Globalübung, 09.07.2014

Nachtrag Klausur 1, Aufgabe 9

$F(x,y) = 4xy$

NB: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$(x_1,y_1) = \left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ $(x_2,y_2) = \left(\frac{-a}{\sqrt{2}}, \frac{-b}{\sqrt{2}}\right)$

$(x_3,y_3) = \left(\frac{a}{\sqrt{2}}, \frac{-b}{\sqrt{2}}\right)$ $(x_4,y_4) = \left(-\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$

$F(x_1,y_1) = F(x_2,y_2) = 2ab$

$F(x_3,y_3) = F(x_4,y_4) = -2ab$

Da $M := \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$ kompakt, d.h.
F nimmt auf M sein Max./Min. an

Aus Vorl. wissen wir, dass Lagrange Max und Min liefert, falls diese existieren,

d.h. $2ab$ ist Max

$-2ab$ ist Min

$\Rightarrow$ F nimmt Max. in $(x_1,y_1), (x_2,y_2)$ an

■

Klausur 2, 2009

Aufgabe 1

(a) Es gilt:

$$\|x\|_\infty = \max_{k=1\ldots n} |x_k| = \sqrt{\max_{k=1\ldots n} |x_k|^2} \leq \ldots \leq \sqrt{\sum_{k=1}^{n} x_k^2} = \|x\|_2$$

$$\|x\|_2 = \left(\sum_{k=1}^{n} x_k^2\right)^{1/2} \leq \left(\sum_{k=1}^{n} \max_{i=1\ldots n} |x_i|^2\right)^{1/2}$$

$$\leq \sqrt{n} \left(\max_{i=1\ldots n} |x_i|^2\right)^{1/2} = \sqrt{n} \max_{i=1\ldots n} |x_i|$$

$$= \sqrt{n} \, \|x\|_\infty$$

(b) durch $\lightning$:

Ang. exist. Norm $\|\cdot\|$ sodass $\|x-y\| = 1$ $\forall x \neq y$

Dann: $1 = \|x-y\| = \|2x-2y\|$

$= 2\|x-y\|$

$= 2$ $\lightning$

Aufgabe 2

(a) Sei $\tilde{d}$ triviale Metrik & $d(x,y) := 2\tilde{d}(x,y)$.

Dann: $d(x,y) = 2 \Leftrightarrow x \neq y$

& $d(x,y) = 0 \Leftrightarrow \tilde{d}(x,y) = 0 \Leftrightarrow x = y$

und: $d(x,y) = 2\tilde{d}(x,y) = 2\tilde{d}(y,x) = d(y,x)$

und: $d(x,y) = 2\tilde{d}(x,y) \leq 2(\tilde{d}(x,z) + \tilde{d}(z,y))$

$= d(x,z) + d(z,y)$

(b)
$$Df = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2x_2 & 0 \\ 0 & 0 & 3x_3^2 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) $f(x,y) = x^3 \Rightarrow \nabla f(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\forall \varepsilon > 0 : f(-\frac{\varepsilon}{2}, 0) < 0 < f(\frac{\varepsilon}{2}, 0)$

$\Rightarrow$ kein Extr. in $(0,0)$

Alternativ: $f(x,y) = x^4 - y^4$

(d) $X = C^0([0,1], \mathbb{R})$, $A = \{f \in X \mid f(0) \geq 0\}$

Sei $(f_n)_{n \in \mathbb{N}}$ Folge in A sd. $f_n \to f$ bzgl. sup-Norm

$\Rightarrow f_n(0) \to f(0)$ in $\mathbb{R}$ da $f_n(0) \geq 0$ $\forall n$

$\Rightarrow f(0) \geq 0 \Rightarrow f \in A \Rightarrow A$ ist abg.

Aus Vorl. folgt $f_n \to f$ glm. & $[0,1]$ kompakt, also f stetig.

Aufgabe 3

$$f(x,y) := \begin{cases} \frac{\cos x \sin y}{x^2+y^4}, & \text{falls } (x,y) \neq (0,0) \\ 0, & \text{falls } (x,y) = (0,0) \end{cases}$$

(a) f <u>nicht</u> stetig:

Sei $(y_n)_{n\in\mathbb{N}}$ Folge in $\mathbb{R}$ $y_n \to 0$, $y_n > 0$

$$f(0,y_n) = \underbrace{\frac{\sin y_n}{y_n}}_{\to 1} \cdot \underbrace{\frac{1}{y_n^3}}_{\to \infty} \longrightarrow \infty$$

$\Rightarrow$ f nicht stetig

(b) $$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-0}{h} = \ldots = \lim_{h\to 0} \frac{\sin h}{h^5} = \infty$$

$\Rightarrow \frac{\partial f}{\partial y}(0,0)$ ex. nicht

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$\blacksquare$

Aufgabe 4

siehe Kl. 1

Aufgabe 5

$F(x,y) := \frac{1}{2}y^3 - 3xy + 2 = 0$

$F(1,2) = 0$

Es gilt: $\frac{\partial F}{\partial x} = -3y$ $\quad \frac{\partial F}{\partial y} = \frac{3}{2}y^2 - 3x$

$\Rightarrow \frac{\partial F}{\partial y}(1,2) = 3 \neq 0$

$F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ sind stetig (insbes. F, DF, stetig)

Satz ü. impl. Fkt.en

$\Rightarrow \exists$ Umg. U von 1 & $u: U \to \mathbb{R}$ st. diffbar

sd. $u(1)=2$ und $F(x, u(x)) = 0 \quad \forall x \in U$

$$\frac{\partial u}{\partial x}(1) = -\left(\frac{\partial F}{\partial y}(1,2)\right)^{-1} \quad \frac{\partial F}{\partial x}(1,2) = 2$$

<u>Aufgabe 6</u>

$f: \mathbb{R}^2 \setminus \{(0,y) \mid y \in \mathbb{R}\} \to \mathbb{R}, \ (x,y) \mapsto e^{x-y} \ln x^2$

$$\frac{\partial f}{\partial x}(x,y) = \left(2\ln x + \frac{2}{x}\right) e^{x-y}$$

$$\frac{\partial f}{\partial y}(x,y) = -2e^{x-y} \ln x$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \left(2\ln x + \frac{4}{x} - \frac{2}{x^2}\right) e^{x-y}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = 2e^{x-y} \ln x$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \left(-2\ln x - \frac{2}{x}\right) e^{x-y}$$

$$f(1,1) = 0 \ ; \quad \frac{\partial f}{\partial x}(1,1) = 2 \ , \ \frac{\partial f}{\partial y}(1,1) = 0 \ ,$$

$$\frac{\partial^2 f}{\partial x^2}(1,1) = 2 \ , \ \frac{\partial^2 f}{\partial y^2}(1,1) = 0 \ , \ \frac{\partial^2 f}{\partial x \partial y}(1,1) = -2$$

$$\Rightarrow T^2_{(1,1)} f(x,y) = 2(x-1) - 2(x-1)(y-1) + (x-1)^2$$

$$T^2_{(1,1)} f(x,y) = f(1,1) + \nabla f(1,1) \cdot \begin{pmatrix} x-1 \\ y-1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x-1 \\ y-1 \end{pmatrix} D^2 f(1,1) \begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$$

<u>Aufgabe 7</u>

$$f(x,y) = (x^2+y^2) e^{x+y}$$

$$\nabla f(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{aligned} 2x + x^2 + y^2 &= 0 \\ 2y + x^2 + y^2 &= 0 \end{aligned}$$

$$\Leftrightarrow \quad x = y$$

$$x = y \Rightarrow (x_1, y_1) = (0,0) \text{ oder } (x_2, y_2) = (-1,-1)$$

$$\frac{\partial^2 f}{\partial x^2} = (2+4x+x^2+y^2)\, e^{x+y}$$

$$\frac{\partial^2 f}{\partial y^2} = (2+4y+x^2+y^2)\, e^{x+y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = (2x+2y+x^2+y^2)\, e^{x+y}$$

$$D^2 f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{pos. def.}$$

$\Rightarrow$ f hat lok. Min in $(0,0)$

$$D^2 f(-1,-1) = e^{-2} \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \quad \text{hat}$$

EW: $-2e^{-2}$ mit EV $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

EW: $2e^{-2}$ mit EV $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

1 pos. und 1 neg. EW $\Rightarrow$ f hat in $(-1,-1)$ weder Max noch Min. ■

Aufgabe 8

$$y'(t) = \frac{y(t)}{t} + e^{y(t)/t} \qquad \Big|\, : t \geq 1$$

$$\frac{y'(t)}{t} - \frac{y(t)}{t^2} = \frac{1}{t}\, e^{y(t)/t}$$

$\tilde{y}(t) := y(t)/t$,

$$\tilde{y}'(t) = \frac{1}{t}\, e^{\tilde{y}(t)} \qquad \Big|\cdot e^{-\tilde{y}(t)}$$

$$\frac{\tilde{y}'(t)}{e^{\tilde{y}(t)}} = \frac{1}{t} \qquad \Big|\int_1^t$$

$$\int_1^t \frac{\tilde{y}'(s)}{e^{\tilde{y}(s)}}\, ds = \int_1^t \frac{1}{s}\, ds$$

$$\Rightarrow \quad -e^{-\tilde{y}(s)}\Big|_1^t = \ln s \Big|_1^t$$

$$-e^{-\tilde{y}(t)} + \underbrace{e^{-\tilde{y}(1)}}_{e^{-y(1)/1} = e^{-1} = \frac{1}{e}} = \ln t$$

$$\exp(-\tilde{y}(t)) = \frac{1}{e} - \ln t$$

$$\tilde{y}(t) = -\ln\left(\frac{1}{e} - \ln t\right)$$

$$\Rightarrow \quad y(t) = -t\ln\left(\frac{1}{e} - \ln t\right)$$

Aufgabe 9

$$y'(t) = 2 \cdot \sqrt{y(t)} \quad , \; y(t) \geq 0 \;\; \& \;\; y(0) = 0$$

$$\frac{y'(t)}{2\sqrt{y(t)}} = 1 \qquad \Big| \int_0^t$$

$$\int_0^t \frac{y'(s)}{2\sqrt{y(s)}}\, ds = \int_0^t 1 \, ds$$

$$\Rightarrow \sqrt{y(s)} \Big|_0^t = t$$

$$\sqrt{y(t)} - \underbrace{\sqrt{y(0)}}_{=0} = t$$

$$y(t) = t^2$$

Klausur 1, Aufgabe 8

$$y'(t) = \left(\frac{1}{t} - 1\right) y(t)$$

$$\int_1^t \frac{y'(s)}{y(s)}\, ds = \int_1^t \left(\frac{1}{s} - 1\right) ds$$

$$\Rightarrow \ln y(s) \Big|_1^t = \ln t - t + 1$$

Da $y(1) = 1$:

$$\ln y(t) = \ln t - t + 1$$

$$y(t) = t e^{1-t}$$