Robust Monolithic - Multigrid FEM Solver for Three Fields Formulation Rising from non-Newtonian Flow Problems

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- Bingham/Viscoplastic flows
- Viscoelastic flows
- Multiphase flows

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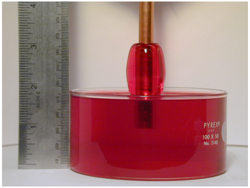
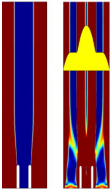
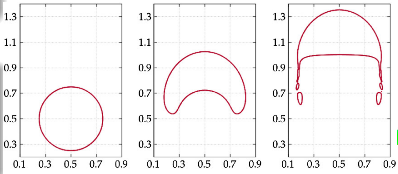
- Requires two inf-sup conditions
  - Velocity-Pressure
  - Stress-Velocity







(a) $t = 0$ (b) $t = 1.5$ (c) $t = 3$

- Model with additional symmetric viscoplastic stress tensor

$$\begin{cases} \left\|\mathbf{D}(\boldsymbol{u})\right\|_{\epsilon} \mathbf{W} - \mathbf{D}(\boldsymbol{u}) = 0 & \text{in } \Omega \\ -\nabla \cdot \left(2\eta \mathbf{D}(\boldsymbol{u}) + \tau_s \mathbf{W}\right) + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g}_D & \text{on } \Gamma_D \end{cases}$$

$$\mathbf{W} = \frac{\mathbf{D}(\boldsymbol{u})}{\left\|\mathbf{D}(\boldsymbol{u})\right\|_{\epsilon}}$$

- $\mathbf{W}$ = new stress tensor
- τ_s = yield stress
- $\mathbf{D}$ = strain rate tensor

- Model with additional elastic stress tensor

- Oldroyd-B ($\gamma = 0$)
- Giesukus ($\gamma = 1$)

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \eta_s \Delta \boldsymbol{u} + \dfrac{\eta_p}{\Lambda} \nabla \cdot e^{\psi} \\ \nabla \cdot \boldsymbol{u} = 0 \\ \dfrac{\partial \psi}{\partial t} + (\boldsymbol{u} \cdot \nabla)\psi - (\Omega\psi - \psi\Omega) - 2B = \dfrac{1}{\Lambda}(e^{-\psi} - \mathbb{I}) - \gamma e^{\psi}(e^{-\psi} - \mathbb{I})^2 \end{cases}$$

- ψ = log conformation tensor
- $\mathbf{D}$ = strain rate tensor
- η_s = solvent viscosity
- η_p = polymer viscosity

Multiphase

- Model with additional multiphase stress tensor

$$
\begin{cases}
\boldsymbol{\tau} = \boldsymbol{\tau}_s + \boldsymbol{\tau}_m \\
\rho(\psi)\left(\dfrac{\partial \boldsymbol{u}}{\partial t} + u \cdot \nabla \boldsymbol{u}\right) - \nabla \cdot \boldsymbol{\tau} + \nabla p = 0 \\
\nabla \cdot \boldsymbol{u} = 0
\end{cases}
$$

$$
\boldsymbol{\tau}_m = -\sigma \left(\frac{\nabla\psi \otimes \nabla\psi}{\|\nabla\psi\|}\right), \quad \boldsymbol{\tau}_s = 2\mu(\psi)\mathbf{D}(\boldsymbol{u})
$$

$$
\begin{cases}
\dfrac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla\varphi - \gamma_{nd}\nabla \cdot \left(\left(\dfrac{\nabla\varphi}{\|\nabla\varphi\|} \cdot \nabla\varphi - 1\right)\dfrac{\nabla\varphi}{\|\nabla\varphi\|}\right) = 0 \\
\dfrac{\partial \psi}{\partial \tau} + \nabla \cdot (\gamma_{nc}\psi(1-\psi)\nabla\varphi) - \nabla \cdot (\gamma_{nd}(\nabla\psi \cdot \nabla\varphi)\ \nabla\varphi) = 0
\end{cases}
$$

- $\boldsymbol{\tau}_m$ = multiphase stress
- $\boldsymbol{\tau}_s$ = viscous stress

Jump Discontinuity

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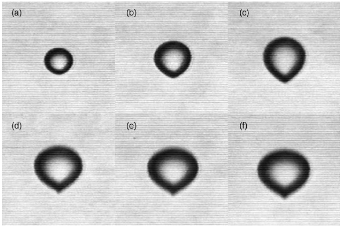
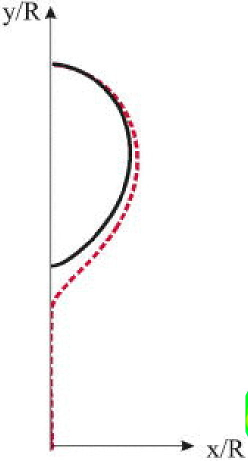
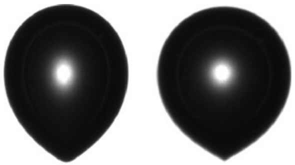
Rising bubble in non-Newtonian fluids

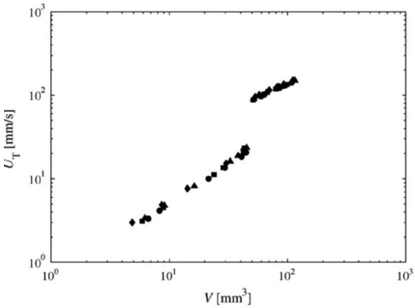
- In viscoelastic liquid
  - Volume exceeds a critical value
  - Rise velocity jump discontinuity
  - shape changes: convex to tear drop





Jump Discontinuity



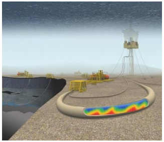


- Prediction: Boundary condition changes from rigid to free
- Jump occurs due to negative wake
- Jump occurs due to shear dependence of the viscosity

Industrial Applications

Bubble Phenomena

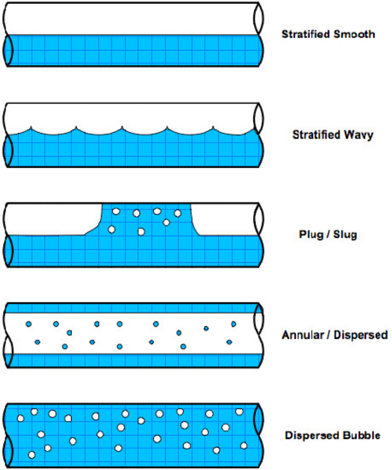
- Pipeline transport application
- Bio reactors
- Combustion engines
- Underwater explosions
- Medicine: Angioplasty
- Food industry
  - Fermentation process
  - Bread and yogurt
- Nature: Bubble fence




Industrial Applications

<u>Bubble Phenomena</u>

- Chemical separator
- Dispersion of gas bubble
- Efficient mass transfer
- Polymer and sludge processes



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Goals to Achieve

- Efficient three fields solver
- Solver: Monolithic multigrid approach
- Constraints free for the choice of finite element spaces

Navier Stokes Equation

Primitive Formulation

$$\begin{cases} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla\cdot 2\eta\mathbf{D}(\boldsymbol{u}) + \nabla p = 0 & \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g}_D & \text{on } \Gamma_D \end{cases} \tag{1}$$

Three Field Formulation

$$\begin{cases} \boldsymbol{\sigma} - \mathbf{D}(\boldsymbol{u}) = 0 & \text{in } \Omega \\ (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla\cdot\Big(2\eta(1-\alpha)\mathbf{D}(\boldsymbol{u}) + 2\eta\alpha\boldsymbol{\sigma}\Big) + \nabla p = 0 & \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g}_D & \text{on } \Gamma_D \end{cases} \tag{2}$$

Three Fields Formulation

Stokes System

$$
\left\{
\begin{aligned}
\boldsymbol{\sigma} - \mathbf{D}(\boldsymbol{u}) &= 0 && \text{in } \Omega \\
-\nabla \cdot \Big(2\eta(1-\alpha)\mathbf{D}(\boldsymbol{u}) + 2\eta\alpha\boldsymbol{\sigma} \Big) + \nabla p &= 0 && \text{in } \Omega \\
\nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega \\
\boldsymbol{u} &= \boldsymbol{g}_D && \text{on } \Gamma_D
\end{aligned}
\right.
\tag{3}
$$

- $\boldsymbol{\sigma}$ = extra stress tensor
- α = parameter
- η = viscosity

Contents

Variational Formulation

- $\mathbb{V} = \boldsymbol{H}_0^1(\Omega) := \left(H_0^1(\Omega)\right)^2 \to$ velocity and dual space $\to \mathbb{V}'$
- $\mathbb{Q} = L_0^2(\Omega) \to$ pressure and dual space $\to \mathbb{Q}'$
- $\mathbb{T} = \left(L^2(\Omega)\right)^4_{\text{sym}} \to$ stress and dual space $\to \mathbb{T}'$

$\mathcal{A}_{\boldsymbol{u}}$ defined on $\mathbb{V} \longrightarrow \mathbb{V}'$

$$\langle \mathcal{A}_{\boldsymbol{u}} \boldsymbol{u}, \boldsymbol{v} \rangle := 2\eta(1-\alpha) \int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})\, dx \tag{4}$$

$\mathcal{A}_{\boldsymbol{\sigma}}$ defined on $\mathbb{T} \longrightarrow \mathbb{T}'$

$$\langle \mathcal{A}_{\boldsymbol{\sigma}} \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle = 2\eta\alpha \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\, dx \tag{5}$$

The associated bilinear forms

$$a_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \mathcal{A}_{\boldsymbol{u}} \boldsymbol{u}, \boldsymbol{v} \rangle, \quad a_{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \langle \mathcal{A}_{\boldsymbol{\sigma}} \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle \tag{6}$$

Variational Formulation

$\mathcal{B}$ and $\mathcal{C}$ defined on $\mathbb{V} \longrightarrow \mathbb{Q}'$ respectively, $\mathbb{V} \longrightarrow \mathbb{T}'$

$$\langle \mathcal{B}\boldsymbol{v}, q \rangle := -\int_\Omega \nabla \cdot \boldsymbol{v} \; q \, dx \tag{7}$$

$$\langle \mathcal{C}\boldsymbol{v}, \boldsymbol{\tau} \rangle := 2\eta\alpha \int_\Omega \boldsymbol{\tau} : \mathbf{D}(\boldsymbol{v}) \, dx \tag{8}$$

Associated bilinear forms $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ defined on $\mathbb{V} \times \mathbb{Q} \longrightarrow \mathbb{R}$ and $\mathbb{V} \times \mathbb{T} \longrightarrow \mathbb{R}$ respectively

$$b(\boldsymbol{v}, q) := \langle \mathcal{B}\boldsymbol{v}, q \rangle \tag{9}$$

$$c(\boldsymbol{v}, \boldsymbol{\tau}) := \langle \mathcal{C}\boldsymbol{v}, \boldsymbol{\tau} \rangle \tag{10}$$

Two-Folds Saddle Point Problem

$$\begin{bmatrix} \mathcal{A}_{\boldsymbol{u}} & \mathcal{C}^{\mathsf{T}} & \mathcal{B}^{\mathsf{T}} \\ \mathcal{C} & -\mathcal{A}_{\boldsymbol{\sigma}} & \mathbf{0} \\ \mathcal{B} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\sigma} \\ p \end{bmatrix} = \begin{bmatrix} RHS_{\boldsymbol{\sigma}} \\ RHS_{\boldsymbol{u}} \\ RHS_p \end{bmatrix} \tag{11}$$

The system (3) has the following weak formulation

$$\begin{cases} -a_{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + c(\boldsymbol{u}, \boldsymbol{\tau}) & = \langle RHS_{\boldsymbol{\sigma}}, \boldsymbol{\tau} \rangle \qquad \forall \boldsymbol{\tau} \in \mathbb{T} \\ a_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{v}, \boldsymbol{\sigma}) + b(\boldsymbol{v}, p) & = \langle RHS_{\boldsymbol{u}}, \boldsymbol{v} \rangle \qquad \forall \boldsymbol{v} \in \mathbb{V} \\ b(\boldsymbol{u}, q) & = \langle RHS_p, q \rangle \qquad \forall q \in \mathbb{Q} \end{cases} \tag{12}$$

Variational Formulation

Introduce the null spaces $\operatorname{Ker}\mathcal{B}$

$$\operatorname{Ker}\mathcal{B} = \{\boldsymbol{v} \in \mathbb{V};\ b(\boldsymbol{v}, q) = 0 \quad \forall q \in \mathbb{Q}\}$$

Let $\mathbb{X} := \operatorname{Ker}\mathcal{B} \times \mathbb{T}$

$$\langle \mathcal{A}(\boldsymbol{u}, \boldsymbol{\sigma}), (\boldsymbol{v}, \boldsymbol{\tau}) \rangle = \langle \mathcal{A}_{\boldsymbol{u}} \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \mathcal{C} \boldsymbol{v}, \boldsymbol{\sigma} \rangle + \langle \mathcal{A}_{\boldsymbol{\sigma}} \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle - \langle \mathcal{C} \boldsymbol{u}, \boldsymbol{\tau} \rangle \tag{13}$$

$$a(\mathcal{U}, \mathcal{V}) = a_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{v}, \boldsymbol{\sigma}) + a_{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) - c(\boldsymbol{u}, \boldsymbol{\tau}) \tag{14}$$

Solvability of Problem

- Find $\mathcal{U} \in \mathbb{X}$ such that:

$$a(\mathcal{U}, \mathcal{V}) = \langle \boldsymbol{f}, \mathcal{V} \rangle \qquad \forall \mathcal{V} \in \mathbb{X} \tag{15}$$

Theorem

Let $\mathbb{X}$ be a Hilbert space and $\boldsymbol{f} \in \mathbb{X}'$, topological dual space of $\mathbb{X}$, and let $a(.,.)$ be a bilinear form on $\mathbb{X}$ satisfying the following hypothesis:
1). $a(\cdot,\cdot)$ is continuous: there exists a constant $C > 0$ such that:

$$a(\mathcal{U}, \mathcal{V}) \leq C \, \|\mathcal{U}\| \, \|\mathcal{V}\| \qquad \forall \mathcal{U}, \mathcal{V} \in \mathbb{X} \tag{16}$$

Solvability of Problem

Theorem (cont...)

2). $a(\cdot,\cdot)$ is $\mathbb{X}$-elliptic: there exists a constant $\beta > 0$ such that:

$$a(\mathcal{V},\mathcal{V}) \geq \beta \left\|\mathcal{V}\right\|^2 \qquad \forall \mathcal{V} \in \mathbb{X} \tag{17}$$

then problem has a unique solution $\mathcal{U} \in \mathbb{X}$.

Remark

Besides from theory of saddle point problem it is easy to show that there exists a unique $p \in Q$ such that $(\boldsymbol{\sigma}, \boldsymbol{u}, p)$ is the unique solution of problem (12)

Solvability of the Problem

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Illustration for $\alpha \neq 1$

$$\begin{aligned} a(\mathcal{V},\mathcal{V}) &= a_{\boldsymbol{u}}(\boldsymbol{v},\boldsymbol{v}) + a_{\boldsymbol{\sigma}}(\boldsymbol{\tau},\boldsymbol{\tau}) \\ &= 2\eta(1-\alpha)\int_\Omega \|\mathbf{D}(\boldsymbol{v})\|^2\, dx + 2\eta\alpha\int_\Omega \|\boldsymbol{\tau}\|^2\, dx \\ &\geq C2\eta(1-\alpha)\|\boldsymbol{v}\|_{\mathbb{V}}^2 + 2\eta\alpha\|\boldsymbol{\tau}\|_{\mathbb{T}}^2 \\ &\geq \tilde{C}\,\|(\boldsymbol{v},\boldsymbol{\tau})\|_X^2 \end{aligned}$$

Remark

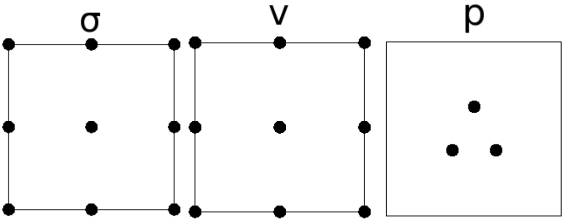
- When $\alpha = 1 \rightarrow$ we can show existence and uniqueness using saddle point theory

Contents

Finite Element Approximation

- Domain $\Omega \subset \mathbb{R}^d \rightarrow$ Quadrilaterals elements K, $\Omega = \text{int}\left(\bigcup_{K \in \mathcal{T}^h} \overline{K}\right)$
- Finite element spaces approaximations

$$\begin{aligned}
\mathbb{T}^h &= \left\{ \boldsymbol{\tau}_h \in \mathbb{M}, \boldsymbol{\sigma}_{h|K} \in Q_2(K) \right\} \\
\mathbb{V}^h &= \left\{ \boldsymbol{v}_h \in \mathbb{V}, \boldsymbol{v}_{h|K} \in Q_2(K) \right\} \\
\mathbb{Q}^h &= \left\{ q_h \in \mathbb{Q}, q_{h|K} \in P_1^{\text{disc}}(K) \right\}
\end{aligned} \tag{18}$$



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Approximation of Stokes problem for finite element spaces $\mathbb{T}^h \subset \mathbb{T}$, $\mathbb{V}^h \subset \mathbb{V}$, $\mathbb{Q}^h \subset \mathbb{Q}$

$$\begin{cases} -a_{\boldsymbol{\sigma}}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + c(\boldsymbol{u}_h, \boldsymbol{\tau}_h) & = \langle RHS_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h \rangle \qquad \forall \boldsymbol{\tau}_h \in \mathbb{T}^h \\ a_{\boldsymbol{u}}(\boldsymbol{u}_h, \boldsymbol{v}_h) + c(\boldsymbol{v}_h, \boldsymbol{\sigma}_h) + b(\boldsymbol{v}_h, p_h) & = \langle RHS_{\boldsymbol{u}}, \boldsymbol{v}_h \rangle \qquad \forall \boldsymbol{v}_h \in \mathbb{V}^h \\ b(\boldsymbol{u}_h, q_h) & = \langle RHS_p, q_h \rangle \qquad \forall q_h \in \mathbb{Q}^h \end{cases} \tag{19}$$

- Two LBB conditions required
  - Firstly, between velocity and stress
  - Secondly, between velocity and pressure

Finite Element Approximation

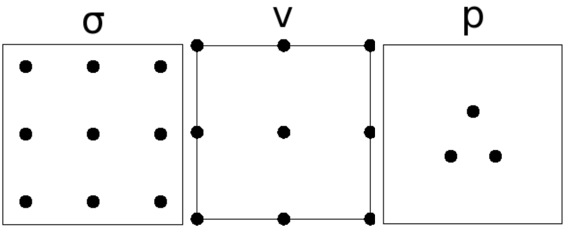
Approximation of Stokes problem for finite element spaces $\mathbb{T}^h \subset \mathbb{T}$, $\mathbb{V}^h \subset \mathbb{V}$, $\mathbb{Q}^h \subset \mathbb{Q}$

$$\begin{cases} -a_{\boldsymbol{\sigma}}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + c(\boldsymbol{u}_h, \boldsymbol{\tau}_h) & = \langle RHS_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h \rangle \qquad \forall \boldsymbol{\tau}_h \in \mathbb{T}^h \\ \quad a_{\boldsymbol{u}}(\boldsymbol{u}_h, \boldsymbol{v}_h) + c(\boldsymbol{v}_h, \boldsymbol{\sigma}_h) + b(\boldsymbol{v}_h, p_h) & = \langle RHS_{\boldsymbol{u}}, \boldsymbol{v}_h \rangle \qquad \forall \boldsymbol{v}_h \in \mathbb{V}^h \\ \quad b(\boldsymbol{u}_h, q_h) & = \langle RHS_p, q_h \rangle \qquad \forall q_h \in \mathbb{Q}^h \end{cases}$$

- Two LBB conditions required
  - Firstly, between velocity and stress ✗
  - Secondly, between velocity and pressure ✓

Stable Element Choice 1

Fortin and Fortin element



- $(\boldsymbol{u}, p)$ spaces $\rightarrow$ LBB compatible
- $\boldsymbol{\sigma}$ and $\mathbf{D}(\boldsymbol{u})$ $\rightarrow$ same FEM discontinuous space

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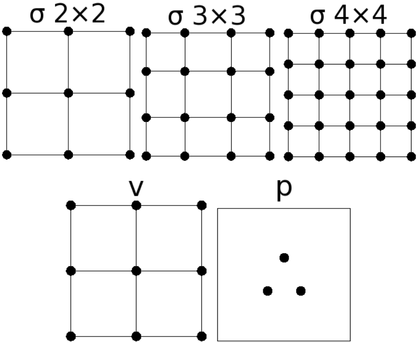
Marchal-Crochet element

- Subcell discretization to enrich the local d.o.f for $\boldsymbol{\sigma}$
- $n_{\boldsymbol{\sigma}} > n_{\boldsymbol{u}} \rightarrow$ condition satisfied



Computational cost is increased due to more d.o.f

EO-FEM Stabilization

Remedy: Jump term addition to ensure element pair is stable

- Penalizing the jump of the solution gradient over E

$$J_{\boldsymbol{u}}(\boldsymbol{u}_h, \boldsymbol{v}_h) = \gamma_{\boldsymbol{u}} \sum_{e \in \mathcal{E}_h} 2\eta\alpha h \int_e [\nabla \boldsymbol{u}_h] : [\nabla \boldsymbol{v}_h] \; d\Omega \tag{20}$$

$$\begin{bmatrix} \mathcal{A}_{\boldsymbol{u}} + J_{\boldsymbol{u}} & \mathcal{C} & \mathcal{B}^{\mathsf{T}} \\ \mathcal{C}^{\mathsf{T}} & -\mathcal{A}_{\boldsymbol{\sigma}} & \mathbf{0} \\ \mathcal{B} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\sigma} \\ p \end{bmatrix} = \begin{bmatrix} RHS_{\boldsymbol{u}} \\ RHS_{\boldsymbol{\sigma}} \\ RHS_p \end{bmatrix} \tag{21}$$

Discrete Problem

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- Find $\mathcal{U}_h \in \mathbb{X}_h$ such that:

$$a(\mathcal{U}_h, \mathcal{V}_h) + j(\mathcal{U}_h, \mathcal{V}_h) = \langle f_h, \mathcal{V}_h \rangle \qquad \forall \mathcal{V}_h \in \mathbb{X}_h \tag{22}$$

$$|\!|\!|\mathcal{V}_h|\!|\!|^2 = \|\mathcal{V}_h\|^2 + j(\mathcal{V}_h, \mathcal{V}_h) \tag{23}$$

Theorem

Let $\mathbb{X}_h$ be a Hilbert space and $f_h \in \mathbb{X}_h'$, topological dual space of $\mathbb{X}$, and let $a(.,.)$ be a bilinear form on $\mathbb{X}_h$ satisfying the following hypothesis:

1). $a(\cdot,\cdot)$ is continuous: there exists a constant $C_h > 0$ such that:

$$a(\mathcal{U}_h, \mathcal{V}_h) \leq C_h \, |\!|\!|\mathcal{U}_h|\!|\!| \, |\!|\!|\mathcal{V}_h|\!|\!| \qquad \forall \mathcal{U}_h, \mathcal{V}_h \in \mathbb{X}_h \tag{24}$$

Solvability of Problem

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Theorem (cont...)

2). There exists a constant $\beta'_h > 0$ such that :

$$\sup_{\mathcal{V}_h \in \mathbb{X}_h} \frac{a(\mathcal{U}_h, \mathcal{V}_h)}{|||\mathcal{V}_h|||} \geq \beta'_h \, |||\mathcal{U}_h||| \qquad \forall \mathcal{U}_h \in \mathbb{X}_h \tag{25}$$

then problem has a unique solution $\mathcal{U}_h \in \mathbb{X}_h$.

Remark

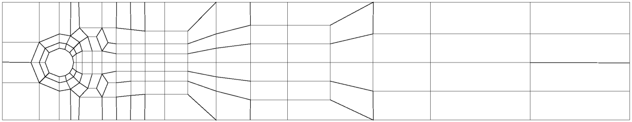
Besides from theory of saddle point problem it is easy to show that there exists a unique $p_h \in Q_h$ such that $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, p_h)$ is the unique solution of problem (19)

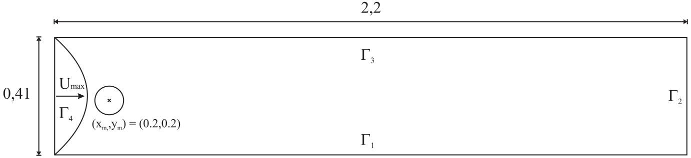
- No constraints on the choice of discrete finite space for stress

Contents

Benchmark Configuration

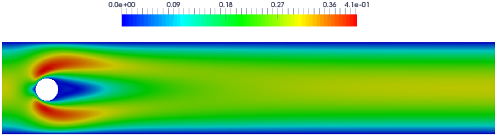
- Inlet: Dirichlet parabolic profile $u(0,y) = \frac{4 \times Uy(0.41 - y)}{(0.41)^2}$
- No-slip at upper and lower walls Γ_1 and Γ_3
- Outlet: Neumann boundary Γ_2
- Kinematic viscosity $\eta = 10^{-3}$
- Characteristic length of cylinder $l_c = 0.1$



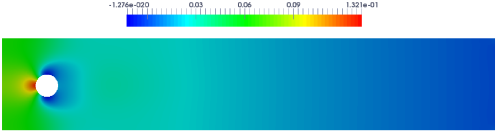


Numerical Results

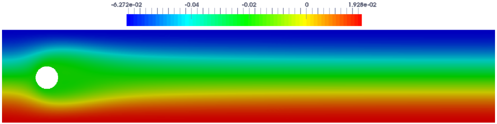
Velocity Magnitude



Pressure



Stream Function



Three fields Stokes vs Stokes solver in primitive variables

Level	Lift	Drag	NL/LL	Lift[1]	Drag	NL/LL
1	0.009498	5.5550	7/2	0.009498	5.5550	9/2
2	0.010601	5.5722	7/2	0.010601	5.5722	9/2
3	0.010616	5.5776	7/2	0.010616	5.5776	9/1
4	0.010618	5.5791	7/1	0.010618	5.5791	8/1
5	0.010619	5.5794	6/2			

Three field solver performance efficient as primitive Stokes solver

Check robustness and consistency!

[1]Damanik. H "FEM Simulation of Non-isothermal Viscoelastic fluids", PhD Thesis

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- Consistency for case $\alpha = 0$

Level	α	Lift	Drag	NL/LL	Lift	Drag	NL/LL
		No Stab.			Stab.		
2	0	0.010601	5.5722	7/2	0.010702	5.5674	7/2
3	0	0.010616	5.5776	7/2	0.010619	5.5757	7/2
4	0	0.010618	5.5791	7/1	0.010617	5.5782	7/2
5	0	0.010619	5.5794	6/2	0.010618	5.5790	6/3

Edge Oriented FEM is consistent

Side effect neither on solution nor on the solver

EO-FEM :Robustness

- Two extreme cases
  - α=0 $\rightarrow$ viscous contribution
  - α=1 $\rightarrow$ no viscous contribution

Level	α	Lift	Drag	NL/LL	Lift	Drag	NL/LL
		No Stab.			Stab.		
2	0	0.010601	5.5722	7/2	0.010702	5.5674	7/2
3	0	0.010616	5.5776	7/2	0.010619	5.5757	7/2
4	0	0.010618	5.5791	7/1	0.010617	5.5782	7/2
5	0	0.010619	5.5794	6/2	0.010618	5.5790	6/3
2	1	———	——	—	0.010588	5.5520	7/2
3	1	———	——	—	0.010600	5.5698	7/2
4	1	———	——	—	0.010612	5.5756	7/2
5	1	———	——	—	0.010617	5.5778	7/3

Robustness w.r.t. problem!

Consistent and grid independent solver

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Robust monolithic multigrid solver for three fields Stokes formulation using EO-FEM:

Advantages

- Taking away the 2nd inf-sup condition (no constraints on the choice of the $\boldsymbol{\sigma}$ space)
- Allowing large class of discretiztation for the stress
- Taking the efficiency of the Stokes solver in primitive variables to Stokes solver in three fields formulation