Bounds on the Pythagoras number and indecomposables in biquadratic fields

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Pythagoras number

- let $\mathcal{O}$ be a commutative ring
- $\sum \mathcal{O}^2 = \left\{ \sum_{i=1}^{n} \alpha_i^2;\ \alpha_i \in \mathcal{O}, n \in \mathbb{N} \right\}$
- $\sum^m \mathcal{O}^2 = \left\{ \sum_{i=1}^{m} \alpha_i^2;\ \alpha_i \in \mathcal{O} \right\}$
- the Pythagoras number of the ring $\mathcal{O}$ is

$$\mathcal{P}(\mathcal{O}) = \inf \left\{ m \in \mathbb{N} \cup \{+\infty\};\ \sum \mathcal{O}^2 = \sum^m \mathcal{O}^2 \right\}.$$

Pythagoras number for orders in totally real fields

K totally real field, $\mathcal{O}_K$ the ring of algebraic integers in K,
$\mathcal{O} \subseteq \mathcal{O}_K$ order of K

- $\mathcal{P}(\mathcal{O}) \leq f(d)$ where f depends on the degree d of K;
  $\mathcal{P}(\mathcal{O}) \leq d+3$ for $2 \leq d \leq 5$
- $\mathcal{P}(\mathcal{O})$ can attain arbitrarily large values (Scharlau, 1980)
- $\mathcal{P}(\mathcal{O}) = 5$ for all orders in real quadratic fields up to finitely many exceptions (Peters, 1973)
- some partial results for cubic fields and real biquadratic fields

Indecomposable integers

$\mathcal{O}_K^+$ the set of all totally positive elements in $\mathcal{O}_K$

Definition

We say that $\alpha \in \mathcal{O}_K^+$ is indecomposable in $\mathcal{O}_K$ if it cannot be written as $\alpha = \beta + \gamma$ for any $\beta, \gamma \in \mathcal{O}_K^+$.

- Only one indecomposable integer in $\mathbb{Z}$, namely 1.
- The element $2 + \sqrt{2}$ is indecomposable in $\mathbb{Q}(\sqrt{2})$.
- Up to multiplication by totally positive units, there are only finitely many indecomposable integers in $\mathcal{O}_K$.
- They can be used to the study of universal quadratic forms or the Pythagoras number.

Results on totally positive indecomposable integers

- We know the precise structure of indecomposable integers in quadratic fields $\mathbb{Q}(\sqrt{D})$, where they can be described using the continued fraction of $\sqrt{D}$ or $\frac{\sqrt{D}-1}{2}$. (Perron, 1913; Dress, Scharlau, 1982)
- some partial results for biquadratic fields (Čech, Lachman, Svoboda, T., Zemková, 2019; Krásenský, T., Zemková, 2020; Man, 2024; Kala, Man, 2025)
- full determination for some families of cubic fields (Kala, T., 2023, T., 2023; Gil-Muñoz, T., 2025)

$\mathfrak{s}$-indecomposable integers

- $\sigma_1, \ldots, \sigma_d$ embeddings of K into $\mathbb{C}$
- $\alpha \in K$, signature of α is

$$\mathfrak{s}(\alpha) = (\mathrm{sgn}(\sigma_1(\alpha)), \ldots, \mathrm{sgn}(\sigma_d(\alpha)))$$

- $\mathcal{O}_K^{\mathfrak{s}}$ the set elements in $\mathcal{O}_K$ with signature $\mathfrak{s}$
- we say that $\alpha \in \mathcal{O}_K^{\mathfrak{s}}$ is $\mathfrak{s}$-indecomposable if it cannot be written as $\alpha = \beta + \gamma$ where $\beta, \gamma \in \mathcal{O}_K^{\mathfrak{s}}$

$\mathfrak{s}$-indecomposable integers

- If there exists a unit in $\mathcal{O}_K$ with signature $\mathfrak{s}$, all $\mathfrak{s}$-indecomposable integers are associated with totally positive indecomposable integers.
- In real quadratic fields, they can be obtained from the continued fraction of $\sqrt{D}$ or $\frac{\sqrt{D}-1}{2}$.
- We know all $\mathfrak{s}$-indecomposable integers for Ennola's family of cubic fields (Kala, Sgallová, T., 2025).

Method for determination of the lower bound on $\mathcal{P}(\mathcal{O})$ using $\mathfrak{s}$-indecomposable integers

Ingredients:

- knowlege of all $\mathfrak{s}$-indecomposable integers in $\mathcal{O}$
- sharp estimates on conjugates of elements in a chosen integral basis and of fundamental units

Method:

- choose $\gamma \in \mathcal{O}$ so that γ can be written as a sum of many squares
- find all $\mathfrak{s}$-indecomposable integers α in $\mathcal{O}$ so that $\gamma \succeq \alpha^2$
- from these $\mathfrak{s}$-indecomposable integers, construct all $\mathfrak{s}$-decomposable integers $\omega \in \mathcal{O}$ so that $\gamma \succeq \omega^2$
- for the finite set of elements ω, discuss how many squares one need to express γ

Results for cubic fields

- $\mathcal{P}(\mathbb{Z}[\rho]) = 6$ where ρ is a root of the polynomial $x^3 - ax^2 - (a+3)x - 1$ with $a \geq 3$ (T., 2023)
- $\mathcal{P}(\mathcal{O}_K) = 6$ where ρ is a root of the polynomial $x^3 - ax^2 - (a+3)x - 1$ with $a > 12$, $a \equiv 3, 21 \pmod{27}$ and $\frac{a^2+3a+9}{27}$ square-free (Gil-Muñoz, T., 2025)
- $\mathcal{P}(\mathbb{Z}[\rho]) = 6$ where ρ is a root of the polynomial $x^3 + (a-1)x^2 - ax - 1$ with $a \geq 4$ (Kala, Sgallová, T., 2025)

Biquadratic fields

- $K = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ where $p, q > 1$ are distinct square-free integers
- different integral bases depending on values $p, q \pmod 4$.

Example

$p \equiv 2 \pmod 4$, $q \equiv 3 \pmod 4$

$$\mathcal{O}_K = \mathbb{Z}\left[1, \sqrt{p}, \sqrt{q}, \frac{\sqrt{p} + \sqrt{\frac{pq}{\gcd(p,q)^2}}}{2}\right]$$

Pythagoras number for biquadratic fields

we have $\mathcal{P}(\mathcal{O}_K) \leq 7$

Theorem (Krásenský, Raška, Sgallová, 2022)

If $p \equiv 2 \pmod 4$, $q \equiv 3 \pmod 4$, $p, q > 7$ and p and q are coprime and square-free, then $\mathcal{P}(\mathcal{O}_K) = 7$.

Theorem (Krásenský, Raška, Sgallová, 2022; He, Hu, 2025+)

Let K be a real biquadratic field containing $\sqrt{2}$ or $\sqrt{5}$. Then $\mathcal{P}(\mathcal{O}_K) \leq 5$.

Pythagoras number for biquadratic fields

Theorem (Krásenský, Raška, Sgallová, 2022)

Let K be a real biquadratic field. Fix a square-free $n > 7$. Then $\mathcal{P}(\mathcal{O}_K) \geq 6$ for all but finitely many totally real biquadratic fields such that $\sqrt{n} \in K$.

Theorem

Let K be a real biquadratic field not containing $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, and $\sqrt{13}$. Then $\mathcal{P}(\mathcal{O}_K) \geq 6$.

Theorem

Let $n \geq 6$ be a rational integer such that $p = (2n-1)(2n+1)$, $q = (2n-1)(2n+3)$ and $r = (2n+1)(2n+3)$ are square-free integers, and let $K = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. Then $\mathcal{P}(\mathcal{O}_K) = 7$.

Norm of indecomposable integers

- for every totally real number field K, there exists a constant B_K such that $N(\alpha) \leq B_K$ for every indecomposable integer $\alpha \in \mathcal{O}_K$ (Brunotte, 1983)
- one can take $B_K = \Delta_K$ where Δ_K is the discriminant of the field K (Kala, Yatsyna, 2023)
- sharper estimates on this bound for real quadratic fields (Dress, Scharlau, 1982; Jang, Kim, 2016; T., Voutier, 2020)
- the best possible bounds for several families of cubic fields (T., 2023; Gil-Muñoz, T., 2025)

Minimal (codifferent) trace of indecomposable integers

- let $\mathcal{O}_K^\vee = \{\delta \in K, \mathrm{Tr}(\alpha\delta) \in \mathbb{Z} \text{ for all } \alpha \in \mathcal{O}_K\}$ be the codifferent of K
- if $\mathcal{O}_K = \mathbb{Z}[\rho]$, f minimal polynomial of ρ

$$\Rightarrow \mathcal{O}_K^\vee = f'(\rho)^{-1}\mathbb{Z}[\rho]$$

- if for $\alpha \in \mathcal{O}_K^+$, there exists $\delta \in \mathcal{O}_K^{\vee,+}$ such that $\mathrm{Tr}(\alpha\delta) = 1$, then α is indecomposable in $\mathcal{O}_K$
- $\min \mathrm{Tr}(\alpha) = \min_{\delta \in \mathcal{O}_K^{\vee,+}} \mathrm{Tr}(\alpha\delta)$ for $\alpha \in \mathcal{O}_K$

Minimal (codifferent) trace of indecomposable integers

- Real quadratic fields: $\min \mathrm{Tr}(\alpha) = 1$ for all $\alpha \in \mathcal{O}_K$ indecomposable (Kala, T., 2023)
- Cubic fields:
  - $\min \mathrm{Tr}(\alpha) = 1$ for all indecomposable integers in the simplest cubic fields with $\mathcal{O}_K = \mathbb{Z}[\rho]$ up to one exception with $\min \mathrm{Tr}(\alpha) = 2$ (Kala, T., 2023)
  - $\min \mathrm{Tr}(\alpha) = 2$ for all indecomposable integers in Ennola's cubic fields up to one exception with $\min \mathrm{Tr}(\alpha) = 1$ (T., 2023)
  - $\min \mathrm{Tr}(\alpha)$ for α indecomposable can attain arbitrarily large values for orders in cubic fields (T., 2023)

- Let $n \geq 6$ be a rational integer such that $p = (2n-1)(2n+1)$, $q = (2n-1)(2n+3)$ and $r = (2n+1)(2n+3)$ are square-free integers, and let $K = \mathbb{Q}(\sqrt{p}, \sqrt{q})$.
- Man (2024) determined all indecomposable integers in K.

Theorem

Let α be an indecomposable integer in $\mathcal{O}_K$. Then

- $N(\alpha) \leq 16n^4 + 64n^3 + 16n^2 - 96n + 36$,
- $\min \operatorname{Tr}(\alpha\delta) \leq 2$.

Thank you for your attention.