Monolithic Newton-Multigrid Solver for Multiphase Flow Problems with Surface Tension

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fakultät für mathematik

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Motivation

Multiphase flows in Nature and Industry

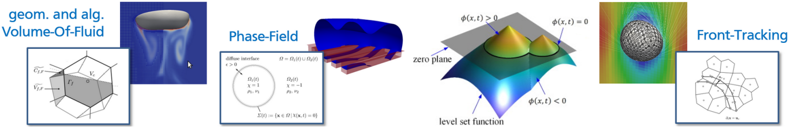
- **Gas-Liquid**
  - Condensation
  - Boiling
  - Spray cooling
  - Biological flows
- **Solid-Liquid**
  - Emulsification
  - Slurries
  - food processing
- **Gas-Solid**
  - Fluidization
  - Coal burners
- **Liquid-Liquid**
  - Petroleum extraction
- **Oil exploration:**

$$\underbrace{Oil - air - water - solid}_{4-phases}$$

Interface Tracking/Capturing

Interface tracking: Immersed boundary,[1] Front tracking[2]

Interface capturing: Volume of fluid,[3] Phase field,[4,5] Level set[6,7]



Continuum surface force (CSF): Interface as a smooth transition[8,9]

Continuum surface stress (CSS): Surface force term as the divergence of the stress tensor[8,10]

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CSF Formulation

The incompressible Navier Stokes equations

$$\begin{cases} \rho\left(\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) - \nabla\cdot 2\mu\mathbf{D}(\boldsymbol{u}) + \nabla p = f, & \textbf{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0, & \textbf{in } \Omega. \end{cases} \tag{1}$$

- ρ = density
- $\mathbf{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$
- $\boldsymbol{u}$ = velocity
- p = pressure
- μ = viscosity
- f = external force

$$f = \kappa\sigma\boldsymbol{n}\delta_\Gamma \tag{2}$$

- κ = curvature
- $\boldsymbol{\sigma}$ = surface tension constant
- $\boldsymbol{n}$ = normal to the interface
- δ_Γ = delta function

CSF Formulation

$$\begin{cases} \rho(\Gamma)\Big(\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\Big) - \nabla\cdot 2\mu(\Gamma)\mathbf{D}(\boldsymbol{u}) + \nabla p = \kappa\sigma\boldsymbol{n}\delta_\Gamma, & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega. \end{cases} \tag{3}$$

- Interface capturing by level set method

$$\frac{\partial \phi}{\partial t} + \boldsymbol{u}\cdot\nabla\phi = 0 \tag{4}$$

- ϕ = level set function

This transport equation can be efficiently solved due to the choice of a smooth level set function

Advantages

- Exact representation of the interface

$$\Gamma = \{x \in \Omega, \phi = 0\} \tag{5}$$

- Provides derived geometrical quantities ($\boldsymbol{n}, \kappa$)

$$\boldsymbol{n} = \frac{\nabla \phi}{\|\nabla \phi\|}, \quad \kappa = -\nabla \cdot \boldsymbol{n} \tag{6}$$

- The surface tension force in terms of the level set function

$$f = \kappa \sigma \boldsymbol{n} \delta_\Gamma(\phi) \tag{7}$$

The signed distance function is the natural choice for the level set!

Problems and Challenges

- **Spurious velocity** Unphysical flows near the interface
  - Unphysical interface movement
  - Misinterpret the flow physics
- **Large surface tension**
  - Spurious velocity may grow
  - Destroy the interface
- **Observed** $\longrightarrow$ explicit/implicit interface representation
- **Explicit treatment of surface tension** Capillary time restriction
- **Explicit reinitialization** Requires perfect interface description

Towards a fully implicit treatment

Curvature-Free Level Set

- Material cut-off function

$$\psi(x,t) = \begin{cases} +\dfrac{1}{2} & \text{if} \quad \phi(x,t) \geq 0 \\ -\dfrac{1}{2} & \text{if} \quad \phi(x,t) < 0 \end{cases} \tag{8}$$

Features

- Enhances the accuracy
- ψ is material characteristic $\longrightarrow$ conservative level set
- Regularization of material cut-off function

$$\psi = \frac{-1}{1 + exp(\frac{\phi}{\epsilon_\psi})} + 0.5 \tag{9}$$

Curvature-Free Level Set

New set of equations for multiphase flow

$$
\begin{cases}
\rho(\psi)\Big(\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\Big) - div\boldsymbol{\tau} + \nabla p = 0, \quad \text{in } \Omega, \\
\nabla\cdot\boldsymbol{u} = 0, \quad \text{in } \Omega, \\
\dfrac{\partial \phi}{\partial t} + \boldsymbol{u}\cdot\nabla\phi = 0, \quad \text{in } \Omega, \\
\psi - \Big(\dfrac{-1}{1+exp(\frac{\phi}{\epsilon_\psi})} + 0.5\Big) = 0, \quad \text{in } \Omega.
\end{cases}
\tag{10}
$$

- $\phi =$ level set function
- $\epsilon_\psi =$ interface thickness
- $\boldsymbol{\tau} = (\boldsymbol{\tau}_s + \boldsymbol{\tau}_m)$ full stress tensor
- $\psi =$ cut-off function

Curvature-Free Level Set

- Viscous stress

$$\boldsymbol{\tau}_s = 2\mu(\psi)\mathbf{D}(\boldsymbol{u})$$

- Multiphase stress

$$\boldsymbol{\tau}_m = -\sigma\left(\frac{\nabla\psi \otimes \nabla\psi}{\|\nabla\psi\|}\right)$$

- Signed distance function needs to satisfy the constraint

$$\|\nabla\phi\| = 1 \quad \Longleftrightarrow \quad \boldsymbol{n} \cdot \nabla\phi = 1$$

- For implicit approach, constraint is imposed

$$\frac{\partial\phi}{\partial t} + \boldsymbol{u} \cdot \nabla\phi - \underbrace{\nabla \cdot (\gamma_{nd}(\boldsymbol{n} \cdot \nabla\phi - 1)\boldsymbol{n})}_{\textit{reinitialization}} = 0$$

- γ_{nd} = penalty parameter

Curvature-Free Level Set[11]

$$
\begin{cases}
\rho(\psi)\Big(\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\Big) - \nabla\cdot\Big(2\mu(\psi)\mathbf{D}(\boldsymbol{u}) + \sigma\Big(\dfrac{\nabla\psi\otimes\nabla\psi}{\|\nabla\psi\|}\Big)\Big) + \nabla p = 0, & \text{in } \Omega,\\
\nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega,\\
\dfrac{\partial\phi}{\partial t} + \boldsymbol{u}\cdot\nabla\phi - \nabla\cdot\Big(\gamma_{nd}\Big(\dfrac{\nabla\phi}{\|\nabla\phi\|}\cdot\nabla\phi - 1\Big)\dfrac{\nabla\phi}{\|\nabla\phi\|}\Big) = 0, & \text{in } \Omega,\\
\psi - \Big(\dfrac{-1}{1+exp(\frac{\phi}{\epsilon_\psi})} + 0.5\Big) = 0, & \text{in } \Omega.
\end{cases}
\tag{11}
$$

Advantages

- Fully implicit and requires less regularity
- Neither $\boldsymbol{n}$ nor κ are explicitly calculated
- No capillary time restriction
- Reinitialization is integrated within the formulation
- Navier-Stokes with homogeneous force terms

The momentum equation gets rid of the CSF force terms

Curvature-Free Cut-off Function

- PDE for the material cut-off function with fictitious time[12]

$$\frac{\partial \psi}{\partial \tau} + \underbrace{\nabla \cdot \Big(\gamma_{nc}\psi(1-\psi)\boldsymbol{n}\Big)}_{\textbf{Conv. normal}} - \underbrace{\nabla \cdot \Big(\gamma_{nd}(\nabla\psi \cdot \boldsymbol{n})\boldsymbol{n}\Big)}_{\textbf{Diff. normal}} = 0 \qquad (12)$$

- **Conv. normal**: nonlinear convection tends to build the Heaviside step function
- **Diff. normal**: normal diffusion control the sharpness of the interface

Curvature-Free Cut-off Function[11]

$$
\begin{cases}
\rho(\psi)\Big(\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\Big) - \nabla\cdot\Big(2\mu(\psi)\mathbf{D}(\boldsymbol{u}) + \boldsymbol{\sigma}\Big(\dfrac{\nabla\psi\otimes\nabla\psi}{\|\nabla\psi\|}\Big)\Big) + \nabla p = 0, & \text{in } \Omega,\\
\nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega,\\
\dfrac{\partial\psi}{\partial t} + \boldsymbol{u}\cdot\nabla\psi + \nabla\cdot\Big(\gamma_{nc}(0.5+\psi)(0.5-\psi)\dfrac{\nabla\psi}{\|\nabla\psi\|}\Big) - \nabla\cdot\Big(\gamma_{nd}\Big(\nabla\psi\cdot\dfrac{\nabla\psi}{\|\nabla\psi\|}\Big)\dfrac{\nabla\psi}{\|\nabla\psi\|}\Big) = 0, & \text{in } \Omega.
\end{cases}
\tag{13}
$$

- γ_{nc} = nonlinear convection
- γ_{nd} = normal diffusion

Advantages

- No need for the level set function
- Fully implicit and requires less regularity
- Neither $\boldsymbol{n}$ nor κ are explicitly calculated
- No capillary time restriction

The momentum equation gets rid of the CSF force terms

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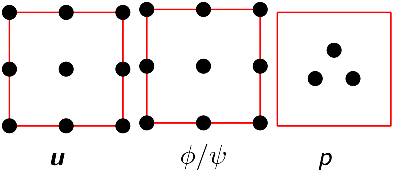
Discretization/Solver

- **Time discretization:** Crank-Nicolson
- **Space discretization:** FEM Q_2, P_1^{disc} for $\boldsymbol{u}$, p and Q_2 for ϕ,ψ



Newton-multigrid solver

- **Nonlinearity:** Treated by Newton solver
- **Linear problem:** Solved by multigrid

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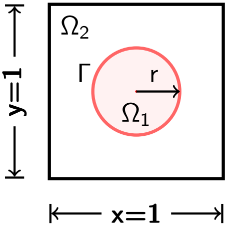
Test Case 1: Static Bubble

- $\Omega = [0,1]^2$ with bubble centred at origin $[0.5, 0.5]$
- $\sigma = \mu = \rho = 1,\ g = 0,\ r = 0.25,\ t_f = 10$

Pressure field should satisfy **Laplace Young law**

$$p_i = p_o + \frac{\sigma}{r} \tag{14}$$

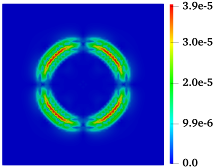


- p_i/p_o = pressure inside/outside the bubble

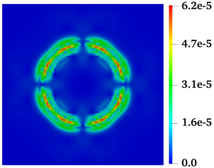
	CSF formulation (3)				S. Turek et al.[13]			
L	$\frac{\lVert p_i - p_o \rVert}{(\sigma/r)}$	$\lVert u - u_h \rVert_0$	$\lVert u - u_h \rVert_{1,h}$	NL/LL	$\frac{\lVert p_i - p_o \rVert}{(\sigma/r)}$	$\lVert u - u_h \rVert_0$	$\lVert u - u_h \rVert_{1,h}$	NL/LL
4	1.040583	1.22×10^{-3}	5.63×10^{-2}	4/1	0.954349	2.61×10^{-3}	2.08×10^{-1}	5/1
5	0.999839	9.58×10^{-5}	1.28×10^{-2}	5/1	0.979682	9.71×10^{-4}	1.54×10^{-1}	5/1
6	1.000969	4.98×10^{-5}	5.26×10^{-3}	5/1	0.992961	3.62×10^{-4}	1.13×10^{-1}	4/1
7	1.000204	2.10×10^{-6}	1.03×10^{-3}	6/1	0.997166	1.38×10^{-4}	8.21×10^{-2}	4/1

Numerical Results:[14] Static Bubble

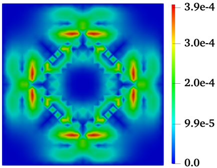
Curvature-Free Level Set | **Curvature-Free Cut-off Material**



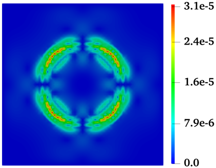
(a) velocity, $\epsilon_\psi = 1.5h$



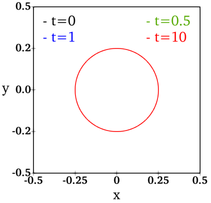
(b) velocity, $\epsilon_\psi = 0.7h$



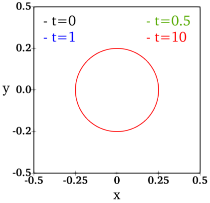
(e) velocity, $\epsilon_\psi = 1.5h$



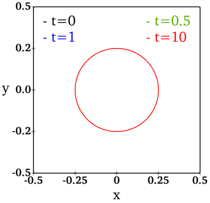
(f) velocity, $\epsilon_\psi = 0.7h$



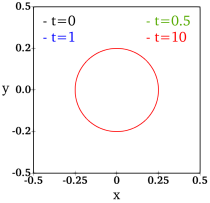
(c) shape, $\epsilon_\psi = 1.5h$



(d) shape, $\epsilon_\psi = 0.7h$



(g) shape, $\epsilon_\psi = 1.5h$

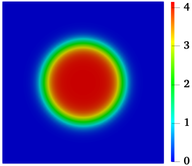


(h) shape, $\epsilon_\psi = 0.7h$

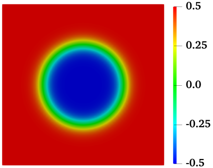
Mass is conserved!

Curvature-Free Level Set: Static Bubble

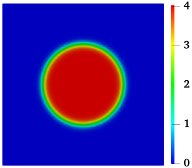
$\epsilon_\psi = 1.5h$ | $\epsilon_\psi = 0.7h$



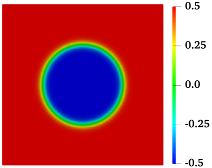
(i) pressure



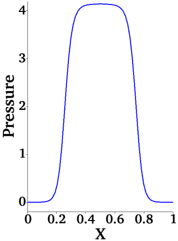
(j) cut-off function

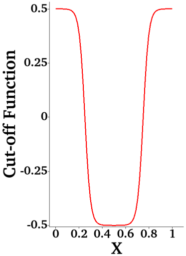


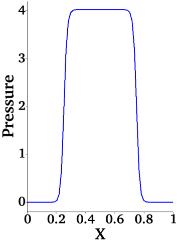
(k) pressure

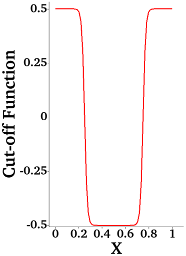


(l) cut-off function



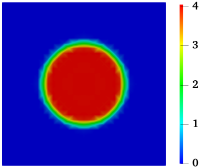




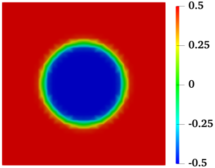


Curvature-Free Cut-off Function: Static Bubble

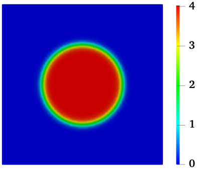
$\epsilon_\psi = 1.5h$ $\qquad$ $\epsilon_\psi = 0.7h$



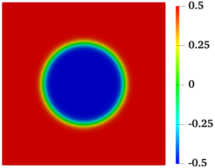
(m) pressure



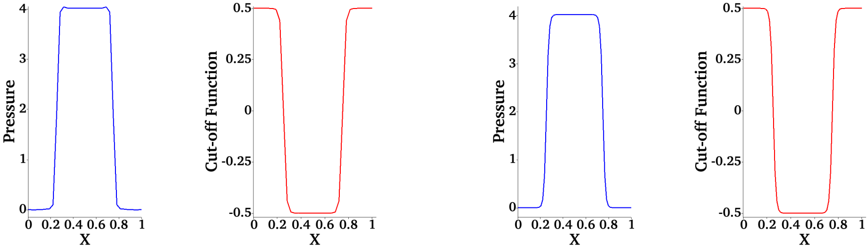
(n) cut-off function



(o) pressure



(p) cut-off function



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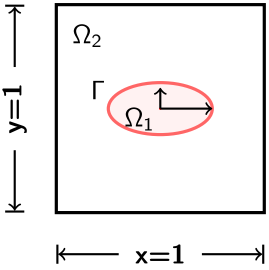
Test Case 2:[14] Oscillating Bubble

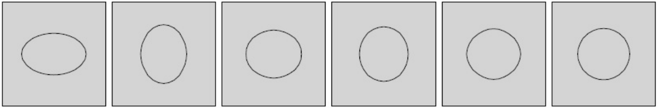
- $\Omega = [0,1]^2$, bubble centred at origin $[0.5, 0.5]$
- $\sigma = \mu = \rho = 1,\ g = 0,\ r_x = 0.25,\ r_y = 0.125$
- $t_f = 100,\ \Delta t = 10^{-2}$



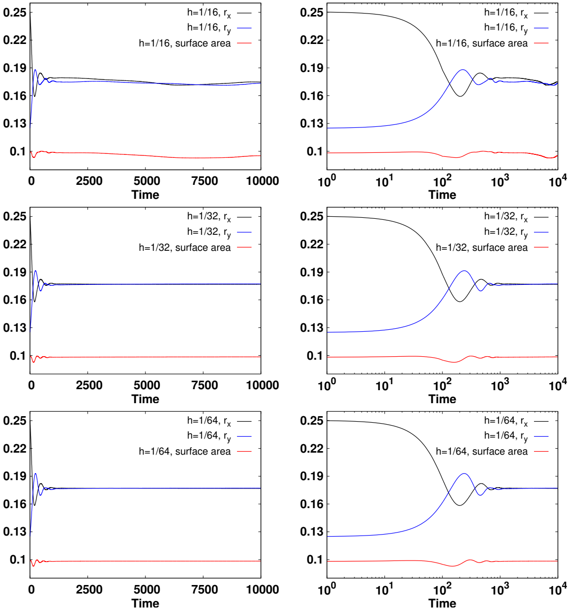
Evaluate

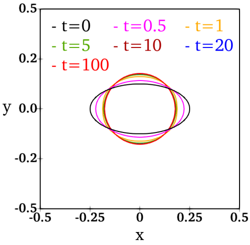
- Mass conservation? $\longrightarrow$ Surface area[14]
- Bubble shape $\longrightarrow$ Interface position[14]

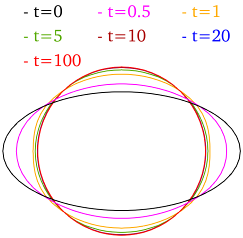
Expected behaviour[15]



Curvature-Free Level Set: Oscillating Bubble

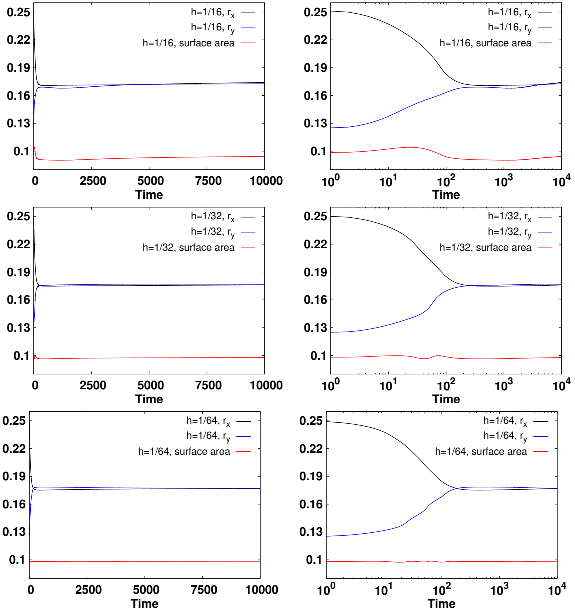


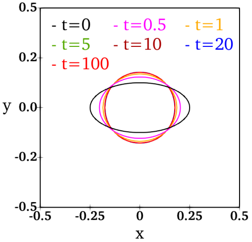




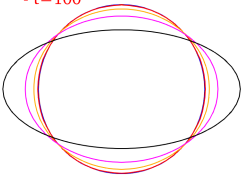
Mass is conserved!

Curvature-Free Cut-off Function: Oscillating Bubble





- t=0 - t=0.5 - t=1
- t=5 - t=10 - t=20
- t=100



Less oscillations,

mass is conserved!

Curvature-Free Cut-off Function

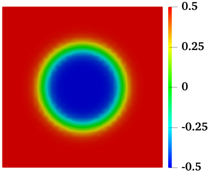
$$\begin{cases} \rho(\psi)\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) - \nabla\cdot\left(2\mu(\psi)\mathbf{D}(\boldsymbol{u}) + \boldsymbol{\sigma}\left(\frac{\nabla\psi\otimes\nabla\psi}{\|\nabla\psi\|}\right)\right) + \nabla p = 0, & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega, \\ \frac{\partial\psi}{\partial t} + \boldsymbol{u}\cdot\nabla\psi + \underbrace{\nabla\cdot\left(\gamma_{nc}(0.5+\psi)(0.5-\psi)\frac{\nabla\psi}{\|\nabla\psi\|}\right)}_{\text{nonlinear convection}} - \underbrace{\nabla\cdot\left(\gamma_{nd}\left(\nabla\psi\cdot\frac{\nabla\psi}{\|\nabla\psi\|}\right)\frac{\nabla\psi}{\|\nabla\psi\|}\right)}_{\text{normal diffusion}} = 0, & \text{in } \Omega. \end{cases}$$

Advantages

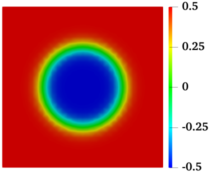
- No need for the level set function
- Fully implicit and requires less regularity
- Neither $\boldsymbol{n}$ nor κ are explicitly calculated
- No capillary time restriction

Investigate the effects of γ_{nc} and γ_{nd}

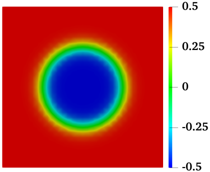
Effects of γ_{nc} and γ_{nd}: Static Bubble



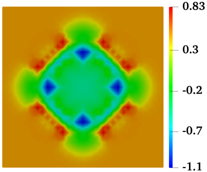
(q) $\gamma_{nc} = 10^{-5}$



(r) $\gamma_{nc} = 10^{-4}$

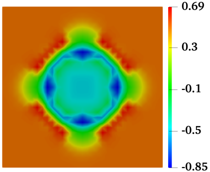


(s) $\gamma_{nc} = 10^{-3}$

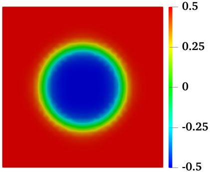


(t) $\gamma_{nc} = 10^{-2}$

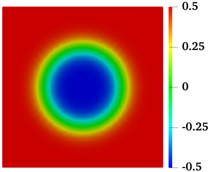
Big $\gamma_{nc} \to$ disturbs ψ range



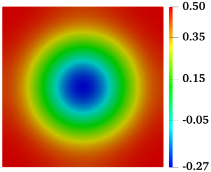
(u) $\gamma_{nd} = 10^{-5}$



(v) $\gamma_{nd} = 10^{-4}$



(w) $\gamma_{nd} = 10^{-3}$



(x) $\gamma_{nd} = 10^{-2}$

ψ range recovers $\to$ suitable value of γ_{nd}

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Benchmark: Rising Bubble[16]

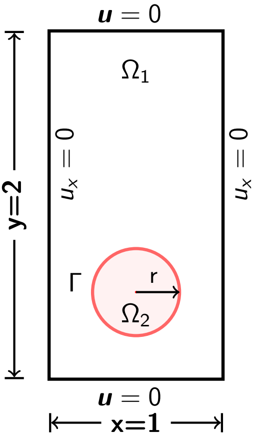
- $\Omega = \Omega_1 \cup \Omega_2$, bubble of radius $r = 0.25$
- centred at origin $[0.5, 0.5]$

Parameters[16]

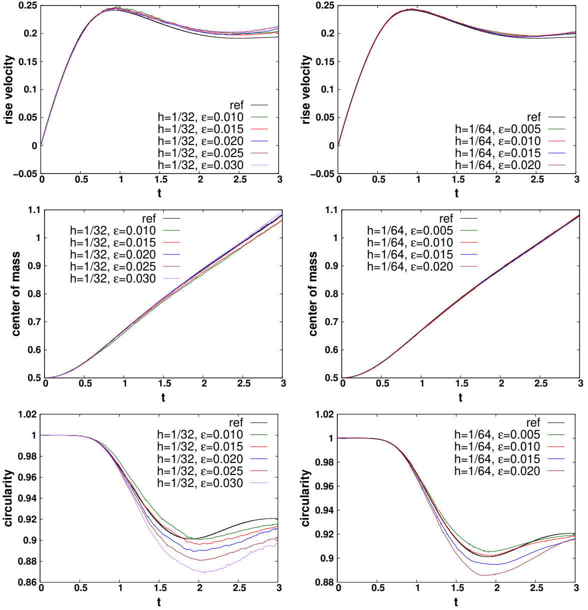
ρ_1	ρ_2	μ_1	μ_2	g	σ	Re	Eo	ρ_1/ρ_2	μ_1/μ_2
1000	100	10	1	0.98	24.5	35	10	10	10

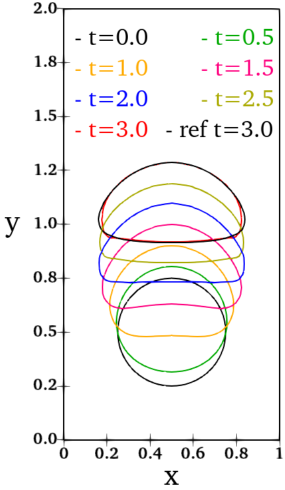
Evaluate

- rise velocity $U_c = \int_{\Omega_2} u dx / \int_{\Omega_2} 1 dx$
- center of mass $X_c = \int_{\Omega_2} u dx / \int_{\Omega_2} 1 dx$
- circularity $c = P_a / P_b = \pi d_a / P_b$
- bubble shape



Curvature-Free Cut-off Function: Rising Bubble





Model accurately validated with TP2D[16,17]!

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Summary

A monolithic Newton-multigrid solver for multiphase flow problems is developed

- Solves velocity, pressure and interface position simultaneously
- Nonlinearity: Treated with a Newton solver
- Linearized system: Solved by geometrical multigrid
- Requires less regularity
- No explicit calculation of $\boldsymbol{n}$ and κ
- No capillary time restriction
- Reinitialization issue is integrated within the formulations

Sr. No.	Problem Formulations	Static Bubble (Test Case 1)	Oscillating Bubble (Test Case 2)	Rising Bubble (Benchmark)
1	Level Set Approach	✓	✓	
2	Level Set with Material Cut-off Function	✓	✓	
3	Curvature-Free Level Set	✓	✓	
4	Curvature-Free Cut-off Material Function	✓	✓	✓

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References I

[1] Charles S Peskin.
Numerical analysis of blood flow in the heart.
Journal of Computational Physics, 25(3):220–252, 1977.

[2] S. O. Unverdi and G. Tryggvason.
A front-tracking method for viscous, incompressible, multi-fluid flows.
Journal of Computational Physics, 100(1):25–37, 1992.

[3] C. W. Hirt and B. D. Nichols.
Volume of fluid (VOF) method for the dynamics of free boundaries.
Journal of Computational Physics, 39(1):201–225, 1981.

[4] D. M. Anderson, G. B. McFadden, and A. A. Wheeler.
Diffuse-interface methods in fluid mechanics.
Annual Review of Fluid Mechanics, 30(1):139–165, 1998.

[5] V. E. Badalassi, H. D. Ceniceros, and S. Banerjee.
Computation of multiphase systems with phase field models.
Journal of Computational Physics, 190(2):371–397, 2003.

[6] S. Osher and J. A. Sethian.
Fronts propagating with curvature-dependent speed: Algorithms based on hamilton-jacobi formulations.
Journal of Computational Physics, 79(1):12–49, 1988.

[7] J. A. Sethian.
Level Set Methods and Fast Marching Methods: Evolving Interfaces in Computational Geometry, Fluid Mechanics, Computer Vision, and Materials Science.
Cambridge monographs on applied and computational mathematics. Cambridge University Press, 1999.

References II

[8] M. Boger.
Numerical Modeling of Compressible Two-Phase Flows with a Pressure-Based Method.
PhD thesis, University of Stuttgart, Institute of Aerodynamics and Gas Dynamics, December 2013.

[9] J. U. Brackbill, D. B. Kothe, and C. Zemach.
A continuum method for modeling surface tension.
Journal of Computational Physics, 100(2):335–354, 1992.

[10] B. Lafaurie, C. Nardone, R. Scardovelli, S. Zaleski, and G. Zanetti.
Modelling merging and fragmentation in multiphase flows with SURFER.
Journal of Computational Physics, 133(1):134–147, 1994.

[11] A. Ouazzi, S. Turek, and H. Damanik.
A curvature–free multiphase flow solver via surface stress–based formulation.
International Journal for Numerical Methods in Fluids, 88(1):18–31, 2018.
https://doi.org/10.1002/fld.4509.

[12] E. Olsson and G. Kreiss.
A conservative level set method for two phase flow.
Journal of Computational Physics, 210(1):225–246, 2005.

[13] S. Turek, A. Ouazzi, and J. Hron.
On pressure separation algorithms (PSEPA) for improving the accuracy of incompressible flow simulations.
International Journal for Numerical Methods in Fluids, 59(4):387–403, 2008.

[14] M. A. Afaq, S. Turek, A. Ouazzi, and A. Fatima.
Monolithic Newton-multigrid solver for multiphase flow problems with surface tension.
Editorial Universitat Politècnica de València, 2021.
ISBN 978-84-9048-969-7.

References III

[15] S. Hysing.
Numerical simulation of immiscible fluids with FEM level set techniques.
PhD thesis, Technische Universität Dortmund, December 2007.

[16] CFD Benchmarking(rising bubble).
http://www.mathematik.tu-dortmund.de/ featflow/en/benchmarks/cfdbenchmarking/bubble.html.

[17] S. Hysing, S. Turek, D. Kuzmin, N. Parolini, E. Burman, S. Ganesan, and L. Tobiska.
Quantitative benchmark computations of two-dimensional bubble dynamics.
International Journal for Numerical Methods in Fluids, 60(11):1259–1288, 2009.

Thank you for your Attention!

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