Totally positive field extensions and Pythagorean index

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Definitions and Notations related to quadratic forms

Throughout this talk by a quadratic form we always mean a *non-degenerate* quadratic form.

Definition (Isotropic quadratic forms)

A quadratic form q defined over F is said to be *isotropic* if there exists a nonzero vector v such that $q(v) = 0$.

Definition (Weakly isotropic quadratic forms)

A quadratic form q defined over F is said to be *weakly isotropic* if there exists $r \in \mathbb{N}$ such that $r \cdot q = \underbrace{q \perp q \perp \cdots \perp q}_{r\text{-times}}$ is isotropic.

Definitions and Notations related to real fields

We collect some definitions and notations.

Definition (Real fields)

A field F is said to be *formally real* or *a real field* if -1 is not a sum of squares in F.

Definition (Semiordering)

A proper subset S of F is said to be a *semiordering* on F if it satisfy the following conditions.

$$1 \in S;\ F^2 \subseteq S;\ S + S \subseteq S,\ S \cup -S = F;\ S \cap -S = \{0\}$$

Definition (Ordering)

An *ordering* P on F is a semiordering with $P \cdot P \subseteq P$.

Totally positive field extensions

Definition (Totally positive field extension)

A field extension K/F is said to be *totally positive* if *every semiordering* of F extends to a semiordering of K.

We have the following criterion due to K. J. Becher [1]

Equivalent criterion

Following statements are equivalent.

1. K/F is a totally positive field extension;
2. if quadratic form q over F becomes isotropic over K then, q is weakly isotropic over F

[1] Becher, *Totally positive extensions and weakly isotropic forms*, Manuscripta Math.

Pythagorean fields

Definition

The field F is said to be *pythagorean* if $F^2 + F^2 \subset F^2$, i.e., $\sum F^2 = F^2$.

Definition

Pythagorean closure There exists a smallest pythagorean subfield of F_{al} that contains F. This field is called the *pythagorean closure* of F and is denoted by F_{py}.

Explicitly, let $\mathcal{F}$ be the family of extensions K/F, $K \subset F^{\mathrm{a}}$, for which there exists a tower

$$F = K_0 \subset K_1 \subset \cdots \subset K_n = K,$$

such that $K_{i+1} = K_i(\sqrt{1 + a_i^2}),$ where $a_i \in K_i$. The compositum of all the fields in $\mathcal{F}$ is F_{py}.

Examples of totally positive field extensions

All fields are assumed to be real fields.

- A quadratic extension $F(\sqrt{d})/F$ is totally positive if and only if d is a sum of squares in F.
- F_{py}/F is totally positive, where F_{py} is the pythagorean closure of F.
- An odd degree field extension is totally positive.
- $F(X)/F$ is totally positive, where $F(X)$ is the rational function field over F in one variable.
- $F((X))/F$ is totally positive, where $F((X))$ is the Laurent series field in one variable over F.

Examples...

- Let K/F be a totally positive field extension. If E/F is a subfield of K/F then, E/F is totally positive. However, in general, K/E need not be totally positive. For instance, consider

$$\mathbb{Q} \subset \mathbb{Q}\left(\sqrt{2}\right) \subset \mathbb{Q}\left(\sqrt[4]{2}\right)$$

- If E/F is a subfield of F_{py}/F then, F_{py}/E is also totally positive.
- If L/E and E/F are both totally positive then, L/F is totally positive.

Example...

The following example is due to K. J. Becher[2]

Function field of weakly isotropic form

The function field of a weakly isotropic quadratic form over F is a totally positive field extension of F.

Proof

Let q be a weakly isotropic quadratic form. So, q is isotropic over F_{py}, and hence, $F_{\mathrm{py}}(q)/F$ is totally positive. As $F(q)/F$ is a subextension of $F_{\mathrm{py}}(q)/F$, we get the result.

[2] *Totally positive extensions and weakly isotropic forms*, Manuscripta Math.

Example...

Generic splitting field

A generic splitting field of an *even* dimensional quadratic form q is totally positive field extension if and only if q is torsion.

Proof

Recall that the generic splitting tower of q is constructed as follows. Let $q_0 = (q)_{\mathrm{an}}$, and $q_i = (q_{i-1})_{\mathrm{an}}$, and let $F_i = F_{i-1}(q_{i-1})$. Thus we get the following tower of fields.

$$F = F_0 \subset F_1 \subset \cdots \subset F_h = K$$

The field K is called the generic splitting field of q.
As $\dim q_h \leq 1$ and $\dim q$ is even, K-anisotropic part of q is zero.
If K/F is totally positive then, we get that $\operatorname{sgn}_P(q) = 0$ for every ordering P of F i.e., q is torsion.
The converse follows from the earlier examples.

A question of K. J. Becher on the pythagorean index

We begin with the definition.

Definition (Pythagorean index)

Let F be a real field and let A be a central simple algebra over F. The pythagorean index of A is

$$\mathrm{pind}(A) = \mathrm{ind}(A \otimes F_{\mathrm{py}}).$$

We recall the following question/conjecture due to K. J. Becher [3].

Question

If K/F is a totally positive field extension and A a central simple algebra over F of exponent 2 then, $\mathrm{pind}(A) = \mathrm{pind}(A \otimes K)$.

In this talk we show that the above question has an affirmative answer when K/F is Galois totally positive and $\mathrm{pind}(A) = 4$.

[3] *Totally positive extensions and weakly isotropic forms*, Manuscripta Math.

We begin with the following lemma.

Lemma

Let K/F be a totally positive field extension. Then, $K_{\mathrm{py}}/F_{\mathrm{py}}$ is totally positive if and only if following condition is satisfied:

$$(*) \quad LE/E \text{ is totally positive for any finite subextensions } L/K \text{ and } E/F \text{ with } L \subset K_{\mathrm{py}} \text{ and } E \subset F_{\mathrm{py}}.$$

Proposition

Let K/F be a totally positive finite Galois extension of formally real fields. Then $KF_{\mathrm{py}}/F_{\mathrm{py}}$ is a totally positive extension.

The proof of the above proposition rely on the theorem of Becher, Leep, and Schubert[4], viz., *a Galois extension is totally positive if and only if it preserves the ordering*.

[4] *Semiorderings and stability index under field extensions*, Israel J. Math.

We now state the theorem.

Theorem

Let K/F be a totally positive Galois extension of formally real fields. Then $K_{\mathrm{py}}/F_{\mathrm{py}}$ is a totally positive extension.

Sketch of the proof.

We first make some reductions.

- It is enough to show that $KF_{\mathrm{py}}/F_{\mathrm{py}}$ is totally positive.
- Any finitely generated field extension L/F_{py} of $KF_{\mathrm{py}}/F_{\mathrm{py}}$ is totally positive.
- We can assume L/F_{py} is finite Galois, and show that L/F_{py} is totally positive.

$\square$

Theorem

Let K/F be a totally positive Galois extension of real fields. Suppose that A is a central simple algebra of exponent 2 and $\mathrm{pind}(A) = 4$. Then, $\mathrm{pind}(A \otimes K) = 4$.

Proof.

Let q be an Albert form associated to the underlying biquaternion division algebra of $A \otimes F_{\mathrm{py}}$. If $\mathrm{pind}(A \otimes K) < 4$ then, q becomes isotropic over K_{py}. As $K_{\mathrm{py}}/F_{\mathrm{py}}$ is totally positive, q is weakly isotropic over F_{py} and hence isotropic over F_{py}, a contradiction. □

Thank you!