Tittle: Positive semidefinite elements and sums of squares in local excellent henselian rings

Abstract: A classical problem in Real Geometry (which goes back to Hilbert's 17th problem) concerns the representation of positive semidefinite elements (psd) of a ring A as sums of squares (sos) of elements of A. If $A = K$ is a field and $\frac{1}{2} \in A$, this problem has always a positive solution (psd=sos) due mainly to Artin-Scheier theory of (formally) real fields, which was developed with a view towards obtaining a solution to Hilbert's 17th problem. However, the situation is in general radically different when A is a (commutative unital) ring (with non-empty real spectrum), because strong dimensional restrictions appear and, apart from (formally) real fields, there are few real rings with the property psd=sos. This makes this type of rings special.

We endow the local rings $(A, \mathfrak{m})$ with the $\mathfrak{m}$-adic topology and denote the set of sums of squares of A with ΣA^2. If A is in addition excellent and henselian, the set $\mathcal{P}(A)$ of psd elements of A is closed in A. Thus, if $\mathcal{P}(A) = \Sigma A^2$, then ΣA^2 is a closed subset of A, that is, it coincides with its closure $\mathrm{Cl}(\Sigma A^2)$. If A is a local excellent ring of real dimension ≥ 3, it is already known that the difference $\mathcal{P}(A) \setminus \mathrm{Cl}(\Sigma A^2) \neq \varnothing$. In addition, if A is a local henselian excellent ring (with non-empty real spectrum) such that $\mathcal{P}(A) = \mathrm{Cl}(\Sigma A^2)$, then A is real reduced and consequently its real and Krull dimensions coincide. Thus, local excellent henselian rings A with the property $\mathcal{P}(A) = \mathrm{Cl}(\Sigma A^2)$ have Krull dimension ≤ 2.

In this talk we determine the local excellent henselian rings A of embedding dimension ≤ 3 such that $\frac{1}{2} \in A$ and $\mathcal{P}(A) = \mathrm{Cl}(\Sigma A^2)$ (and surprisingly only Klein's singularities (and their limits) appear). If ΣA^2 is a closed subset of A (which happens for instance if the Pythagoras number of A is finite), this provides a full characterization of the local excellent henselian rings A of embedding dimension ≤ 3 such that $\frac{1}{2} \in A$ and $\mathcal{P}(A) = \Sigma A^2$.