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An Unfitted Diffuse Interface Method for Elliptic Interface Problems

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lehrstuhl für angewandte
mathematik und numerik

fakultät für
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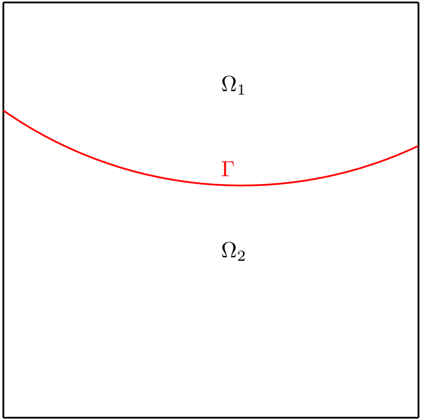
Interface description

Lipschitz domain $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2 \subset \mathbb{R}^d$



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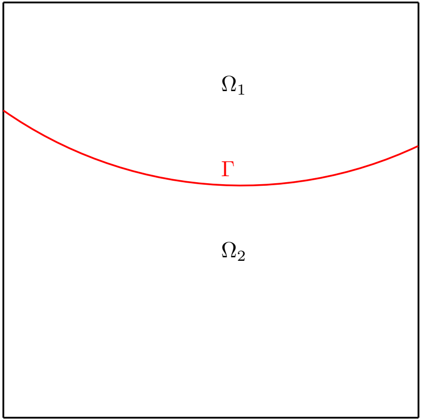

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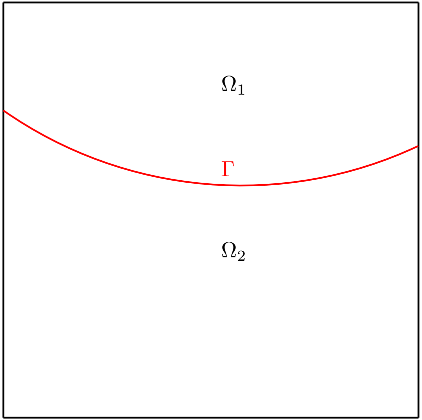
Level set function $\phi \in C(\bar{\Omega})$

$\Omega_1 = \{\mathbf{x} \in \Omega : \phi(\mathbf{x}) > 0\}$

$\Gamma = \{\mathbf{x} \in \Omega : \phi(\mathbf{x}) = 0\}$

$\Omega_2 = \{\mathbf{x} \in \Omega : \phi(\mathbf{x}) < 0\}$



Elliptic interface problem

Strong form

$$
\begin{aligned}
-\nabla \cdot (\mu \nabla u) &= f \quad \text{in } \Omega, \\
u &= \bar{u} \quad \text{on } \partial\Omega, \\
[\![u]\!] &= 0 \quad \text{on } \Gamma, \\
[\![\mu \nabla u]\!] &= 0 \quad \text{on } \Gamma
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$$

$$
[\![u]\!] = u_1 - u_2 \text{ and } [\![\mu \nabla u]\!] = (\mu_1 \nabla u_1 - \mu_2 \nabla u_2) \cdot \mathbf{n}
$$

Discrete weak form

$$\int_{\Omega_{1,h}} \mu_1 \nabla u_{1,h} \cdot \nabla w_{1,h} + \int_{\Omega_{2,h}} \mu_2 \nabla u_{2,h} \cdot \nabla w_{2,h}$$

$$= \int_{\Omega_{1,h}} f_1 w_{1,h} + \int_{\Omega_{2,h}} f_2 w_{2,h}.$$

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$$- \int_{\Gamma_h} [\![u_h]\!]\{\mu \nabla w_h\} - \int_{\Gamma_h} \{\mu \nabla u_h\}[\![w_h]\!]$$

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$$\{\mu \nabla u_h\} = (\kappa_1 \mu_1 \nabla u_{1,h} + \kappa_2 \mu_2 \nabla u_{2,h}) \cdot \mathbf{n} \qquad \text{with} \qquad \kappa_i|_{K^e} = \frac{|K^e \cap \Omega_{i,h}|}{|K^e|}$$

Splitting into sub-domain problems

$$\int_{\Omega_{1,h}} \mu_1 \nabla u_{1,h} \cdot \nabla w_{1,h} = \int_{\Omega_{1,h}} f_1 w_{1,h},$$

$$\int_{\Omega_{2,h}} \mu_2 \nabla u_{2,h} \cdot \nabla w_{2,h} = \int_{\Omega_{2,h}} f_2 w_{2,h}$$

Splitting into sub-domain problems

$$\int_{\Omega_{1,h}} \mu_1 \nabla u_{1,h} \cdot \nabla w_{1,h} + \int_{\Gamma_h} \alpha(u_{1,h} - u_{2,h}) w_{1,h}$$

$$= \int_{\Omega_{1,h}} f_1 w_{1,h},$$

$$\int_{\Gamma_h} \alpha(u_{1,h} - u_{2,h})(w_{1,h} - w_{2,h})$$

$$\int_{\Omega_{2,h}} \mu_2 \nabla u_{2,h} \cdot \nabla w_{2,h} - \int_{\Gamma_h} \alpha(u_{1,h} - u_{2,h}) w_{2,h}$$

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Projection-based stabilization I

Neumann penalty terms

$$s_{1,\Omega_{2,h}}(u_{1,h}, w_{1,h}) = \int_{\Omega_{2,h}} \mu_1(\nabla u_{1,h} - \mathbf{g}_1) \cdot \nabla w_{1,h},$$

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Lumped-mass L^2-projection

$$\mathbf{g}_{j,k} = \frac{\int_{\Omega_h} \nabla u_{h,k} \varphi_j}{\int_{\Omega_h} \varphi_j}$$

Stabilized split form

$$\int_{\Omega_{1,h}} \mu_1 \nabla u_{1,h} \cdot \nabla w_{1,h} + \int_{\Gamma_h} \alpha (u_{1,h} - u_{2,h}) w_{1,h} - \int_{\Gamma_h} (u_{1,h} - u_{2,h})(\kappa_1 \mu_1 \nabla w_{1,h}) \cdot \mathbf{n}$$

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Projection-based stabilization II

Problem: Stabilization matrix is not positive semi-definite

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Matrix properties: positive semi-definite with bounded eigenvalues

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Integration over sharp interface:

Diffuse interface formulation

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- piecewise-linear approximation to the zero level set

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Approximation of surface integrals by volume integrals:

- Use regularized delta functions
- Extrapolate solution values and normal derivatives

Extrapolation using level sets

To be defined: constant extensions of values and normal derivatives

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Main steps:

1. closest-point search
2. evaluation of solution and normal derivative
3. constant extrapolation

Extrapolation using level sets

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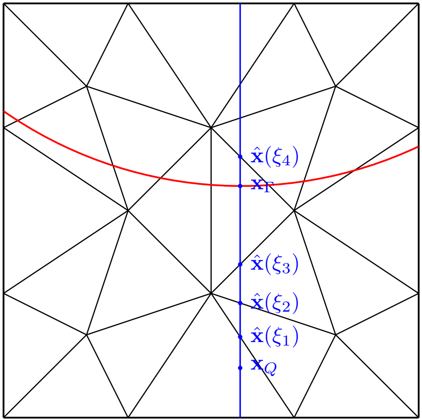
Requirements: simplicity, efficiency, accuracy

Closest point search

Interface pointer

$$\mathbf{n}_Q := -\nabla \phi_h(\mathbf{x}_Q)$$



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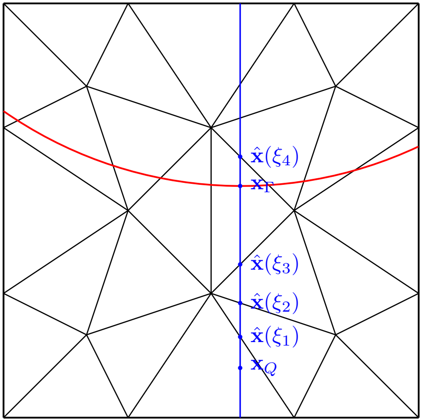
Exact distance function ϕ

$$\mathbf{x}_\Gamma := \mathbf{x}_Q + \phi(\mathbf{x}_Q)\mathbf{n}_Q$$



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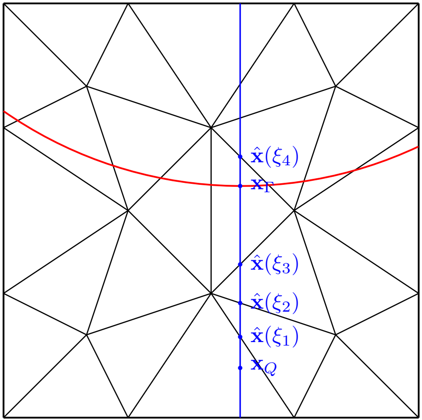
Numerical approximation ϕ_h

$$\hat{\mathbf{x}}(\xi) = \mathbf{x}_Q + \xi \mathrm{sign}(\phi_h(\mathbf{x}_Q))\mathbf{n}_Q, \quad \xi \in \mathbb{R}$$

$$\phi_h(\mathbf{x}_\Gamma) = 0 \text{ at } \mathbf{x}_\Gamma = \hat{\mathbf{x}}(\xi_\Gamma)$$



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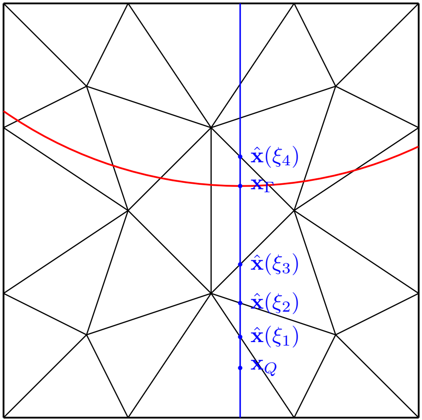
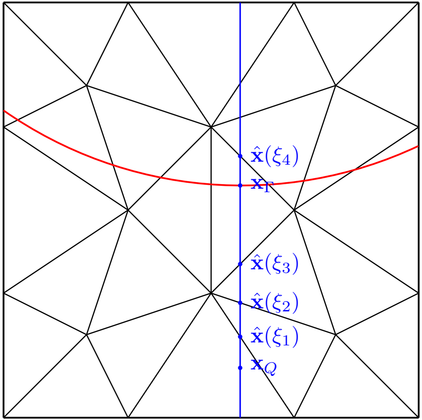
$\Rightarrow$ simple line search



Search algorithm

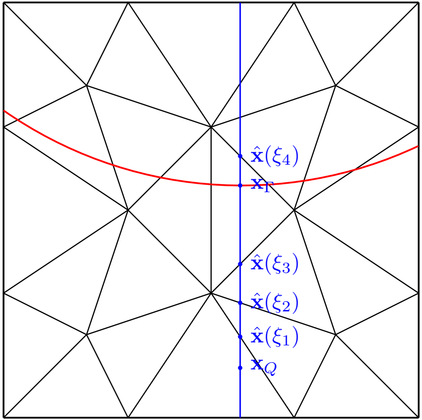
- Set $\xi_0 = 0$



Closest point search

Search algorithm

- Set $\xi_0 = 0$
- For $i > 1$: Find next intersection $\hat{\mathbf{x}}(\xi_i)$, $\xi_i > \xi_{i-1}$ of $\hat{\mathbf{x}}(\xi)$ with boundary of a mesh cell boundary



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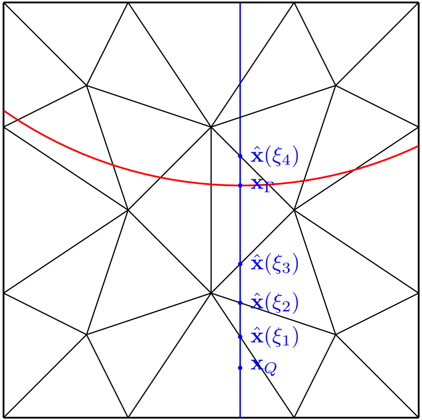

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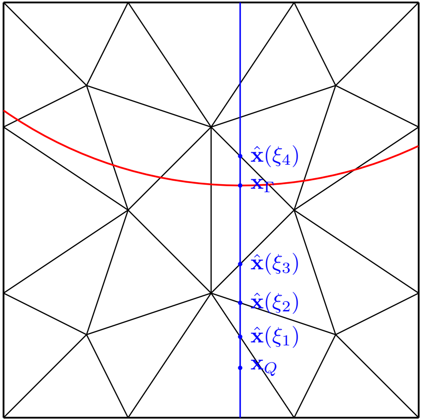

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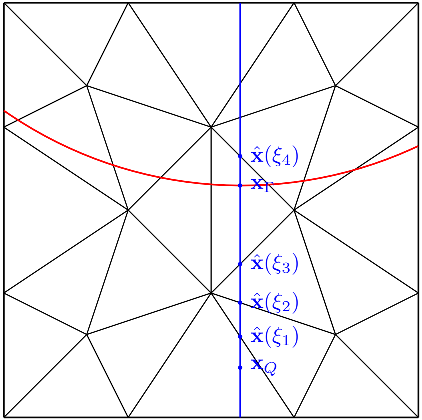
- Solve linear/quadratic equation to find root $\xi_\Gamma \in [\xi_{m-1}, \xi_m]$ of $\phi(\hat{\mathbf{x}}(\xi))$
- Set $\mathbf{x}_\Gamma = \hat{\mathbf{x}}(\xi_\Gamma)$



Diffuse interface split form

$$\int_{\Omega_{1,h}} \mu_1 \nabla u_{1,h} \cdot \nabla w_{1,h} + \int_{\Gamma_h} \alpha(u_{1,h} - u_{2,h}) w_{1,h}$$
$$- \int_{\Gamma_h} (u_{1,h} - u_{2,h})(\kappa_1 \mu_1 \nabla w_{1,h}) \cdot \mathbf{n}$$
$$- \int_{\Gamma_h} (\kappa_1 \mu_1 \nabla u_{1,h} + \kappa_2 \mu_2 \nabla u_{2,h}) \cdot \mathbf{n} w_{1,h}$$
$$+ s_{1,\Omega_h}(u_{1,h}, w_{1,h}) = \int_{\Omega_{1,h}} f_1 w_{1,h}$$

Diffuse interface split form

$$\int_{\Omega_{1,h}} \mu_1 \nabla u_{1,h} \cdot \nabla w_{1,h} + \int_{\Omega_h} \alpha(U_{1,h} - U_{2,h}) W_{1,h} \delta_\epsilon(\phi_h) |\nabla \phi_h|$$

$$- \int_{\Gamma_h} (u_{1,h} - u_{2,h})(\kappa_1 \mu_1 \nabla w_{1,h}) \cdot \mathbf{n}$$

$$- \int_{\Gamma_h} (\kappa_1 \mu_1 \nabla u_{1,h} + \kappa_2 \mu_2 \nabla u_{2,h}) \cdot \mathbf{n} w_{1,h}$$

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Diffuse interface split form

$$
\begin{aligned}
\int_{\Omega_{1,h}} \mu_1 \nabla u_{1,h} \cdot \nabla w_{1,h} &+ \int_{\Omega_h} \alpha (U_{1,h} - U_{2,h}) W_{1,h} \delta_\epsilon(\phi_h) |\nabla \phi_h| \\
&- \int_{\Omega_h} (U_{1,h} - U_{2,h}) (\bar{\kappa}_1 \mu_1 \nabla W_{1,h}) \cdot \mathbf{n} \delta_\epsilon(\phi_h) |\nabla \phi_h| \\
&- \int_{\Gamma_h} (\kappa_1 \mu_1 \nabla u_{1,h} + \kappa_2 \mu_2 \nabla u_{2,h}) \cdot \mathbf{n} w_{1,h} \\
&+ s_{1,\Omega_h}(u_{1,h}, w_{1,h}) = \int_{\Omega_{1,h}} f_1 w_{1,h}
\end{aligned}
$$

Diffuse interface split form

$$\begin{aligned}\int_{\Omega_{1,h}} \mu_1 \nabla u_{1,h} \cdot \nabla w_{1,h} &+ \int_{\Omega_h} \alpha(U_{1,h} - U_{2,h}) W_{1,h} \delta_\epsilon(\phi_h) |\nabla \phi_h| \\ &- \int_{\Omega_h} (U_{1,h} - U_{2,h})(\bar{\kappa}_1 \mu_1 \nabla W_{1,h}) \cdot \mathbf{n} \delta_\epsilon(\phi_h) |\nabla \phi_h| \\ &- \int_{\Omega_h} (\bar{\kappa}_1 \mu_1 \nabla U_{1,h} + \bar{\kappa}_2 \mu_2 \nabla U_{2,h}) \cdot \mathbf{n} W_{1,h} \delta_\epsilon(\phi_h) |\nabla \phi_h| \\ &+ s_{1,\Omega_h}(u_{1,h}, w_{1,h}) = \int_{\Omega_{1,h}} f_1 w_{1,h}\end{aligned}$$

Diffuse interface split form

$$
\begin{aligned}
\int_{\Omega_{2,h}} \mu_2 \nabla u_{2,h} \cdot \nabla w_{2,h} &- \int_{\Omega_h} \alpha (U_{1,h} - U_{2,h}) W_{2,h} \delta_\epsilon(\phi_h) |\nabla \phi_h| \\
&- \int_{\Omega_h} (U_{1,h} - U_{2,h})(\bar{\kappa}_2 \mu_2 \nabla W_{2,h}) \cdot \mathbf{n} \delta_\epsilon(\phi_h) |\nabla \phi_h| \\
&+ \int_{\Omega_h} (\bar{\kappa}_1 \mu_1 \nabla U_{1,h} + \bar{\kappa}_2 \mu_2 \nabla U_{2,h}) \cdot \mathbf{n} W_{2,h} \delta_\epsilon(\phi_h) |\nabla \phi_h| \\
&+ s_{2,\Omega_h}(u_{2,h}, w_{2,h}) = \int_{\Omega_{2,h}} f_2 w_{2,h}
\end{aligned}
$$

Smooth test problem:

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$$\Omega = (0,1) \qquad \Gamma = \{0.51\}$$

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Non-smooth test problem:

Numerical examples

Smooth test problem:

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Non-smooth test problem:

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Non-smooth test problem:

$$\Omega = (0,1) \qquad \Gamma = \{0.51\}$$

$$\mu_1 = 0.5 \qquad \mu_2 = 3 \qquad f_1 = 1 \qquad f_2 = 1$$

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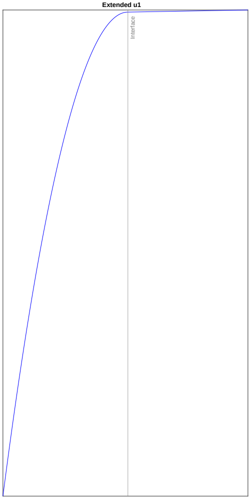
Non-smooth test problem:

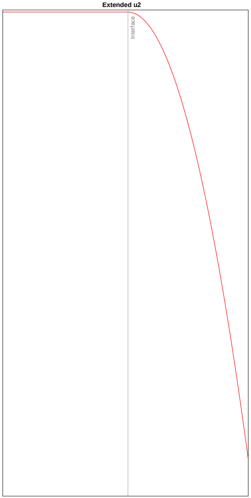
$$\Omega = (0,1) \qquad \Gamma = \{0.51\}$$

$$\mu_1 = 0.5 \qquad \mu_2 = 3 \qquad f_1 = 1 \qquad f_2 = 1$$

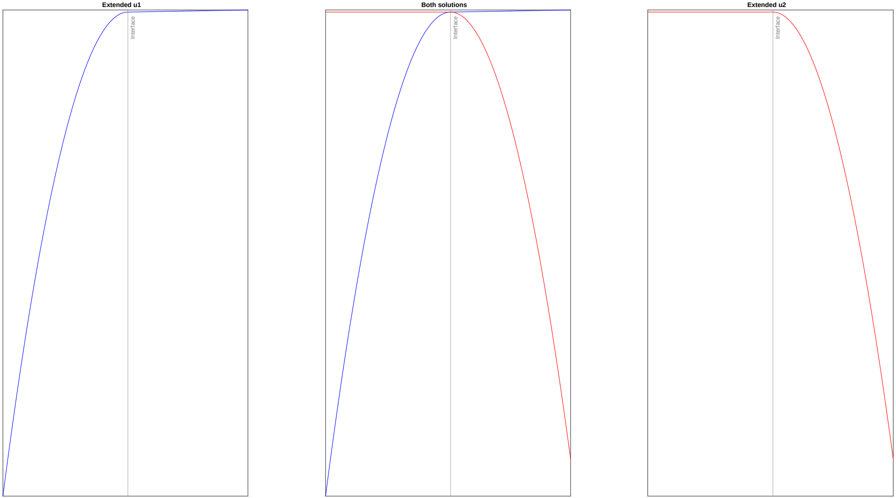
$$u_1(x) = \frac{9}{14}(x - 0.01) - (x - 0.01)^2 \qquad u_2(x) = \frac{5}{84} + \frac{9}{84}(x - 0.01) - \frac{(x - 0.01)^2}{6}$$

Sharp interface - smooth solution

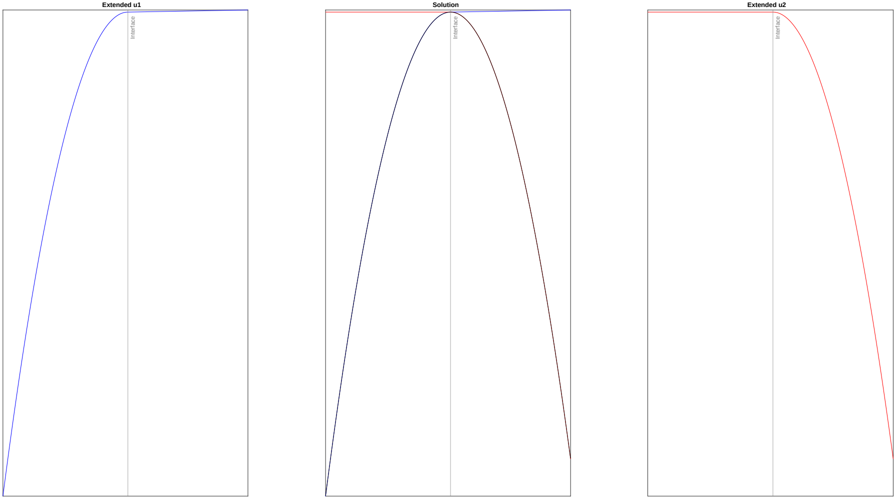




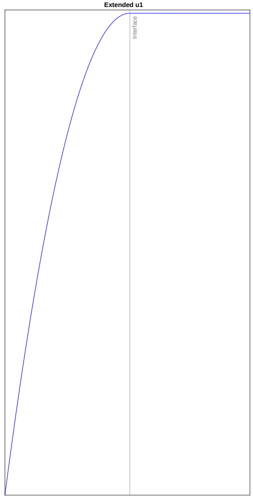
Sharp interface - smooth solution

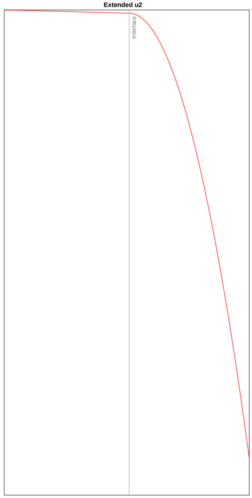


Sharp interface - smooth solution

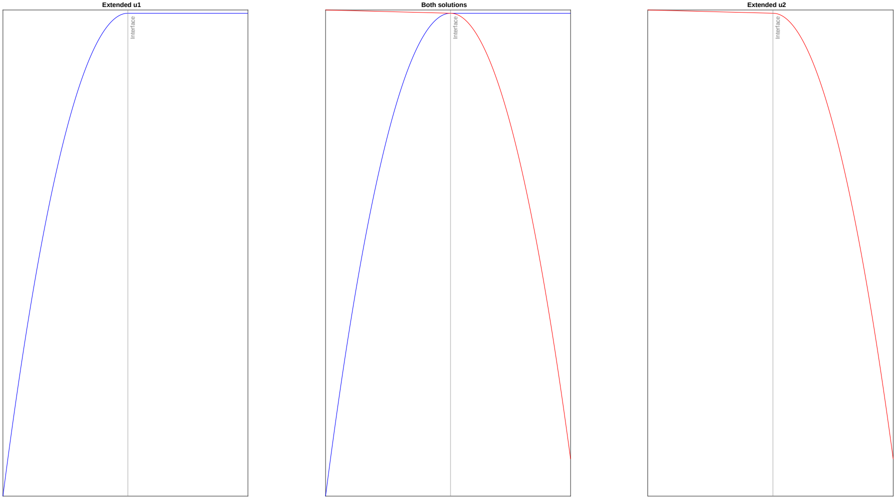


Diffuse interface - smooth solution

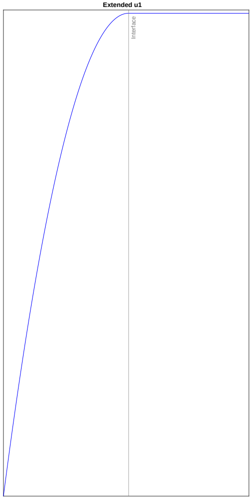


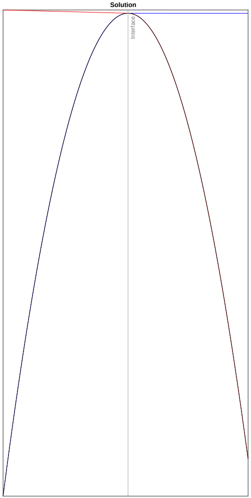


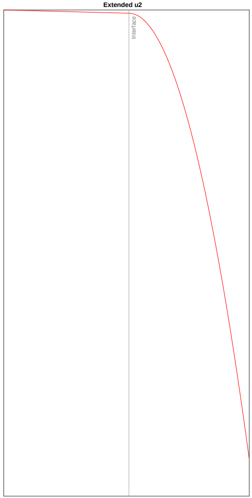
Diffuse interface - smooth solution



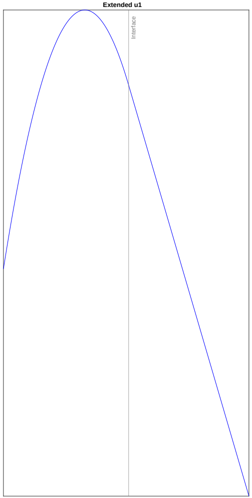
Diffuse interface - smooth solution

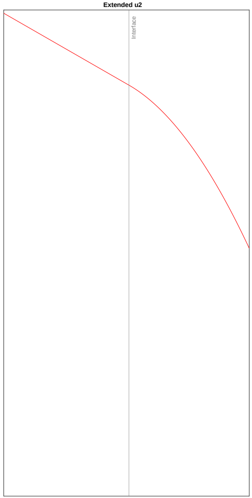




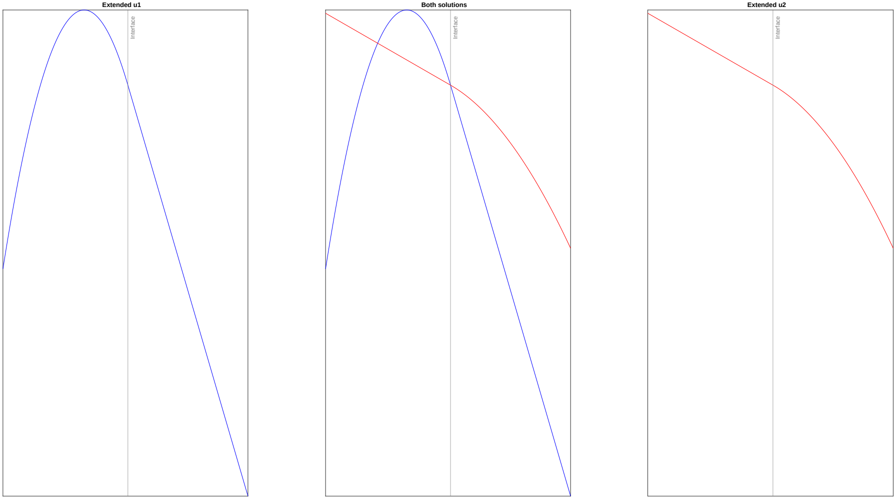


Sharp interface - non-smooth solution

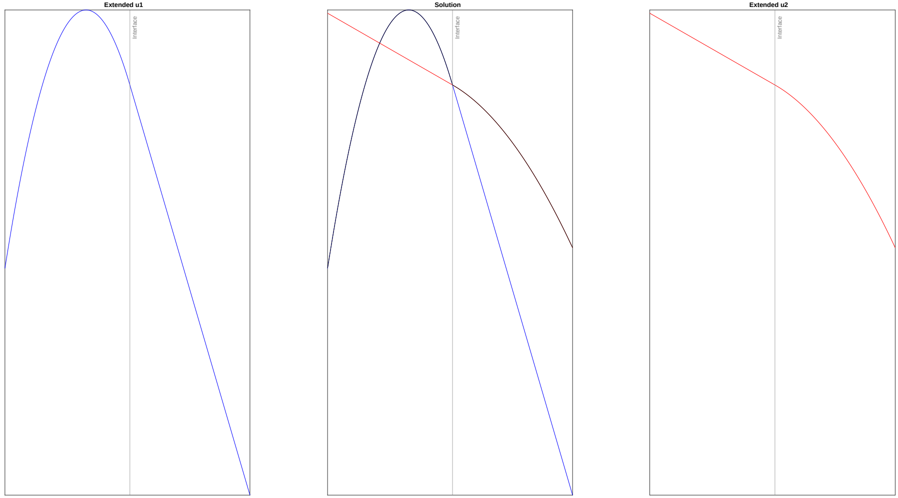




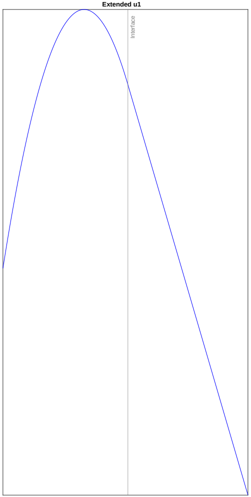
Sharp interface - non-smooth solution

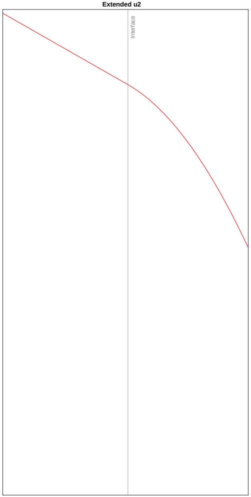


Sharp interface - non-smooth solution

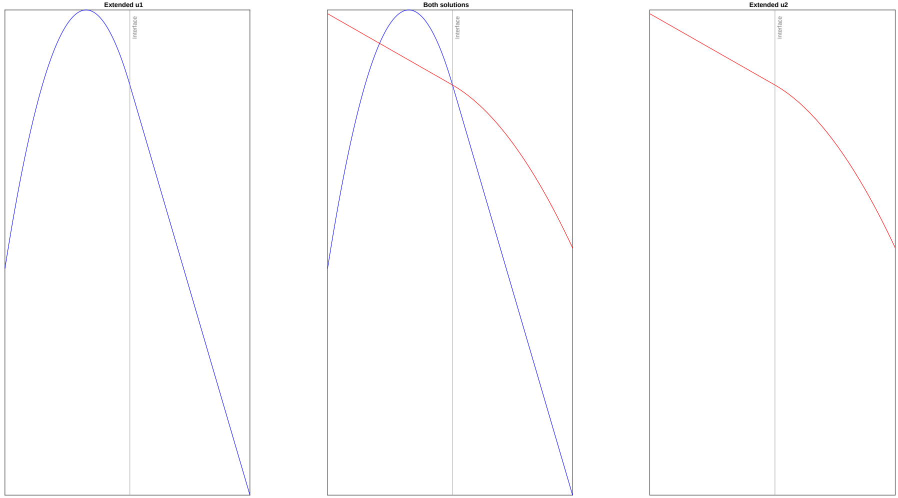


Diffuse interface - non-smooth solution





Diffuse interface - non-smooth solution



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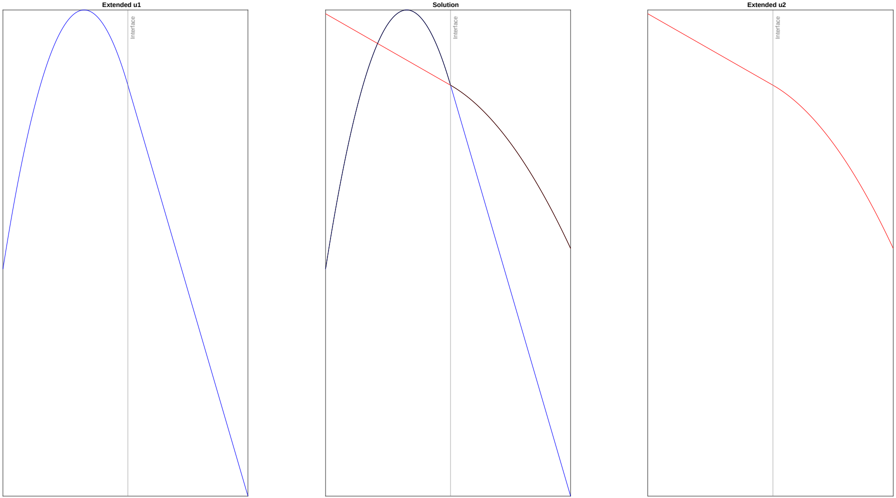

Grid convergence - non-smooth solution

h^{-1}	sharp	EOC	diffuse	EOC
32	4.96e-04		4.95e-04	
64	1.29e-04	1.94	1.28e-04	1.98
128	3.21e-05	2.01	3.17e-05	2.01
256	8.27e-06	1.96	8.06e-06	1.98
512	1.94e-06	2.09	1.87e-06	2.11
1024	5.03e-07	1.95	4.54e-07	2.04
2048	1.30e-07	1.95	1.06e-07	2.10
4096	2.93e-08	2.15	1.90e-08	2.48
8192	7.40e-09	1.99	4.21e-09	2.17

Conclusions

Summary

- parameter-free stabilization using Neumann penalties
- approximation of surface integrals by volume integrals
- fast closest-point search algorithm

Summary

- parameter-free stabilization using Neumann penalties
- approximation of surface integrals by volume integrals
- fast closest-point search algorithm

Outlook

- theoretical studies of proposed approach
- extension to moving boundary problems
- application to PDE systems

References

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- P. Hansbo and A. Hansbo, An unfitted finite element method, based on Nitsche's method, for elliptic interface problems. *Comput. Methods Appl. Mech. Engrg.* **191** (2002) 5537–5552.

Thank you for your attention!

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