-degenerate Hermitian form H. Let A be an orthogonal subalgebra of $\mathfrak{A}$.

(a) $[A] \in \mathrm{Br}_2(K, F)$.

(b) $\mathrm{disc}(H) = \mathrm{disc}_K([A])^m \,\mathrm{disc}(\iota_{|A}) \in F^\times / N_{K/F}(K^\times)$.

Proof: Choose suitable orthogonal basis and use formula above for each orthogonal summand.

Fixed algebras of certain group automorphisms are orthogonal subalgebras.

Fixed algebras

- $\chi \in \mathrm{Irr}^o(G)$, $\rho : G \to \mathrm{GL}_{2m}(K)$ representation affording χ
- $\alpha \in \mathrm{Aut}(G)$ with $\alpha^2 = 1$ and $\chi \circ \alpha = \overline{\chi}$.
- $A = \langle \rho(g) + \rho(\alpha(g)) : g \in G \rangle_F = \mathrm{Fix}(\alpha)$ is orthogonal subalgebra
- $\Sigma^-(A) = \langle \rho(g) + \rho(\alpha(g)) - (\rho(g^{-1}) + \rho(\alpha(g^{-1}))) : g \in G \rangle_F$
- $X \in \Sigma^-(A) \cap A^\times$ then the α-discriminant of χ is

$$\mathrm{disc}^\alpha(\chi) = (-1)^m \det(X)(F^\times)^2.$$

Theorem

$\mathrm{disc}(\chi) = \mathrm{disc}^\alpha(\chi)\,\mathrm{disc}_K([A])^m$.

$\mathrm{disc}^\alpha(\chi)$ is a square class of F
hence can be obtained by enough reductions modulo primes
$\mathrm{disc}_K([A])$ is obtained from Schur indices of the induced character of χ to the semidirect product $G : \langle \alpha \rangle$.

Conclusion

- Computed the orthogonal discriminants of the characters in

$$\mathrm{Irr}^+(G) = \{\chi \in \mathrm{Irr}(G) \mid \mathrm{ind}(\chi) = +, \chi(1) \text{ even } \}.$$

for the ATLAS groups up to HN.
(joint with Richard Parker and Thomas Breuer)

- Computed the unitary discriminants of the characters in

$$\mathrm{Irr}^o(G) = \{\chi \in \mathrm{Irr}(G) \mid \mathrm{ind}(\chi) = +, \chi(1) \text{ even } \}.$$

for the ATLAS groups up to HN. (joint with David Schlang)

$\Sigma^-(G) := \langle g - g^{-1} \mid g \in G \rangle \leq \mathbb{Z}G$.

χ orthogonally stable $\Leftrightarrow$ there is $X \in \Sigma^-(G)$, $\det(\rho(X)) \neq 0$ for any representation ρ affording χ. Then

$$\det(\chi) = \det(\rho(X))(\mathbb{Q}(\chi)^\times)^2.$$

Parker's conjecture

$X \in \Sigma^-(G) \Rightarrow \nu(\det(\rho(X)))$ even for all dyadic valuations ν.