Sums of squares in function fields of curves over $\mathbb{R}((t_1))\dots((t_n))$

Gonzalo Manzano-Flores

Oberseminar Algebra und Geometrie

Abstract

Ernst Witt showed in 1934 that every sum of squares is a sum of two squares in every algebraic function field F over $\mathbb{R}$ (equivalently, the pythagoras number of F denoted by $p(F)$, is equal to 2). It is a natural question whether we can bound $p(F)$ also for algebraic function fields over a field with properties similar to $\mathbb{R}$. For example, we can consider the case where the base field K is a hereditarily pythagorean field, such as $K_n = \mathbb{R}((t_1))\dots((t_n))$, for some $n \in \mathbb{N}$.

Consider an algebraic function field F/K_n. It was shown by J. Van Geel, K. Becher and D. Grimm that $2 \leq p(F) \leq 3$ and that $G(F) = (\sum F^2)^\times/(F^2+F^2)^\times$ is a finite group, which controls the failure of a sum of squares to be a sum of 2 squares. It is natural to ask for the precise value of $p(F)$ and for the size of $G(F)$. In this talk, I will put the focus on algebraic function fields of the form $F = K_n(X)(\sqrt{f})$, where $f \in K_n[X]$ is a square-free polynomial (the function field of the hyperelliptic curve $C : Y^2 = f(X)$). Taking $g \in \mathbb{N}$ such that $\deg(f) = 2g+1$ or $\deg(f) = 2g+2$ (the genus of F/K_n) I will show that $|G(F)| \leq 2^{n(g+1)}$ and I will give some examples indicating that this bound is optimal.