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Differential Geometry explained easily: A new teaching concept

Abstract

This article describes a new method to teach differential geometry in a way which is more intuitive, more appealing to students, and which can help students to understand crucial concepts of differential geometry better.

Introduction

Differential geometry (DG) is a topic which is popular among mathematics students, including those studying mathematics education. However, until now, it was commonly thought that DG is suitable only for those students who are well advanced in their studies, and too difficult for others. This is because the traditional way of teaching DG requires several semesters of analysis and linear algebra. This discourages students who are less strong in these subjects from taking a DG course. Also, some students who are interested in DG might not be able to attend all of the required prerequisite courses; e.g., this regularly applies to students who are not studying mathematics as their main major.

Here I propose a new method of teaching DG which uses fewer prerequisites but nonetheless allows to teach several crucial and interesting topics. In particular, the following concepts can easily be taught this way: geodesics, curvature, and the Gauss-Bonnet theorem.

Note that the method proposed here does not merely help struggling students succeed in learning some DG concepts (although the method does accomplish that), but it is also a method which is valuable for stronger students since it illustrates some important and useful aspects of geometry which remain hidden in standard DG courses.

The method contains the following elements: visual demonstrations; metric geometry; curvature bounds; applications (including the Gauss-Bonnet theorem); and several standard topics of curves and surfaces.

Thoughts on visual demonstrations in differential geometry

DG is highly suitable for dealing with shapes that can actually be seen, and the author strongly believes that a course on DG should contain many visual demonstrations of such shapes. Computer software is available that allows to show and interactively manipulate curves and surfaces and will provide the course with interesting visual experiments in the classroom. These should be part of the course.

A note for readers of this article who are about to write a book or course notes on DG: a book should also contain high-quality graphics of the presented material wherever possible. Sadly, in the past it was common to limit book illustrations to the occasional sketch. Nowadays, easy-to-use software is available for making high-quality illustrations. Books and classes have slightly different requirements on software (classroom teaching requires interaction and quick response, whereas for a picture in a book, fine resolution is important), but software available for both tasks.

Teaching differential geometry via metric geometry

Metric geometry is an easy but powerful method to understand concepts from DG. It was developed at the St. Petersburg school of mathematics by A. D. Alexandrov. Metric geometry uses just the notion of *metric space* (i.e., distance function); this is an easy definition in a few lines. From this, various mathematical concepts can be derived. However, it does not use the properties of the underlying space unless derived from the axioms of the distance function. This is important: E.g. when studying Euclidean geometry, it is customary to formulate Euclid's axioms and derive properties from those. This can be a bit tedious. The axioms of a metric space are probably an easier set of axioms to work with.

Geodesics

Geodesics are the most important type of curves in DG. In Euclidean space, they are straight lines. On a sphere, geodesics are *great circles*. On a (smooth) curved surface, they are curves which are “as straight as possible”. They also have something to do with minimizing length (more precisely, are shortest connections between points on them, as long as those points are close together). However, even the precise definition of this (intuitive) concept of “as straight as possible” is cumbersome in traditional DG, because the traditional definition requires the *covariant derivative*, which in turn requires the *Levi-Civita connection*, which requires very good knowledge of differentiation. This makes the classical definition appear rather late in DG textbooks, and this hinders students from understanding this important class of curves in the beginning of the course.

On the other hand, defining a geodesic with metric geometry is very easy. The aforementioned notion of minimizing length between (nearby) points on a curve is easily defined; all that is required is to define the length of a curve via distance of its points (and a limit). This allows the topic of geodesics to be introduced early in the DG course.

Curvature

Curvature is one of the most important concepts in DG. Unfortunately, the curvature of a manifold is usually defined in a technical way which makes it extremely difficult for students to grasp its meaning. In Riemannian geometry, first a *curvature tensor* is defined. This is problematic for students who do not yet have a thorough understanding of linear algebra. Moreover, this tensor has many components, and only a few of them are understood. To get from this tensor to more meaningful concepts of curvature, such as actual numbers, another nontrivial method is needed to derive the *sectional curvature(s)*. As a result, even on very simple surfaces it is not easy to figure out even very basic properties of the curvature, such as its sign. From a teaching perspective, this is very problematic.

For (2-dimensional) surfaces embedded in 3-dimensional Euclidean space, there is an easier method to define curvature, namely via *principal curvatures*. This approach is already noticeably more intuitive. However, to understand this definition still requires good knowledge of linear algebra.

The author has studied which method of explaining curvature to students works best and has found that yet another method works much better in a classroom setting. The method deals with curvature bounds in metric spaces. This is explained in the following.

Using just the definition of distances, it is possible to derive angles of (some) intersecting curves. Neither tangent vectors nor differentiation is required.

Next, for any interior metric space, it is possible to compare each triangle (i.e. 3 points and the geodesics joining them) to a triangle in the Euclidean plane with the same side lengths. If the angles of the triangle in the metric space all have angles not bigger than those of the *comparison triangle*, then the former is defined to be of non-positive curvature. Similarly, if (in small triangles) all angles are not smaller, then the curvature is defined to be non-negative. This is a very short and intuitive definition.

If in the preceding paragraph the Euclidean space is replaced by a sphere, then the aforementioned upper or lower curvature bound is not zero but the curvature of the sphere. And if it is replaced by hyperbolic space, then the curvature bound is a negative number. By changing the radius of the sphere, its curvature can be made any positive number (it is actually the inverse of the radius). Students do not have to be able to compute the curvatures of spheres; all that is required is to understand that increasing the radius will decrease the curvature. Something similar applies to hyperbolic space, hence providing the concept of arbitrary curvature bounds.

Introducing hyperbolic space is more difficult that defining the sphere. The author found that the intuitive geometric approach with the *Poincare disc* (and its isometries) works well in a classroom setting.

The Gauss-Bonnet theorem

One of the most advanced topics covered in one semester of DG is the theorem of Gauss-Bonnet. It is a theorem about (2-dimensional) surfaces which nicely connects the concepts of geodesics, angles, and curvature. With the methods described here, it is possible to cover this theorem (including proof) in a DG course where the students are not necessarily strong when dealing with analysis or linear algebra. This can be done with the following procedure:

- Prove the Gauss-Bonnet theorem on the sphere (of radius 1). There is a nice intuitive proof for that which is very visual and very short. Students like it. Then explain how the quantities curvature, area, length, and angle change when the radius of the sphere varies. This is easy.

- Prove the Gauss-Bonnet theorem in hyperbolic space. This can be done with intuitive (although not at all trivial) observations of properties of hyperbolic isometries. Then, just as for the sphere, explain how quantities change when the geometry is rescaled.

- This proves the theorem for constant curvature surfaces. It is not just as abstract proof, but students will really understand it.

- Using integration, the theorem can be derived for surfaces with variable curvature.

Note

This method is not intended to do away with classical DG. All topics traditionally taught can still be taught after using this method, and at that point students will benefit from the insight gained.

References

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