Efficient numerical and algorithmic realization of a pressure Poisson complement solver for the incompressible Navier-Stokes equations in FEAT3

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Overview

Introduction

non-stationary incompressible Navier-Stokes equations

$$u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f, \qquad (-)\nabla \cdot u = 0 \quad \text{in } \Omega \times [0, T]$$

with initial and boundary conditions

- domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$
- time limit T
- velocity u
- pressure p
- kinematic viscosity ν
- rhs f (external source)

Introduction

weak formulation

$$\int_\Omega \partial_t u \cdot v + \int_\Omega (u \cdot \nabla u) \cdot v - \nu \int_\Omega \Delta u \cdot v + \int_\Omega \nabla p \cdot v = \int_\Omega f \cdot v$$

partial derivative

$$-\nu \int_\Omega \Delta u \cdot v = \nu \int_\Omega \nabla u : \nabla v - \nu \int_{\partial\Omega} (\nabla u \cdot n) \cdot v$$
$$\int_\Omega \nabla p \cdot v = -\int_\Omega p \nabla \cdot v + \int_{\partial\Omega} p v \cdot n$$

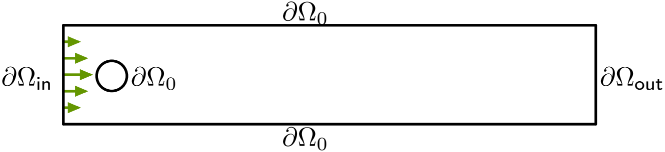
discretization

$$(\partial_t u_h, v_h) + (u_h \cdot \nabla u_h, v_h) + \nu(\nabla u_h, \nabla v_h) - (p_h, \nabla \cdot v_h) = (f, v_h)$$
$$(-)(q_h, \nabla \cdot u_h) = 0$$

Introduction

boundary conditions

Given area $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with boundary $\partial\Omega = \partial\Omega_{\text{in}} \cup \partial\Omega_0 \cup \partial\Omega_{\text{out}}$.



- On $\partial\Omega_D := \partial\Omega_{\text{in}} \cup \partial\Omega_0$ dirichlet boundary are given:

$$u_{|\partial\Omega_{\text{in}}} := u_{\text{in}} \quad \text{and} \quad u_{|\partial\Omega_0} := 0$$

- On $\partial\Omega_{\text{out}}$ „do nothing“ $\Rightarrow -\nu \int_{\partial\Omega} (\nabla u \cdot n) \cdot v + \int_{\partial\Omega} pv \cdot n = 0$

„do nothing“

leave the solution and the test space free on that portion of the boundary

Introduction

matrix formulation

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \partial_t u_h \\ 0 \end{bmatrix} + \begin{bmatrix} K(u_h) + \nu L & B \\ (+) - B^T & 0 \end{bmatrix} \begin{bmatrix} u_h \\ p_h \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

mass matrix	$M_{ij} := (v_h^i, v_h^j)$
transport matrix	$K(u)_{ij} := (u_h \cdot \nabla v_h^j, v_h^i)$
laplacian matrix	$L_{ij} := (\nabla v_h^j, \nabla v_h^i)$
gradient matrix	$B_{ij} := -(q_h^j, \nabla \cdot v_h^i)$ (divergence matrix B^T)
right hand side	$F_i := (f, v_h^i)$

time stepping techniques (theta-scheme)

$$\begin{bmatrix} S(u^l) & kB \\ B^T & 0 \end{bmatrix} \begin{bmatrix} u^l \\ p^l \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix}$$

- step size $k := \Delta t$
- $S(u^l) := \alpha M + k\theta(K(u^l) + \nu L)$
- $g := Mu^{l-1} - k(1-\theta)(K(u^{l-1}) + \nu L) + k\theta F^l + k(1-\theta)F^{l-1}$

Structure of (almost) all solvers

$$\begin{bmatrix} S(u^l) & kB \\ B^T & 0 \end{bmatrix} \begin{bmatrix} u^l \\ p^l \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{aligned} S(u)u + kBp &= g \\ B^T u &= 0 \end{aligned}$$

Galerkin schemes

local MPSC

Outer: N nonlinear steps

Inner: L DPM (**D**iscrete **P**rojection **M**ethod) steps (Oseen)

- $L = 1$ → **CC**: **C**oupled solution by **C**oupled solver
- $L > 1$ → **CP**: **C**oupled solution by **P**rojection solver

Projection schemes

global MPSC

Outer: 1 DPM steps

Inner: N nonlinear steps (Burgers)

→ **PP**: **P**rojection solution by **P**rojection solver

All versions of CC, CP, PP lead to the „same“ solutions.

Essential difference between the solvers

key ideas of MPSC approaches

Re-interpretation of Navier-Stokes solvers (Chorin, VanKan, Uzawa, etc.) as „incomplete solvers“ for discrete saddle-point problems.

Galerkin schemes

local MPSC (Multilevel Pressure Schur Complement)

- **Fully coupled** Newton-like solver as outer nonlinear procedure
- Solve exactly on **subsets/patches** and perform an outer Block–Gauß-Seidel/Jacobi iteration as smoother

Projection schemes

global MPSC (Multilevel Pressure Schur Complement)

- Outer **decoupling** of velocity and pressure
- Newton-like schemes for Momentum equations
- Multigrid solver for all scalar subproblems

Essential difference between the solvers

Galerkin schemes

CC (**C**oupled solution by **C**oupled solver)

- „direct solvers“ for stationary (generalized) Navier-Stokes equations
- fully implicit character
  $\Rightarrow$ the **most accurate and robust** time stepping schemes
  $\Rightarrow$ only variants which allow a rigorous a **posteriori error control**
- **large time steps** to reach a desired accuracy
- **very expensive costs** for one time step

CP (**C**oupled solution by **P**rojection solver)

- **cost can be diminished** by
  - weakening the threshold parameters
  - applying only a fixed small number of nonlinear steps

$\Rightarrow$ **accuracy and robustness behaviour may be weakened**

Essential difference between the solvers

Projection schemes

PP (**P**rojection solution by **P**rojection solver)

- applied to nonstationary flows only
- rigorous error control in time is not clear
- **„exact“ treatment of the nonlinearity**
- depending on the last pressure iterate
  (→ smaller time steps)
- **smaller and much cheaper time steps**
  (compared to Galerkin schemes)
- resulting solutions satisfy the continuity equation,
  but the discrete momentum equation only approximately

for **fully nonstationary flows** with **dominating convective** term and on **complex domains**, this approach is a **favourized** one

PP formulation

- PP (**P**rojection solution by **P**rojection solver)
  *global MPSC (**M**ultilevel **P**ressure **S**chur **C**omplement)*
  1. a decoupling step for u and p as outer iteration
  2. compute a velocity field without taking into account incompressibility
  3. perform a pressure correction, which is a projection back to the subspace of divergence free vector fields

pressure Schur complement:

$$S(u)u + kBp = g \qquad u = S^{-1}(u)g - kS^{-1}(u)Bp$$

$$B^T u = 0 \qquad 0 = B^T S^{-1}(u)g - kB^T S^{-1}(u)Bp$$

pressure Schur complement

A scalar equation that contains the pressure:

$$\underbrace{B^T S^{-1}(u)B}_{=:\tilde{P}} p = \underbrace{\frac{1}{k} B^T S^{-1}(u)g}_{=:f_p}$$

PP formulation - perform only once per time step

Equation for u ('Burgers')

solve for u^l :

$$S(u^l)u^l = g - kBp^{l-1}$$

Equation for p

pressure correction with a suitable preconditioner C:

$$p^l = p^{l-1} + C^{-1} \underbrace{\left(\frac{1}{k} B^T S^{-1}(u^l) g - B^T S^{-1}(u^l) Bp \right)}_{\text{residual (pressure Schur complement)}}$$

$$Cp^l = Cp^{l-1} + \frac{1}{k} B^T \underbrace{\left(S^{-1}(u^l) g - k S^{-1}(u^l) Bp^l \right)}_{\text{definition of } u^l}$$

$$= Cp^{l-1} + \frac{1}{k} B^T u^l$$

convergence $\quad Cp = Cp + \frac{1}{k} B^T u^l \quad \Longrightarrow B^T u = 0$

Construction of globally defined additive preconditioning operators

$$A := B^T S^{-1}(u) B \qquad \text{with} \quad S(u) = \alpha M + k\theta(\nu L + K(u))$$

additive approach: construct **„optimal“** operators for the limit cases

$$C^{-1} := \alpha_R A_R^{-1} + \alpha_D A_D^{-1} + \alpha_K A_K^{-1}$$

- A_R is an „optimal“ (reactive) preconditionier for $B^T M^{-1} B$

  (divergence-free L^2-projection)

- A_D is an „optimal“ (diffusive) preconditionier for $B^T L^{-1} B$

  (Stokes-equation)

- A_K is an „optimal“ (convective) preconditionier for $B^T K^{-1}(u) B$

  (incompressible Euler equation)

„optimal“

- partial preconditioners were direct solvers with respect to the underlying subproblem
- resulting convergence behaviour is independent of
  - outer parameters
  - underlying mesh

The „reactive“ preconditioner for $B^T M^{-1} B$

- M is already diagonal by construction
  - finite difference approach
  - nonconforming triangular finite elements
- Otherwise lumping

$$A_R := P := B^T M_l^{-1} B$$

- An (almost) exact solver / preconditioner A_R for small time steps
  $$S(u) = \alpha M_l + k\theta(\nu L + K(u)) \quad \longrightarrow \quad \alpha M_l \quad \text{for } k \to 0$$
- A flexible treatment of pressure boundary conditions on discrete level
- Highly efficient multigrid solvers for applying A_R^{-1}
- Very compact matrices A_R in independence of the spatial discretization

The „reactive“ preconditioner - Pressure Poisson problem

Poisson problem

$$-\Delta q = rhs$$

matrix formulation

- $rhs = -\nabla \cdot \nabla q = \nabla \cdot v$
- $v = -\nabla q$

$$\begin{bmatrix} I & \nabla \\ \nabla\cdot & 0 \end{bmatrix} \begin{bmatrix} v \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ rhs \end{bmatrix} \quad \backsim \quad \begin{bmatrix} M_l & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} v \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ f_p \end{bmatrix}$$

Pressure Poisson problem

solve for q: $(P := B^T M_l^{-1} B)$

$$Pq = f_p \qquad (-\Delta q = rhs)$$

- P calculated only once in a preprocessing step (or if the spatial mesh has changed)
- P arises from a mixed formulation $\Rightarrow$ even piecewise constant ansatz functions are allowed

The „diffusive“ preconditioner for $B^T L^{-1} B$

The inverse discrete Laplacian L^{-1} and also $B^T L^{-1} B$ are **full** matrices!
(At least for all finite difference/element/volumen approaches.)

Idea:

$$\nabla \cdot \Delta^{-1} \nabla \backsim I$$

In the finite element context:

$$B^T L^{-1} B \backsim M_p$$

$$A_D := M_p$$

- All numerical tests show, that indeed $A_D := M_p$ is sufficient.
- Absolutely robust against all variations of parameters and the shape of the mesh in the pure Stokes case.
- Leads to an **improved convergence rate** in the pressure update!

The „convective“ preconditioner for $B^T K^{-1}(u) B$

The inverse transport matrix $K^{-1}(u)$ and $B^T K^{-1}(u) B$ are **full** matrices!

- continuous construction

$$\nabla \cdot (U \cdot \nabla)^{-1} \nabla \backsim (\tilde{U} \cdot \nabla)?$$

- discrete construction

$$\begin{bmatrix} ILU(S) & B \\ B^T & 0 \end{bmatrix} \Leftrightarrow B^T ILU(S)^{-1} B \quad \text{instead of} \quad \begin{bmatrix} S & B \\ B^T & 0 \end{bmatrix} \Leftrightarrow B^T S^{-1} B$$

- poor condition number: $\mathcal{O}(h^{-1}) - \mathcal{O}(h^{-2})$
- sensitive to mesh anisotropies
- complete solution process is almost so expensive as for the original system

- new techniques ...

PP Algorithm

Start with: $u_0 := 0$. **Given:** Iterate p^{l-1}.
Perform:

1. **('Burgers') Solve:** for an intermediate velocity $\tilde{u}$:

   $S(\tilde{u})\tilde{u} = g - kBp^{l-1}$ *VanKan*

   $S(\tilde{u})\tilde{u} = g$ *Chorin*

2. **Calculate:** the right hand side f_p for the pressure Poisson problem:

   $$f_p = \frac{1}{k}B^T\tilde{u} \qquad \left(= \frac{1}{k}B^T S^{-1}[g - kBp^{l-1}] = \text{ residual } (p^{l-1}) \right)$$

3. **('Pressure Poisson') Solve:** for q:

   $$Pq = f_p \qquad (P := B^T M_l^{-1} B)$$

4. **Update:** new pressure p^l:

   $p^l = p^{l-1} + \alpha_R q + \alpha_D M_{p_l}^{-1} f_p$ *VanKan*

   $p^l = \alpha_R q + \alpha_D M_{p_l}^{-1} f_p$ *Chorin*

5. **Update:** new velocity u^l to satisfy the incompressibility constraint:

   $$u^l = \tilde{u} - kM_l^{-1}Bq$$

PP Algorithm - Burgers equation

Burgers equation with given Iterate p^{l-1} and u^{l-1}

$$S(\tilde{u})\tilde{u} = \underbrace{g - kBp^{l-1}}_{=:f} \text{ (VanKan)} \qquad \text{or} \qquad S(\tilde{u})\tilde{u} = \underbrace{g}_{=:f} \text{ (Chorin)}$$

fixed point iteration

1. Calculate nonlinear residual d^n:

$$d^n = f - S(\tilde{u}^n)\tilde{u}^n$$

2. Solve an auxiliary subproblem for y^n:

$$S(\tilde{u}^n)y^n = d^n$$

3. Update $\tilde{u}$ via the auxiliary solution y^n:

$$\tilde{u}^{n+1} = \tilde{u}^n + y^n$$

full fixed point iteration

- set $\tilde{u}^0 := u^{l-1}$
- use N nonlinear steps

extrapolate previous time step

- linear extrapolation of solution in time:

$$\tilde{u}^0 := 2u^{l-1} - u^{l-2}$$

- one nonlinear step

PP solver configurations

fixed point loop

- relative tolerance $(\sim 10^{-2})$

velocity solver - solver a

- Richardson with Jacobi Smoother
- Richardson-Multigird with Jacobi-Smoother (or SOR)
  - Coarse-Grid Solver: Richardson-Multigird with Jacobi-Smoother (50 iterations)
- max. iteration (50)
- relative tolerance $(\sim 10^{-2})$
- smooth steps (2)
- smooth damp (0.7)
  non newtonian fluid (0.3)

pressure solver - solver s

- PCG with Jacobi Smoother
- PCG-Multigird with Jacobi-Smoother
  - Coarse-Grid Solver: UMFPACK (exact solver)
- max. iteration (100)
- absolute tolerance $(\sim 10^{-10})$
- relative tolerance $(\sim 10^{-2})$
- smooth steps $(16, (32))$
- smooth damp (~ 0.9)

scaling parameters

- $\alpha \in [0,1]$ (1)
- $\alpha_R \in [0,1]$ (1)
- $\alpha_D \leq k\theta\nu$ $(k\theta\nu)$

DFG flow around cylinder benchmark 2D-1, laminar case

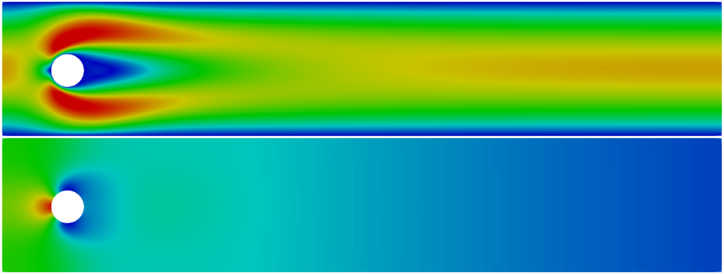


Figure 1: *Velocity and pressure profile for $Re = 20$.*

Drag:	5.57953523384
Lift:	0.01061894815
Pressure difference:	0.11752016697

post processing - pressure

- do nothing
- midpoint
  $p_{\text{post}} = 0.5(p^l + p^{l-1})$
- linear extrapolation
  $p_{\text{post}} = p^l + 0.5(p^l - p^{l-1})$

DFG flow around cylinder benchmark 2D-2 and 2D-3

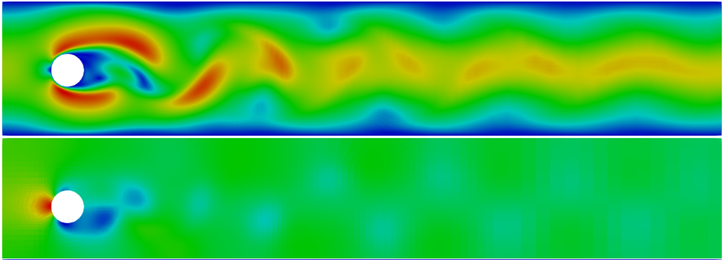


Figure 2: *Velocity and pressure profile for* $Re = 100$.

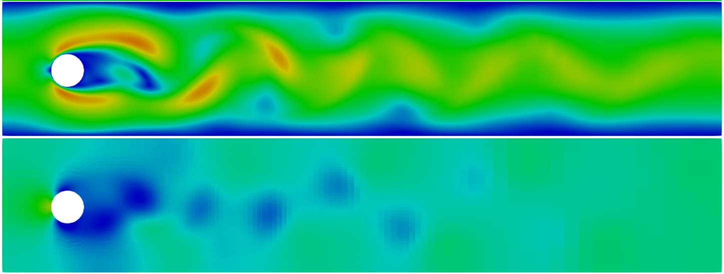
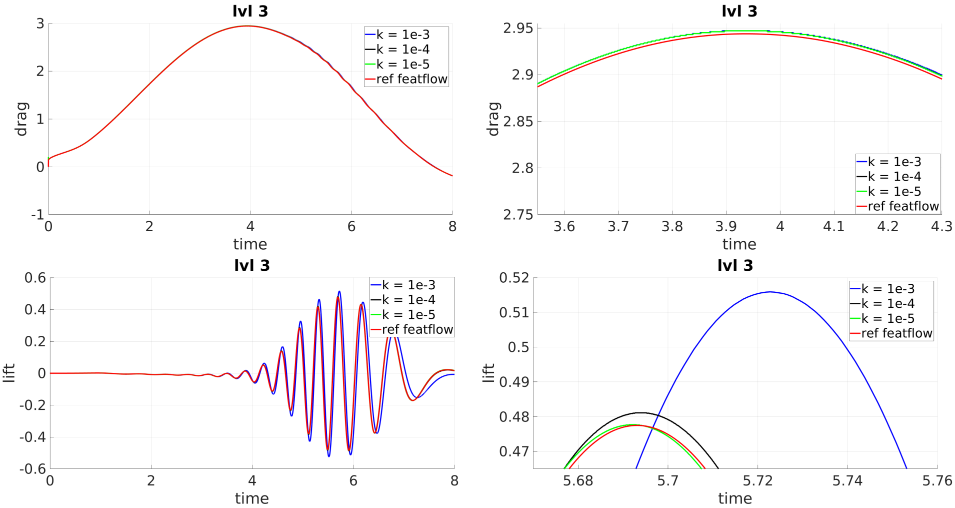


Figure 3: *Velocity and pressure profile for* $Re \in [0, 100]$.

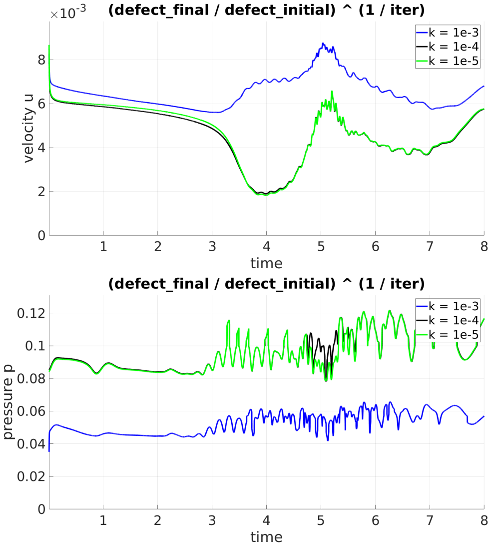
DFG flow around cylinder benchmark 2D-3, fixed time interval



Convergence history

Burgers:
$(\backsim 10^{-3})$

Pressure Poisson:
$(\backsim 0.1)$



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