PDE based visualization of nonstationary flows

Using an **n**onlinear **a**nisotropic **t**ransport-**d**iffusion (NATD) method for flow visualization

Jens F. Acker, Stefan Turek

Institute for Applied Mathematics
University of Dortmund

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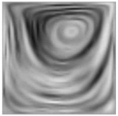

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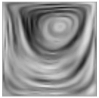


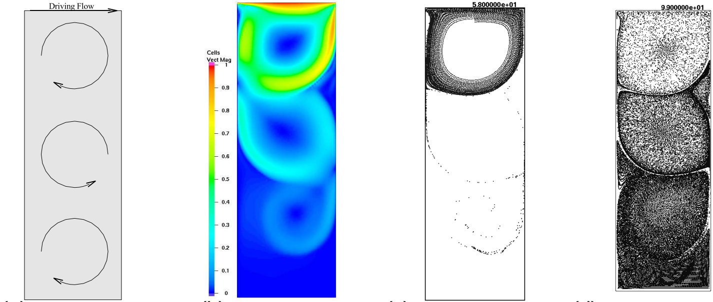
Motivation

Visualizing vortices and material flow of complicated instationary flows is difficult and a current topic of various research groups.
Plotting vector fields or the stream function might give some insights for stationary flow fields, but are often misleading for instationary fields.
It would be desirable to find a visualization method with following properties:

- Showing the flow as a ‘whole’ with all interesting features in view.
- Depiction of the ‘material flow’ in a Lagrangian sense (as with **particle tracing**).
- Easily extendable for 3D applications.

Problems

Attempts with particle tracing shift the problems to the proper placement of particle sources and show only the flow currently containing particles. This causes its own set of problems.



(a) Flow configuration (b) Velocity magnitude (c) Particle Tracing (top insertion) (d) Particle Tracing (full insertion)

Figure: Snapshots for 'Driven cavity'

Ideas

The ideas behind the NATD method can be summed up as

- Smooth a noise field tangentially to a given flow field.
- Smooth perpendicular to this flow depending on the local gradient of the current solution to generate ‘clustered’ flow lines. The amount of diffusion is determined by a function also used in Perona-Malik models.
- Transport the solution with the flow.

Since this diffusion creates a ‘feature scale space’ that gets coarser with time several solutions started at different times have to be computed and blended together to keep the visual output of a desired feature scale.

The PDE to be solved

The nonlinear anisotropic transport diffusion (NATD) model (of Rumpf/Preusser)

$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho - \operatorname{div}\left(\mathsf{A}(\mathbf{v}, \nabla \rho) \nabla \rho\right) = f(\rho)$$

with (external) CFD flow field $\mathbf{v}(\mathbf{x}, t)$

$$\mathsf{A}(\mathbf{v}, \nabla \rho) := B(\mathbf{v})^T \begin{pmatrix} \alpha(\|\mathbf{v}\|) & 0 \\ 0 & G(\nabla \rho, \mathbf{v}) \mathsf{Id}_{d-1} \end{pmatrix} B(\mathbf{v})$$

$$\alpha(\|\mathbf{v}\|) := \frac{\beta^2 \max\left(\|\mathbf{v}\|^2, \|\mathbf{v}_{min}\|^2\right) \tau}{2} \quad G(\nabla \rho, \mathbf{v}) := \begin{cases} \frac{\epsilon}{1 + c\|\nabla \rho\|^2} & \text{if } \|\mathbf{v}\| > 0 \\ \delta & \text{if } \|\mathbf{v}\| = 0 \end{cases}$$

$$f(\rho) := \phi \cdot \left((2\rho - 1) - (2\rho - 1)^3\right) \qquad \delta := \frac{\beta^2 \|\mathbf{v}_{min}\|^2 \tau}{2}$$

Remark: $\|\mathbf{v}_{min}\|$, τ, ϕ, c, β and ϵ are nonnegative real parameters, τ is the used timestep.

Numerical Discretization

In space a robust and accurate bilinear FEM discretization is used. The time discretization is done by linearization of the PDE and using implicit Euler (Crank-Nicolson is also possible).

The discretized variational formulation reads

$$(u_{n+1},\psi) + \tau(v_{n+1}\cdot\nabla u_{n+1},\psi) + \tau(A(v_{n+1},\nabla u_n)\nabla u_{n+1},\nabla\psi) = \tau(f(u_n),\psi) + (u_n,\psi) \quad \forall\psi\in\mathcal{V}$$

with the test function space $\mathcal{V}$ and test functions ψ.

<u>Remark:</u> Treatment of the nonlinearity by using Newton or higher order time-stepping schemes does not give as much improvement as to compensate for the higher processing time/complexity.

Numerical & algorithmic aspects I

Problem: Transport term causes numerical instabilities
$\Rightarrow$ Stabilization needed!

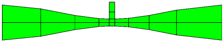
Diffusion operator already split in flow direction
$\Rightarrow$ Use streamline diffusion (SD) by modifying $\alpha(\cdot)$
Additionally SD is ‘2nd’ order!

Alternative methods that could also be used are

- Upwind diffusion (UPD). *Only 1st order!*
- Total variation diminishing (TVD)
- Internal penalty/Edge-oriented stabilization.

Numerical & algorithmic aspects II

Anisotropic diffusion is equivalent to isotropic diffusion on an anisotropic deformed mesh. The problems encountered solving this kind of PDE are therefore similar to those on very anisotropic meshes. $\Rightarrow$ Solvers that are good for anisotropic meshes should also be useful for anisotropic diffusion problems. See also the diploma thesis of Michael Köster for further information.



level	elements	nodes
1	20	34
2	80	107
3	320	373
4	1280	1385
5	5120	5329
6	20480	20897
7	81920	82753

Table: Listing of number of nodes and cells for different mesh refinement levels

level	$\tau = 0.001$	#it	$\tau = 0.002$	#it	$\tau = 0.003$	#it
5	0.08753	6	0.08345	6	0.12957	7
6	0.08973	6	0.11818	7	0.19461	9
7	0.20600	9	0.37173	15	**0.50550**	21

level	$\tau = 0.001$	#it	$\tau = 0.002$	#it	$\tau = 0.003$	#it
5	0.00001	1	0.00008	2	0.00030	2
6	0.00003	2	0.00043	2	0.00985	3
7	0.00027	2	0.06115	5	**0.60475**	28

level	$\tau = 0.001$	#it	$\tau = 0.002$	#it	$\tau = 0.003$	#it
5	0.00001	1	0.00001	2	0.00008	2
6	0.00001	1	0.00059	2	0.00275	3
7	0.00001	1	0.00641	3	**0.05587**	5

Table: Comparing convergence rates for different refinement levels, time-steps and solvers (preconditioned (ADITRIGS) BiCG-stab (top), MG with ADITRIGS smoother, 8 smoothing steps, F-cycle (middle), preconditioned (MG/ADITRIGS) BiCG-stab (bottom). In each case the noise field was replaced with sine waves and 6 digits accuracy had to be gained.

$\Rightarrow$ **BiCG-stab preconditioned with MG/ADITRIGS allows larger timesteps τ.**

Numerical & algorithmic aspects III

<u>Problem for unstructured grids:</u>

No fixed element size

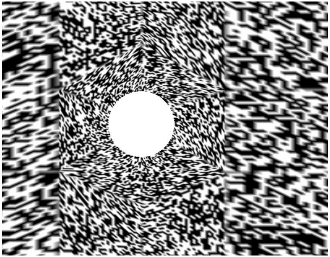
$\Rightarrow$ Frequency of noise not uniform

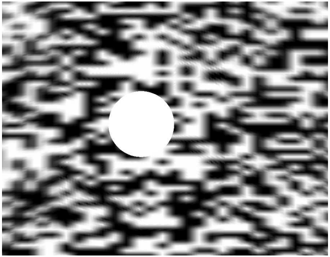
$\Rightarrow$ Underlying mesh can be 'seen' in solution!

<u>Solution:</u>

Create noise textures and map those to the grid.

$\Rightarrow$ Underlying mesh 'disappears'!





Numerical & algorithmic aspects IV

As mentioned before, the NATD method needs to blend different solutions representing various feature scales together to create a solution at ‘stabilized’ feature scale.
There are different possibilities for such a ‘blending’ of solutions that are all based on different decompositions of 1.

- Trigonometric functions, especially sine functions.
  $\Rightarrow C^\infty$ transitions between solutions.
  <u>Drawbacks:</u>
  - No good solutions for the ‘startup phase’.
  - One solution is dominant while the other solutions are repressed.
- Bernstein polynomials, especially B-Splines.
  $\Rightarrow$ Solutions for the ‘startup phase’ can be increasing the order of the polynomials and the contribution to the blended solution is more evenly spread.
  <u>Drawback:</u> The smoothness of transitions is reduced.

In the following examples B-Spline based blending is used!

Circular flow

For a more analytical view of the behavior of the NATD method, we define an artificial stationary circular flowfield on an unstructured mesh by scaling the tangential to each point by a sine function depending on the distance of that point from the center.

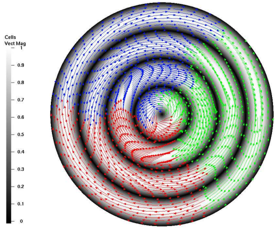


Figure: Circular flowfield. Flow strength and direction additionally visualized by vectors.

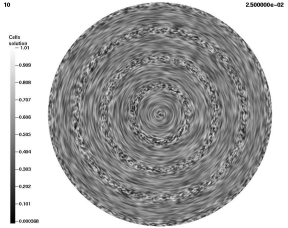


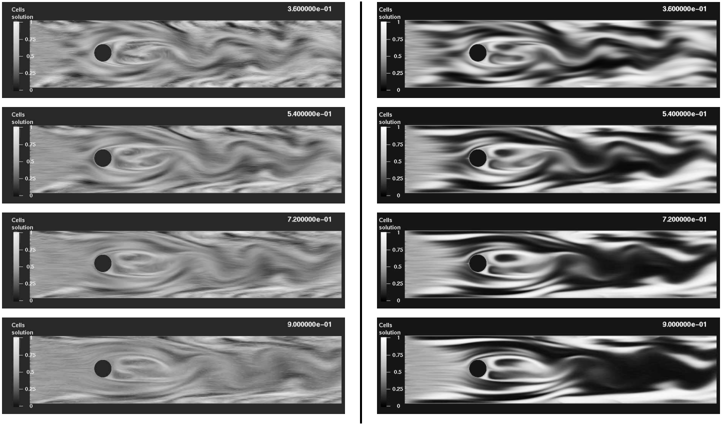
Figure: Solution after 10 timesteps (no blending).

Settings: $\tau = 0.0025$, $\epsilon = 0.01$, $c = 300$, $\phi = 5$, #cells=589824, #nodes=590593.

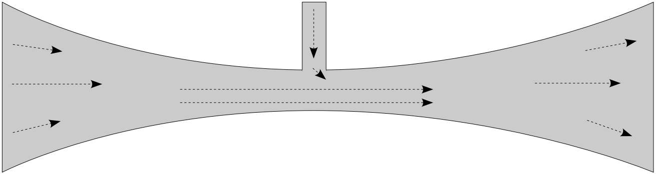
Vortex shedding behind a cylinder

Problem: *Vortex shedding behind a cylinder.*

Both columns show the same flow, but with slightly different enhancement and blending settings.



The Venturi pipe



Venturi pipe

Inventor: Giovanni Battista Venturi (18th-19th-century)

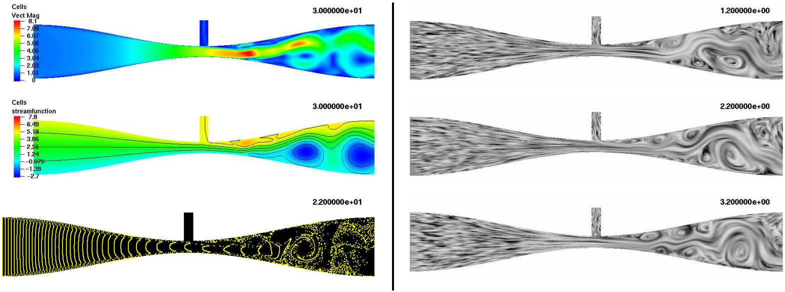
Purpose: Creating suction/low pressure by using a fast flow (*Bernoulli's principle*)

Usage example: Draining sailing boats by their own movements

Pictures/Movies

Visualization of the complex vortices and material flow given by this flow is prone with difficulties. Neither of the Eulerian methods using streamfunction or flow magnitude are showing much. Even the Lagrangian method of particle tracing has its problems.

The NATD method gives much better results, as can be seen at the right picture column. Even small vortices are identifiable!



Summary & Outlook

Summary

- Visualization technique for time-dependent CFD flow fields on unstructured grids.
- An accurate, robust and efficient solver.
- Handling of noise on unstructured grids.
- Blending of solutions.

Outlook

- Improvements in controlling diffusion strengths and directions for vector based solutions. See also the diploma thesis of David Tschumperlé (University of Nice-Sophia Antipolis).
- Use of higher order fully implicit time-stepping schemes.
- Replacement of streamline diffusion with better stabilization methods.
- 3D!

Bibliography

J. F. Acker, J. Hron, T. Preusser, M. Rumpf, and S. Turek.
FEM techniques for multiscale visualization of time–dependent flow fields.
Technical reports of the Institute of Applied Mathematics, Nr. 314, University of Dortmund, 2006. Also submitted to International Journal of Computational Science and Engineering (IJCSE) on January 17th.

D. Bürkle, T. Preusser, and M. Rumpf.
Transport and anisotropic diffusion in time-dependent flow visualization.
In *Proceedings Visualization '01*, 2001.

M. Köster.
Robuste Mehrgitter-Krylowraum-Techniken für FEM-Verfahren.
Diploma thesis, University of Dortmund, 2004.

G. Matthies and L. Tobiska.
The streamline–diffusion method for conforming and nonconforming finite elements of lowest order applied to convection–diffusion problems.
Computing, 66:343–364, 2001.

T. Preusser and M. Rumpf.
An adaptive finite element method for large scale image processing.
Journal of Visual Comm. and Image Repres., 11:183–195, 2000.

J. Weickert.
Anisotropic diffusion in image processing.
Teubner, 1998.