Monolithic Newton-Multigrid Solver for Multiphase Flow Problems with Surface Tension

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Contents

Contents

Motivation

Multiphase flows in Nature and Industry

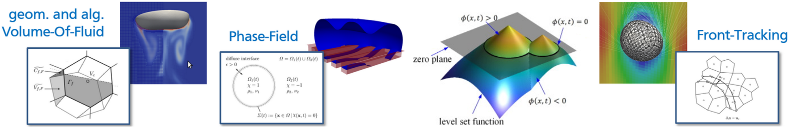
- **Gas-Liquid**
  - Condensation
  - Boiling
  - Spray cooling
  - Biological flows
- **Solid-Liquid**
  - Emulsification
  - Slurries
  - Food processing
- **Gas-Solid**
  - Fluidization
  - Coal burners
- **Liquid-Liquid**
  - Petroleum extraction
- **Oil exploration:**

$$\underbrace{Oil - air - water - solid}_{4-phases}$$

Interface Tracking/Capturing

Interface tracking: Immersed boundary,[1] Front tracking[2]

Interface capturing: Volume of fluid,[3] Phase field,[4,5] Level set[6,7]



Continuum surface force (CSF): Interface as a smooth transition[8,9]

Continuum surface stress (CSS): Surface force term as the divergence of the stress tensor[8,10]

Contents

CSF Formulation

The incompressible Navier Stokes equations

$$\begin{cases} \rho\Big(\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\Big) - \nabla\cdot 2\mu\mathbf{D}(\boldsymbol{u}) + \nabla p = f, & \textbf{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0, & \textbf{in } \Omega. \end{cases} \tag{1}$$

- ρ = density
- $\mathbf{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$
- $\boldsymbol{u}$ = velocity
- p = pressure
- μ = viscosity
- f = external force

$$f = \kappa\sigma\boldsymbol{n}\delta_\Gamma \tag{2}$$

- κ = curvature
- $\boldsymbol{\sigma}$ = surface tension constant
- $\boldsymbol{n}$ = normal to the interface
- δ_Γ = delta function

CSF Formulation

$$
\begin{cases}
\rho(\Gamma)\left(\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) - \nabla\cdot 2\mu(\Gamma)\mathbf{D}(\boldsymbol{u}) + \nabla p = \kappa\sigma\boldsymbol{n}\delta_\Gamma, & \text{in } \Omega, \\
\nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega.
\end{cases}
\tag{3}
$$

- Interface capturing by level set method

$$
\frac{\partial \phi}{\partial t} + \boldsymbol{u}\cdot\nabla\phi = 0 \tag{4}
$$

- $\phi =$ level set function

This transport equation can be efficiently solved due to the choice of a smooth level set function

Advantages

- Exact representation of the interface

$$\Gamma = \{x \in \Omega, \phi = 0\} \tag{5}$$

- Provides derived geometrical quantities ($\boldsymbol{n}, \kappa$)

$$\boldsymbol{n} = \frac{\nabla \phi}{\|\nabla \phi\|}, \quad \kappa = -\nabla \cdot \boldsymbol{n} \tag{6}$$

- The surface tension force in terms of the level set function

$$f = \kappa \sigma \boldsymbol{n} \delta_\Gamma(\phi) \tag{7}$$

The signed distance function is the natural choice for the level set!

Problems and Challenges

- **Spurious velocity**[11] Unphysical flows near the interface
  - Unphysical interface movement
  - Misinterpret the flow physics
- **Large surface tension**
  - Spurious velocity may grow
  - Destroy the interface
- **Observed** $\longrightarrow$ explicit/implicit interface representation
- **Explicit treatment of surface tension** Capillary time restriction[12]
- **Explicit reinitialization** Requires perfect interface description

Problems and Challenges: Capillary Time Step

- Dominant limitation on time step in simulations of interfacial flows with surface tension
- Time-explicit discretization of the surface tension leads to stability constraint

Mathematical expression

- Same density: $\Delta t_h^{cap} < \sqrt{\frac{\rho h^3}{\sigma}}$
- Different densities: $\Delta t_h^{cap} < \sqrt{\frac{(\rho_1+\rho_2)h^3}{4\pi\sigma}}$
- Capillary time step $\propto$ to mesh size $h^{\frac{3}{2}}$

Towards a fully implicit treatment

Curvature-Free Level Set

- Material cut-off function

$$\psi(x,t) = \begin{cases} +\frac{1}{2} & \text{if} \quad \phi(x,t) \geq 0 \\ -\frac{1}{2} & \text{if} \quad \phi(x,t) < 0 \end{cases} \tag{8}$$

Features

- Enhances the accuracy
- ψ is material characteristic $\longrightarrow$ conservative level set
- Regularization of material cut-off function

$$\psi = \frac{-1}{1 + exp(\frac{\phi}{\epsilon_\psi})} + 0.5 \tag{9}$$

Curvature-Free Level Set

New set of equations for multiphase flow

$$\begin{cases} \rho(\psi)\Big(\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\Big) - div\boldsymbol{\tau} + \nabla p = 0, & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega, \\ \dfrac{\partial \phi}{\partial t} + \boldsymbol{u}\cdot\nabla\phi = 0, & \text{in } \Omega, \\ \psi - \Big(\dfrac{-1}{1+exp(\frac{\phi}{\epsilon_\psi})} + 0.5\Big) = 0, & \text{in } \Omega. \end{cases} \tag{10}$$

- $\phi =$ level set function
- $\epsilon_\psi =$ interface thickness
- $\boldsymbol{\tau} = (\boldsymbol{\tau}_s + \boldsymbol{\tau}_m)$ full stress tensor
- $\psi =$ cut-off function

Curvature-Free Level Set

- Viscous stress

$$\boldsymbol{\tau}_s = 2\mu(\psi)\mathbf{D}(\boldsymbol{u})$$

- Multiphase stress

$$\boldsymbol{\tau}_m = -\sigma\left(\frac{\nabla\psi \otimes \nabla\psi}{\|\nabla\psi\|}\right)$$

- Signed distance function needs to satisfy the constraint

$$\|\nabla\phi\| = 1 \quad \Longleftrightarrow \quad \boldsymbol{n}\cdot\nabla\phi = 1$$

- For implicit approach, constraint is imposed

$$\frac{\partial\phi}{\partial t} + \boldsymbol{u}\cdot\nabla\phi - \underbrace{\nabla\cdot\left(\gamma_{nd}(\boldsymbol{n}\cdot\nabla\phi - 1)\boldsymbol{n}\right)}_{\textit{reinitialization}} = 0 \tag{11}$$

- γ_{nd} = penalty parameter

Curvature-Free Level Set[13]

$$
\begin{cases}
\rho(\psi)\Big(\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\Big) - \nabla\cdot\Big(2\mu(\psi)\mathbf{D}(\boldsymbol{u}) + \sigma\Big(\dfrac{\nabla\psi\otimes\nabla\psi}{\|\nabla\psi\|}\Big)\Big) + \nabla p = 0, & \text{in } \Omega,\\
\nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega,\\
\dfrac{\partial\phi}{\partial t} + \boldsymbol{u}\cdot\nabla\phi - \nabla\cdot\Big(\gamma_{nd}\Big(\dfrac{\nabla\phi}{\|\nabla\phi\|}\cdot\nabla\phi - 1\Big)\dfrac{\nabla\phi}{\|\nabla\phi\|}\Big) = 0, & \text{in } \Omega,\\
\psi - \Big(\dfrac{-1}{1+exp(\frac{\phi}{\epsilon_\psi})} + 0.5\Big) = 0, & \text{in } \Omega.
\end{cases}
\tag{12}
$$

Advantages

- Fully implicit and requires less regularity
- Neither $\boldsymbol{n}$ nor κ are explicitly calculated
- No capillary time restriction
- Reinitialization is integrated within the formulation
- Navier-Stokes with homogeneous force terms

The momentum equation gets rid of the CSF force terms

Curvature-Free Cut-off Function

- PDE for the material cut-off function with fictitious time[14]

$$\frac{\partial \psi}{\partial \tau} + \underbrace{\nabla \cdot \Big(\gamma_{nc}\psi(1-\psi)\boldsymbol{n}\Big)}_{\textbf{Conv. normal}} - \underbrace{\nabla \cdot \Big(\gamma_{nd}(\nabla\psi \cdot \boldsymbol{n})\boldsymbol{n}\Big)}_{\textbf{Diff. normal}} = 0 \qquad (13)$$

- **Conv. normal**: nonlinear convection tends to build the Heaviside step function
- **Diff. normal**: normal diffusion control the sharpness of the interface

Curvature-Free Cut-off Function[13]

$$\begin{cases} \rho(\psi)\Big(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\Big) - \nabla\cdot\Big(2\mu(\psi)\mathbf{D}(\boldsymbol{u}) + \boldsymbol{\sigma}\Big(\frac{\nabla\psi\otimes\nabla\psi}{\|\nabla\psi\|}\Big)\Big) + \nabla p = 0, & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega, \\ \frac{\partial\psi}{\partial t} + \boldsymbol{u}\cdot\nabla\psi + \nabla\cdot\Big(\gamma_{nc}(0.5+\psi)(0.5-\psi)\frac{\nabla\psi}{\|\nabla\psi\|}\Big) - \nabla\cdot\Big(\gamma_{nd}\Big(\nabla\psi\cdot\frac{\nabla\psi}{\|\nabla\psi\|}\Big)\frac{\nabla\psi}{\|\nabla\psi\|}\Big) = 0, & \text{in } \Omega. \end{cases} \tag{14}$$

- γ_{nc} = nonlinear convection
- γ_{nd} = normal diffusion

Advantages

- No need for the level set function
- Fully implicit and requires less regularity
- Neither $\boldsymbol{n}$ nor κ are explicitly calculated
- No capillary time restriction

Using Coupez[15] and Olsson and Kreiss[14] the two step method converted to one step method

Weak Formulation

- Spaces: $\mathbb{V} = \boldsymbol{H}_0^1(\Omega) := \left(H_0^1(\Omega)\right)^2, \quad \mathbb{Q} = L_0^2(\Omega), \quad \mathbb{M} = H^1(\Omega)$

$$\underbrace{\int_\Omega \rho(\psi)\boldsymbol{u}\boldsymbol{v}\,dx}_{\langle \mathcal{M}_{\boldsymbol{u}}\boldsymbol{u},\boldsymbol{v}\rangle} + \underbrace{\int_\Omega \rho(\psi)\boldsymbol{w}\cdot\nabla\boldsymbol{v}\,dx}_{\langle \mathcal{N}_{\boldsymbol{u}}(\boldsymbol{w})\boldsymbol{u},\boldsymbol{v}\rangle} + \underbrace{\int_\Omega 2\mu(\psi)\mathbf{D}(\boldsymbol{u}):\mathbf{D}(\boldsymbol{v})\,dx}_{\langle \mathcal{L}_{\boldsymbol{u}}\boldsymbol{u},\boldsymbol{v}\rangle}$$

$$+ \underbrace{\int_\Omega \boldsymbol{\tau}_m(\psi):\nabla\boldsymbol{v}\,dx}_{\langle \mathcal{C}\psi,\boldsymbol{v}\rangle} - \underbrace{\int_\Omega p\nabla\cdot\boldsymbol{v}\,dx}_{\langle \mathcal{B}^T\boldsymbol{u},\boldsymbol{q}\rangle} = 0 \quad \text{in } \Omega$$

$$\underbrace{\int_\Omega q\;\nabla\cdot\boldsymbol{u}\,dx}_{\langle \mathcal{B}\boldsymbol{u},q\rangle} = 0 \quad \text{in } \Omega$$

$$\underbrace{\int_\Omega \psi\,\xi\,dx}_{\langle \mathcal{M}_\psi\psi,\xi\rangle} + \underbrace{\int_\Omega \boldsymbol{u}\cdot\nabla\psi\,\xi\,dx}_{\langle \mathcal{N}_\psi(\boldsymbol{u})\psi,\xi\rangle} + \underbrace{\int_\Omega \mathcal{F}(\psi)\nabla\xi\,dx}_{\langle \mathcal{N}_\psi(\nabla\psi)\psi,\xi\rangle} + \underbrace{\int_\Omega \mathcal{G}(\psi)\nabla\xi\,dx}_{\langle \mathcal{L}_\psi(\nabla\psi)\psi,\xi\rangle} = 0 \quad \text{in } \Omega$$

- $\mathcal{F}(\psi) = (0.5+\psi)(0.5-\psi)\frac{\nabla\psi}{\|\nabla\psi\|}$
- $\mathcal{G}(\psi) = -\left(\nabla\psi\cdot\frac{\nabla\psi}{\|\nabla\psi\|}\right)\frac{\nabla\psi}{\|\nabla\psi\|}$

Weak Formulation

$$\begin{bmatrix} \mathcal{A}_{\boldsymbol{u}} & \mathcal{C} & \mathcal{B}^{\mathsf{T}} \\ \mathbf{0} & \mathcal{A}_{\psi} & \mathbf{0} \\ \mathcal{B} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \psi \\ p \end{bmatrix} = \begin{bmatrix} rhs_{\boldsymbol{u}} \\ rhs_{\psi} \\ rhs_{p} \end{bmatrix} \tag{15}$$

- $\mathcal{A}_{\boldsymbol{u}} = \mathcal{M}_{\boldsymbol{u}} + \mathcal{L}_{\boldsymbol{u}} + \mathcal{N}_{\boldsymbol{u}}(\boldsymbol{u})$
- $\mathcal{A}_{\psi} = \mathcal{M}_{\psi} + \mathcal{N}_{\psi}(\boldsymbol{u}) + \gamma_{nc}\mathcal{N}_{\psi}(\nabla\psi) + \gamma_{nd}\mathcal{L}_{\psi}(\nabla\psi)$

For $\mathcal{U} = (\boldsymbol{u}, \psi)$ and $\mathcal{V} = (\boldsymbol{v}, \xi)$

$$\begin{aligned} a(\mathcal{U}, \mathcal{V}) =& \langle \mathcal{A}_{\boldsymbol{u}}\boldsymbol{u}, \boldsymbol{v}\rangle + \langle \mathcal{A}_{\psi}\psi, \xi\rangle + \langle \mathcal{C}\psi, \boldsymbol{v}\rangle, \\ b(\mathcal{U}, q) =& b(\boldsymbol{u}, q). \end{aligned} \tag{16}$$

Find $(\mathcal{U}, p) \in \mathbb{Y} \times \mathbb{Q}$ s.t.

$$\begin{cases} a(\mathcal{U}, \mathcal{V}) + b(\mathcal{V}, p) = 0 & \forall \mathcal{V} \in \mathbb{Y}, \\ b(\mathcal{U}, q) \qquad\qquad\quad\;\; = 0 & \forall q \in \mathbb{Q}. \end{cases} \tag{17}$$

Contents

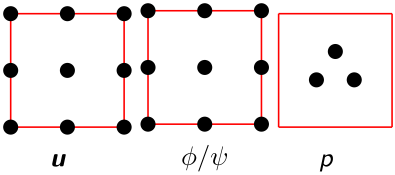
Discretization/Solver

- **Time discretization:** Crank-Nicolson
- **Space discretization:** Finite element method

$$\mathbb{V}^h = \left\{ \boldsymbol{v}_h \in \mathbb{V}, \boldsymbol{v}_{h|K} \in (Q_2(K))^2 \right\}$$

$$\mathbb{M}^h = \left\{ \varphi_h \in \mathbb{M}, \varphi_{h|K} \in Q_2(K) \right\}$$

$$\mathbb{Q}^h = \left\{ q_h \in \mathbb{Q}, q_{h|K} \in P_1^{\mathrm{disc}}(K) \right\}$$



Newton-multigrid solver

- **Nonlinearity:** Newton solver
- **Linear problem:** Multigrid

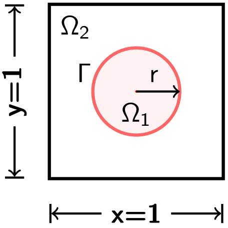
Contents

Test Case 1: Static Bubble

- $\Omega = [0,1]^2$ with bubble centred at origin $[0.5, 0.5]$
- $\sigma = \mu = \rho = 1,\ g = 0,\ r = 0.25,\ t_f = 10$

Pressure field should satisfy **Laplace Young law**

$$p_i = p_o + \frac{\sigma}{r} \qquad (18)$$

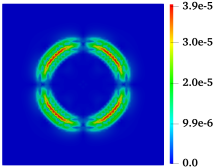


- p_i/p_o = pressure inside/outside the bubble

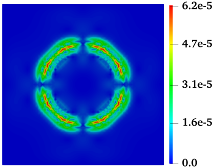
	CSF formulation (3)				S. Turek et al.[16]			
L	$\frac{\lVert p_i - p_o \rVert}{(\sigma/r)}$	$\lVert u - u_h \rVert_0$	$\lVert u - u_h \rVert_{1,h}$	NL/LL	$\frac{\lVert p_i - p_o \rVert}{(\sigma/r)}$	$\lVert u - u_h \rVert_0$	$\lVert u - u_h \rVert_{1,h}$	NL/LL
4	1.040583	1.22×10^{-3}	5.63×10^{-2}	4/1	0.954349	2.61×10^{-3}	2.08×10^{-1}	5/1
5	0.999839	9.58×10^{-5}	1.28×10^{-2}	5/1	0.979682	9.71×10^{-4}	1.54×10^{-1}	5/1
6	1.000969	4.98×10^{-5}	5.26×10^{-3}	5/1	0.992961	3.62×10^{-4}	1.13×10^{-1}	4/1
7	1.000204	2.10×10^{-6}	1.03×10^{-3}	6/1	0.997166	1.38×10^{-4}	8.21×10^{-2}	4/1

Numerical Results:[17] Static Bubble

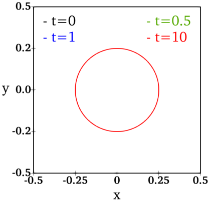
Curvature-Free Level Set



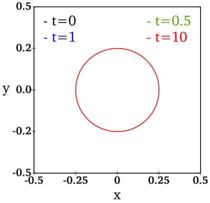
(a) velocity, $\epsilon_\psi = 1.5h$



(b) velocity, $\epsilon_\psi = 0.7h$

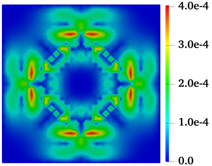


(c) shape, $\epsilon_\psi = 1.5h$

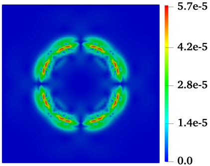


(d) shape, $\epsilon_\psi = 0.7h$

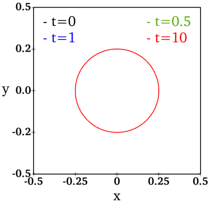
Curvature-Free Cut-off Material



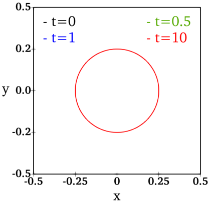
(e) velocity, $\epsilon_\psi = 1.5h$



(f) velocity, $\epsilon_\psi = 0.7h$



(g) shape, $\epsilon_\psi = 1.5h$

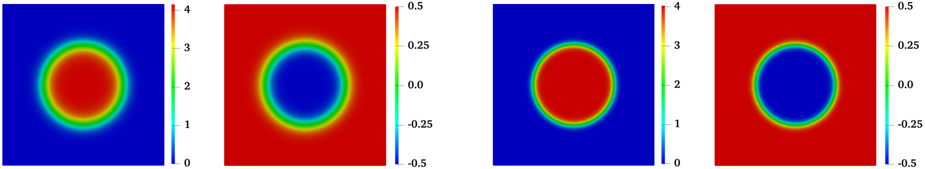


(h) shape, $\epsilon_\psi = 0.7h$

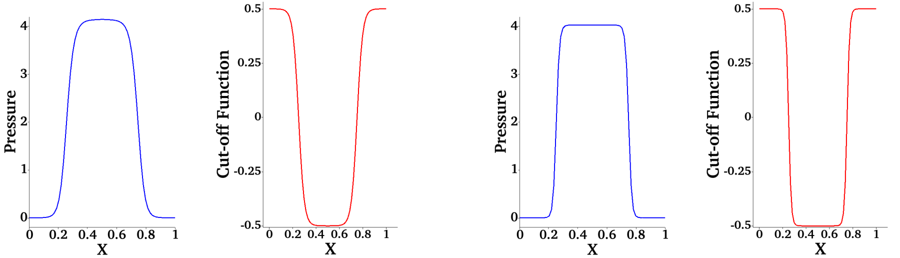
Mass is conserved!

Curvature-Free Level Set: Static Bubble

$\epsilon_\psi = 1.5h$ $\epsilon_\psi = 0.7h$

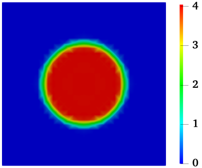


(a) pressure (b) cut-off function (c) pressure (d) cut-off function

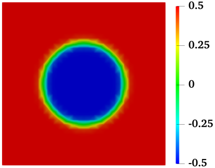


Curvature-Free Cut-off Function: Static Bubble

$\epsilon_\psi = 1.5h$

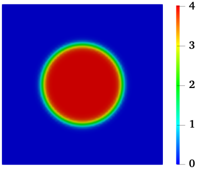


(a) pressure

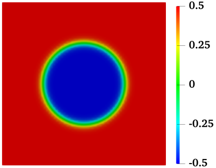


(b) cut-off function

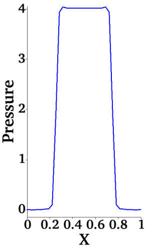
$\epsilon_\psi = 0.7h$

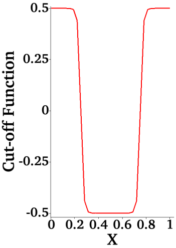


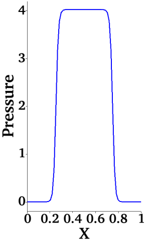
(c) pressure

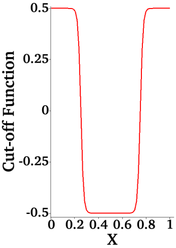


(d) cut-off function





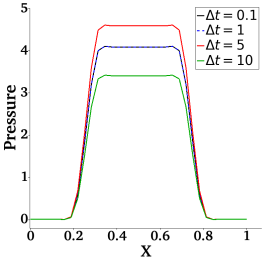




Capillary Time Step Restriction: Static Bubble

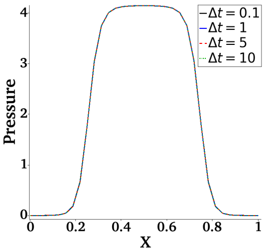
- In CSF: Some upper bounds on $\Delta t \longrightarrow \Delta t_h^{cap} < \sqrt{\frac{\rho h^3}{\sigma}}$
- Static bubble: $\rho = 10^4$, $\sigma = 1$, $h = 1/32 \longrightarrow \Delta t_h^{cap} = 0.55$

Level Set Approach



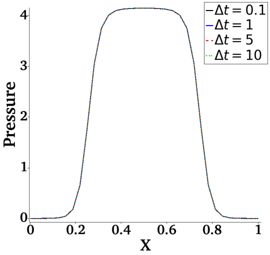
Young's Law not satisfied

Curvature free Level Set



Young's Law satisfied

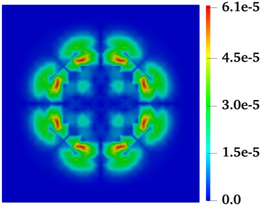
Curvature free Cut-off



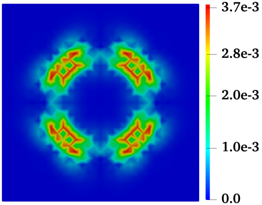
Young's Law satisfied

No capillary time step restriction for CSS!

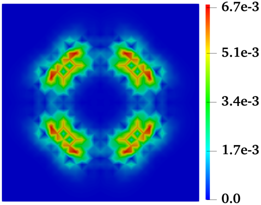
Large Surface Tension Effects: Static Bubble



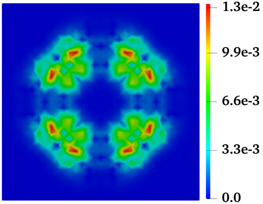
(a) $\sigma = 1$



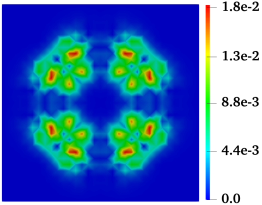
(b) $\sigma = 10$



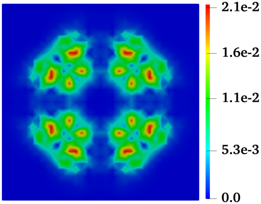
(c) $\sigma = 20$



(d) $\sigma = 50$



(e) $\sigma = 75$



(f) $\sigma = 100$

Spurious velocity magnitude increased!

Curvature-Free Cut-off Function

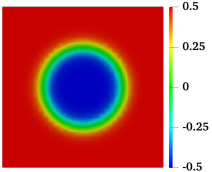
$$\begin{cases} \rho(\psi)\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) - \nabla\cdot\left(2\mu(\psi)\mathbf{D}(\boldsymbol{u}) + \boldsymbol{\sigma}\left(\frac{\nabla\psi\otimes\nabla\psi}{\|\nabla\psi\|}\right)\right) + \nabla p = 0, & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega, \\ \frac{\partial\psi}{\partial t} + \boldsymbol{u}\cdot\nabla\psi + \underbrace{\nabla\cdot\left(\gamma_{nc}(0.5+\psi)(0.5-\psi)\frac{\nabla\psi}{\|\nabla\psi\|}\right)}_{\text{nonlinear convection}} - \underbrace{\nabla\cdot\left(\gamma_{nd}\left(\nabla\psi\cdot\frac{\nabla\psi}{\|\nabla\psi\|}\right)\frac{\nabla\psi}{\|\nabla\psi\|}\right)}_{\text{normal diffusion}} = 0, & \text{in } \Omega. \end{cases}$$

Advantages

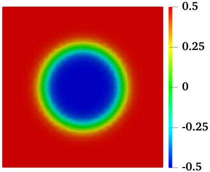
- No need for the level set function
- Fully implicit and requires less regularity
- Neither $\boldsymbol{n}$ nor κ are explicitly calculated
- No capillary time restriction

Investigate the effects of γ_{nc} and γ_{nd}

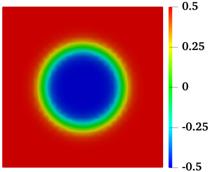
Effects of γ_{nc} and γ_{nd}: Static Bubble



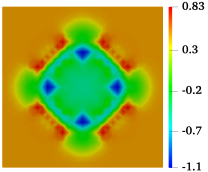
(a) $\gamma_{nc} = 10^{-5}$



(b) $\gamma_{nc} = 10^{-4}$

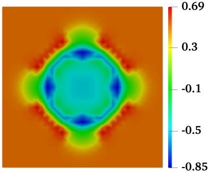


(c) $\gamma_{nc} = 10^{-3}$

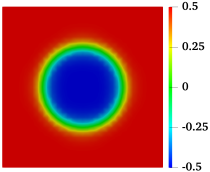


(d) $\gamma_{nc} = 10^{-2}$

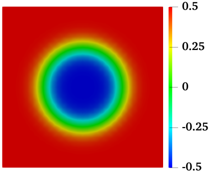
Big $\gamma_{nc} \rightarrow$ disturbs ψ range



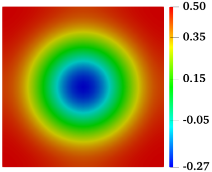
(e) $\gamma_{nd} = 10^{-5}$



(f) $\gamma_{nd} = 10^{-4}$



(g) $\gamma_{nd} = 10^{-3}$



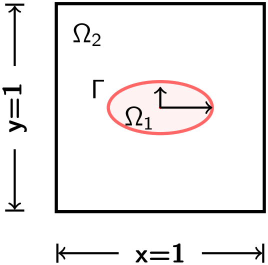
(h) $\gamma_{nd} = 10^{-2}$

ψ range recovers $\rightarrow$ suitable value of γ_{nd}

Contents

Test Case 2:[17] Oscillating Bubble

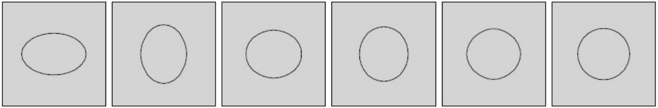
- $\Omega = [0,1]^2$, bubble centred at origin $[0.5, 0.5]$
- $\sigma = \mu = \rho = 1,\ g = 0,\ r_x = 0.25,\ r_y = 0.125$
- $t_f = 100,\ \Delta t = 10^{-2}$



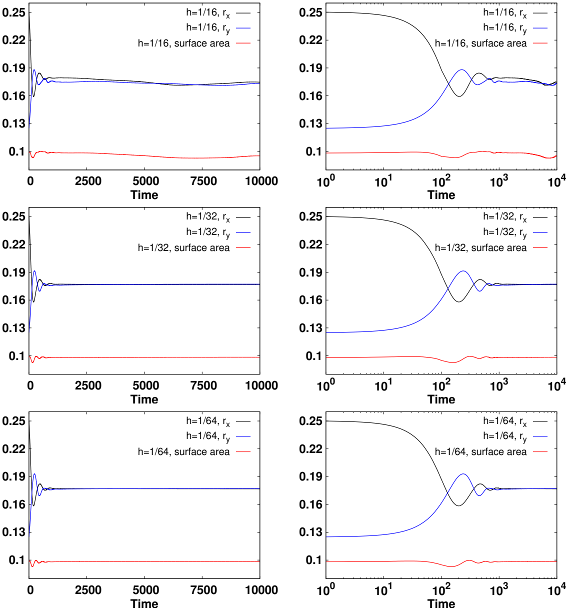
Evaluate

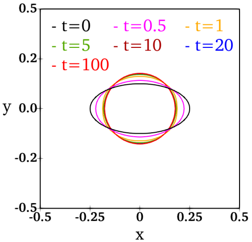
- Mass conservation? $\longrightarrow$ Surface area[17]
- Bubble shape $\longrightarrow$ Interface position[17]

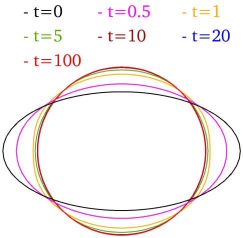
Expected behaviour[18]



Curvature-Free Level Set: Oscillating Bubble

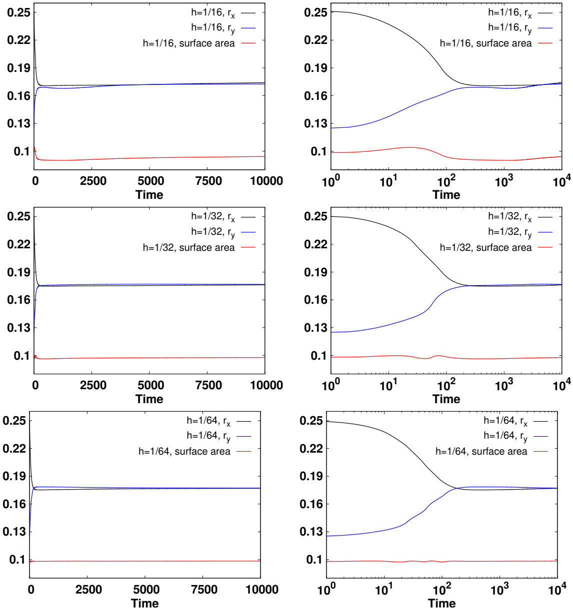


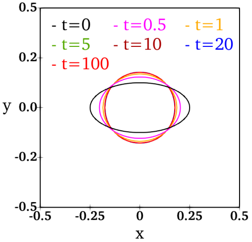


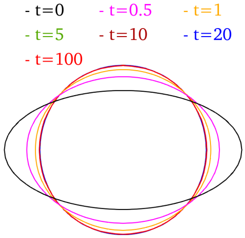


Mass is conserved!

Curvature-Free Cut-off Function: Oscillating Bubble







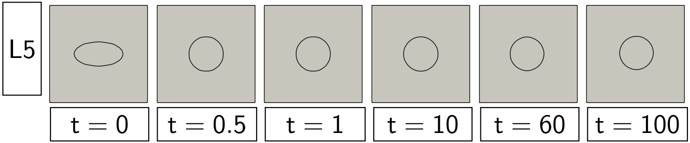
Less oscillations,

mass is conserved!

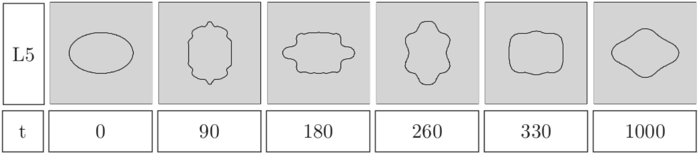
Capillary Time Step Restriction: Oscillating Bubble

- In CSF: Some upper bounds on $\Delta t \longrightarrow \Delta t_h^{cap} < \sqrt{\frac{\rho h^3}{\sigma}}$
- Oscillating bubble: $\rho = 1,\ \sigma = 1,\ h = 1/32 \longrightarrow \Delta t_h^{cap} = 0.0055$

Mesh level	L4	L5	L6
Δt_h^{cap}	0.0156	0.0055	0.0019

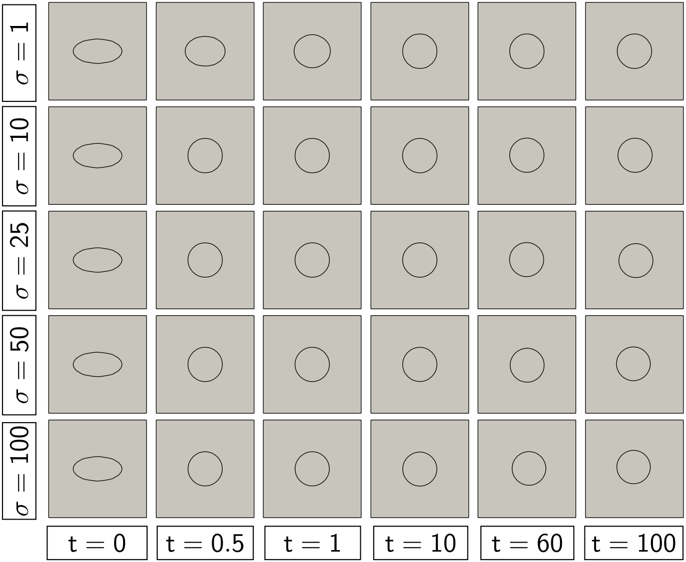


$\Delta t = 0.1$, no capillary time step restriction for CSS!

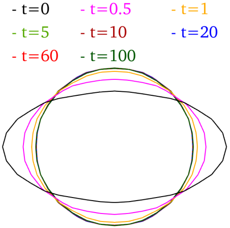


Behaviour due to capillary time step restriction[18]

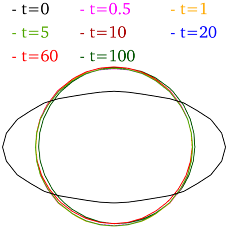
Large Surface Tension Effects: Oscillating Bubble



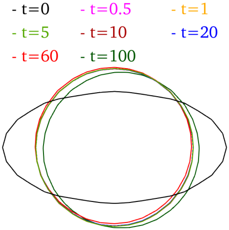
Large Surface Tension Effects: Oscillating Bubble



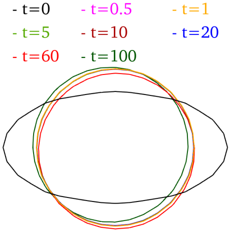
(a) $\sigma = 1$



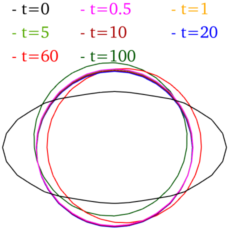
(b) $\sigma = 10$



(c) $\sigma = 25$



(d) $\sigma = 50$



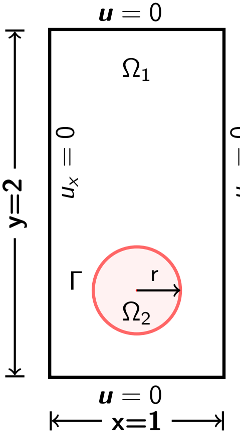
(e) $\sigma = 100$

Spurious velocity magnitude increased and interface position is effected!

Contents

Test: Rising Bubble

- $\Omega = \Omega_1 \cup \Omega_2$, bubble of radius $r = 0.2$
- centred at origin $[0.5, 0.5]$



Parameters[18]

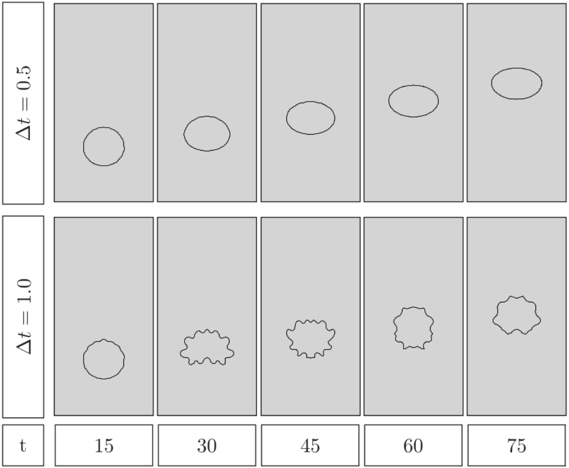
ρ_1	ρ_2	μ_1	μ_2	g	σ
10^4	10^3	1	1	-8×10^{-4}	0.5

- In CSF: Some upper bounds on Δt

$$\Delta t_h^{cap} < \sqrt{\frac{(\rho_1 + \rho_2)h^3}{4\pi\sigma}}$$

Capillary Time Step Restriction: Rising Bubble

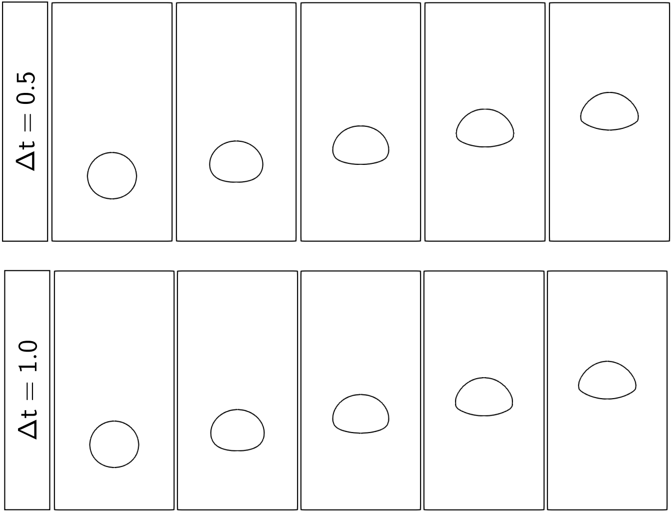
- In CSF: Some upper bounds on Δt $\longrightarrow$ $\Delta t_h^{cap} < \sqrt{\frac{(\rho_1+\rho_2)h^3}{4\pi\sigma}}$
- Rising bubble: $\sigma = 0.5,\ h = 1/80$ $\longrightarrow$ $\Delta t_h^{cap} = 0.056$(S. Hysing 2007)



Severe oscillations pollute the solution![18]

Capillary Time Step Restriction: Rising Bubble

- Rising bubble: $\sigma = 0.5,\ h = 1/64 \longrightarrow \Delta t_h^{cap} = 0.081$



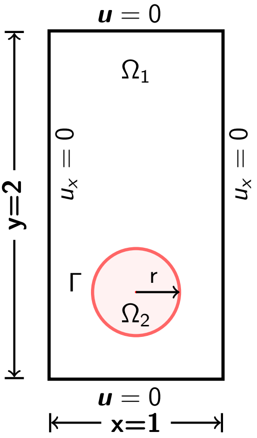
(a) $t = 15$ (b) $t = 30$ (c) $t = 45$ (d) $t = 60$ (e) $t = 75$

No capillary time step restriction for CSS!

Contents

Benchmark: Rising Bubble[19]

- $\Omega = \Omega_1 \cup \Omega_2$, bubble of radius $r = 0.25$
- centred at origin $[0.5, 0.5]$



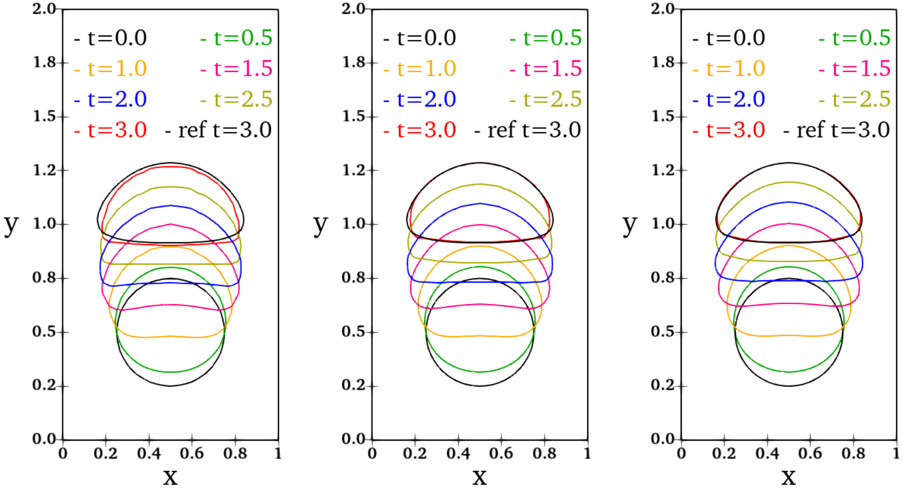
Parameters[19]

ρ_1	ρ_2	μ_1	μ_2	g	σ	Re	Eo	ρ_1/ρ_2	μ_1/μ_2
1000	100	10	1	0.98	24.5	35	10	10	10

Evaluate

- rise velocity $\boldsymbol{u} = \int_{\Omega_2} u dx / \int_{\Omega_2} 1 dx$
- center of mass $X = \int_{\Omega_2} u dx / \int_{\Omega_2} 1 dx$
- circularity $\not{c} = P_a / P_b = \pi d_a / P_b$
- bubble shape

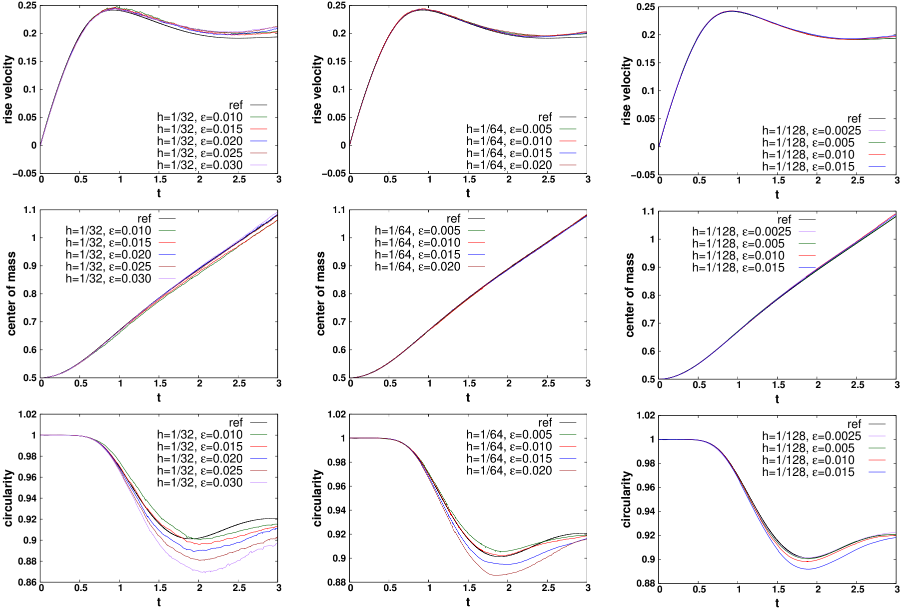
Curvature-Free Cut-off Function: Rising Bubble



(a) $h = 1/32, \epsilon = 0.015$ (b) $h = 1/64, \epsilon = 0.010$ (c) $h = 1/128, \epsilon = 0.005$

Good conformity with TP2D(Hysing et al.[20]), experiment(Clift. et al.[21])

Curvature-Free Cut-off Function: Rising Bubble



Curvature-Free Cut-off Function: Rising Bubble

h	ϵ	$\not{c}_{min}$	$t\|_{\not{c}=\not{c}_{min}}$	$\boldsymbol{u}_{max}$	$t\|_{\boldsymbol{u}=\boldsymbol{u}_{max}}$	$X(t=3)$
1/32	0.015	0.8946	2.170	0.2468	0.940	1.0650
1/64	0.010	0.9022	1.955	0.2443	0.925	1.0822
1/128	0.005	0.9010	1.890	0.2426	0.915	1.0815
TP2D[20]		0.9013	1.9041	0.2417	0.9213	1.0813
FreeLIFE[20]		0.9011	1.8750	0.2421	0.9313	1.0799
MooNMD[20]		0.9013	1.9000	0.2417	0.9239	1.0817
Hosseini et al.[22]	0.005	0.9013	1.9200	0.2420	0.9200	1.0794

- ϵ: Interface thickness
- $\not{c}_{min}$: Min. circularity
- $t|_{\not{c}=\not{c}_{min}}$: Min. circularity at t
- $\boldsymbol{u}_{max}$: Max. velocity
- $t|_{\boldsymbol{u}=\boldsymbol{u}_{max}}$: Max. velocity at t
- $X(t=3)$: Bubble center

The results are in good aggrement with Hysing et al.[20] Hosseini et al.[22]

Curvature-Free Cut-off Function: Rising Bubble

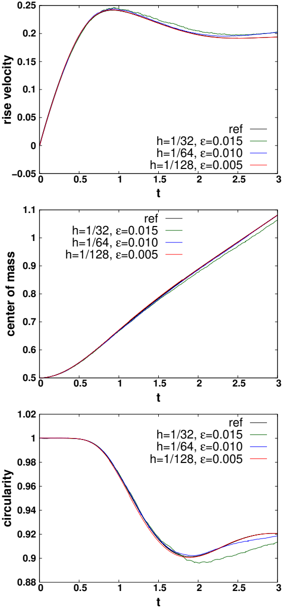
$$\|e\|_1 = \frac{\sum_{t=1}^{N} |q_{t,ref} - q_t|}{\sum_{t=1}^{N} |q_{t,ref}|}, \|e\|_2 = \left(\frac{\sum_{t=1}^{N} |q_{t,ref} - q_t|^2}{\sum_{t=1}^{N} |q_{t,ref}|^2}\right)^{1/2}, \|e\|_\infty = \frac{max_t|q_{t,ref} - q_t|}{max_t|q_{t,ref}|}.$$

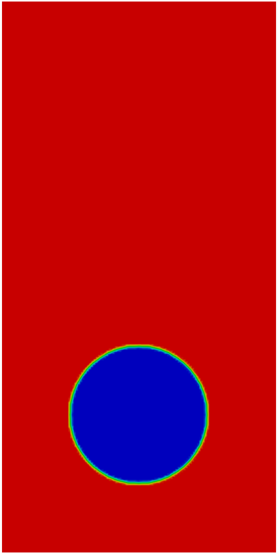
$$EOC_{(.)} = \frac{log(\|e_{i-1}\|_{(.)} / \|e_i\|_{(.)})}{log(h_{i-1}/h_i)}.$$

q	h	ϵ	$\|e\|_1$	EOC_1	$\|e\|_2$	EOC_2	$\|e\|_\infty$	EOC_∞
X	1/32	0.015	0.0098		0.0121		0.0170	
	1/64	0.010	0.0024	2.0280	0.0029	2.0423	0.0039	2.1335
	1/128	0.005	0.0007	1.7776	0.0009	1.6881	0.0012	1.7004
$\boldsymbol{u}$	1/32	0.015	0.0252		0.0268		0.0377	
	1/64	0.010	0.0130	0.9549	0.0158	0.7623	0.0341	0.1448
	1/128	0.005	0.0035	1.8931	0.0045	1.8119	0.0102	1.7412
$\not c$	1/32	0.015	0.0044		0.0063		0.0123	
	1/64	0.010	0.0016	1.4594	0.0021	1.5849	0.0045	1.4507
	1/128	0.005	0.0007	1.1926	0.0009	1.2224	0.0018	1.3219

Order of convergence between 1 and approx. 2 in all norms

Curvature-Free Cut-off Function: Rising Bubble





Contents

Summary

A monolithic Newton-multigrid solver for multiphase flow problems is developed

- Solves velocity, pressure and interface position simultaneously
- Nonlinearity: Treated with a Newton solver
- Linearized system: Solved by geometrical multigrid
- Requires less regularity
- No explicit calculation of $\boldsymbol{n}$ and κ
- No capillary time restriction
- Reinitialization issue is integrated within the formulations

Sr. No.	Problem Formulations	Static Bubble (Test Case 1)	Oscillating Bubble (Test Case 2)	Rising Bubble (Benchmark)
1	Level Set Approach	✓	✓	
2	Level Set with Material Cut-off Function	✓	✓	
3	Curvature-Free Level Set	✓	✓	
4	Curvature-Free Cut-off Material Function	✓	✓	✓

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Thank you for your attention!

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