A finite element method for fluid-structure interaction problems with large deformations

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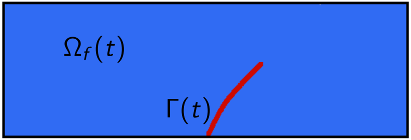
Goal: simulate the motion of a thin elastic leaflet Γ immersed in an incompressible, viscous, and Newtonian fluid. We assume the leaflet undergoes large displacements.



We would like to use a method that has the following advantages:

- The interface and problem specific features (hydrodynamic forces, pressure discontinuities etc.) can be resolved very accurately
  $\Rightarrow$ typical of interface tracking methods such as ALE methods
- Flexibility in handling large displacements of Γ
  $\Rightarrow$ typical of interface capturing methods such as level set methods

Fluid equations: The fluid is governed by the incompressible Navier-Stokes equations

$$\rho_f \frac{\partial \mathbf{u}}{\partial t} + \rho_f \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{f}_f \qquad \text{in } \Omega_f(t) \times (0, T)$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega_f(t) \times (0, T)$$

$\mathbf{u}$: fluid velocity

$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\epsilon(\mathbf{u})$: Cauchy stress tensor

p: fluid pressure

$\epsilon(\mathbf{u}) = \frac{(\nabla \mathbf{u}) + (\nabla \mathbf{u})^T}{2}$: strain rate tensor

The fluid domain changes in time → ALE formulation

Fluid equations: The fluid is governed by the incompressible Navier-Stokes equations

$$\rho_f \frac{\partial \mathbf{u}}{\partial t}\Big|_{x_0} + \rho_f(\mathbf{u} - \mathbf{w}) \cdot \nabla \mathbf{u} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{f}_f \qquad \text{in } \Omega_f(t) \times (0, T)$$

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$\mathbf{w}$: ALE velocity

$\frac{\partial \mathbf{u}}{\partial t}\Big|_{x_0}$: ALE time derivative

Structure equation: the leaflet is modeled as an inextensible beam with negligible torsional effects[1]

$$\rho_s \ddot{\mathbf{x}} + EI\mathbf{x}'''' = \mathbf{f}_\Gamma, \quad \text{with } |\mathbf{x}'| = 1, \quad \text{on } (0,T) \times [0,L].$$

ρ_s: linear density

$\mathbf{x}$: position

$\dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial t}$: time derivative

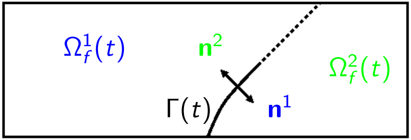
$\mathbf{x}' = \frac{\partial \mathbf{x}}{\partial s}$: arc length derivative

EI: flexural stiffness

L: beam length

[1] dos Santos, Gerbeau, Bourgat, *A partitioned fluid-structure algorithm for elastic thin valves with contact*, CMAME (2008)

The leaflet ideally separates $\Omega_f(t)$ into two subdomains $\Omega_f^1(t)$ and $\Omega_f^2(t)$ and it deforms due to the contact force exerted by the fluid.



- Adherence $\Rightarrow$ Continuity of velocities

$$\mathbf{u} = \dot{\mathbf{x}} \quad \text{on } \Gamma(t);$$

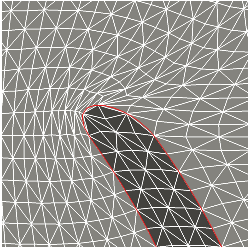
- Action-Reaction principle $\Rightarrow$ Continuity of stresses

$$\mathbf{f}_\Gamma = -\boldsymbol{\sigma}^1\mathbf{n}^1 - \boldsymbol{\sigma}^2\mathbf{n}^2 \quad \text{on } \Gamma(t).$$

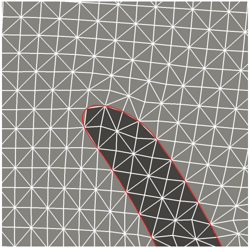
We are interested in having a triangulation that is at every time:

- aligned with Γ
- of "optimal" quality

We use a mesh optimization technique with an additional constraint to enforce the alignment of the edges of the resulting triangulation with the interface.[2]



Standard ALE



Extended ALE

[2] Basting, Weismann, *A hybrid level set - front tracking finite element approach for fluid-structure interaction and two-phase flow applications*, JCP (2013)

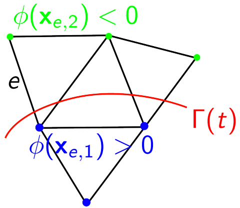
Let $\phi : [0, T] \times \Omega \to \mathbb{R}$ be a continuous level set function:

$$\begin{aligned} \Omega_f^{1/2}(t) &= \{\mathbf{x} \in \Omega : \phi(t, \mathbf{x}) \gtrless 0\}, \\ \Gamma(t) &= \{\mathbf{x} \in \Omega : \phi(t, \mathbf{x}) = 0\}. \end{aligned}$$

A triangulation $\mathcal{T}$ is called linearly aligned with $\Gamma(t)$ if for all edges e we have:

$$\phi(\mathbf{x}_{e,1})\phi(\mathbf{x}_{e,2}) \geq 0$$

where $\mathbf{x}_{e,1}$ and $\mathbf{x}_{e,2}$ are the endpoints of e.



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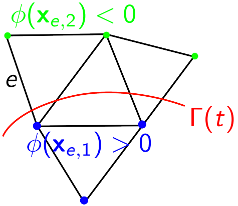

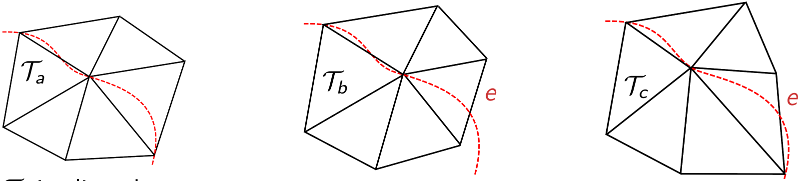

$\mathcal{T}_a$ is aligned

$\mathcal{T}_b$ is NOT aligned

$\mathcal{T}_c$ is aligned

Starting from an initial triangulation $\mathcal{T}$ of Ω, we want to find an optimal triangulation $\mathcal{T}^*$ resulting from a mesh deformation χ^*:

$$\mathcal{T}^* = \chi^*(\mathcal{T}).$$

Deformation χ^* is:

- piecewise affine
- orientation preserving
- globally continuous
- optimal in the sense it is the argument for which a certain functional $\mathcal{F}$ attains its minimum value:

$$\mathcal{F}(\chi^*) = \min \mathcal{F}(\chi).$$

Assumption: $\mathcal{F}$ can be represented by a sum of weighted, element-wise contributions F_T:

$$\mathcal{F}(\chi) = \sum_{T \in \mathcal{T}} \mu_T F_T(\chi)$$

[3] Rumpf, *A variational approach to optimal meshes*, Numer. Math. (1996)

Let R_T denote the affine reference mapping from the optimally deformed simplex T^* to T.

A classical example of function F_T is given by

$$F_T(\chi) = (\|\nabla R_T(\chi)\|^2 - 2)^2 + \det(\nabla R_T(\chi)) + \frac{1}{\det(\nabla R_T(\chi))}$$

- $\|\nabla R_T(\chi)\|^2$ measures the change of edge lengths
- the second term measures the change in area
- the third term rules out deformations with vanishing determinant

With this technique, we obtain optimal, non-degenerate triangulations (i.e., no self intersection occurs), and local mesh quality control.

Price to pay: $\mathcal{F}$ is highly non-linear, non-convex, and global minimizers may be non-unique.

Aligned triangulations can be characterized using a single scalar constraint.

A deformed triangulation is linearly aligned if and only if

$$0 = c(\chi) = \sum_{e \in \chi(\mathcal{T})} \mathcal{H}\big(\phi(\mathbf{x}_{e,1})\phi(\mathbf{x}_{e,2})\big),$$

where

$$\mathcal{H}(z): \quad \begin{cases} > 0 & \text{for } z < 0 \\ = 0 & \text{for } z \geq 0. \end{cases}$$

An optimal, level set aligned triangulation is obtained from the nonlinear constrained optimization problem

$$\min \mathcal{F}(\chi) \quad \text{s.t.} \quad c(\chi) = 0.$$

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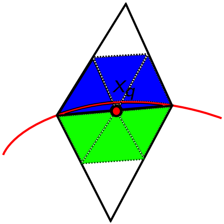
We make use of isoparametric elements equipped with additional degrees of freedom located at the edges.

To obtain a piecewise quadratic approximation of $\Gamma(t)$, we adopt a two-tier procedure:

1. Get a linearly aligned triangulation and a discrete interface Γ_h.
2. Move each quadratic node $\mathbf{x}_q \in \Gamma_h$ along the (linear) normal onto the zero level set.



Remark: to reduce computational costs, the mesh optimization is performed only in a box bounding the leaflet.

At every time t^{n+1}, the Dirichlet-Neumann (DN) algorithm iterates over the fluid and structure subproblems until convergence.

Dirichlet-Neumann algorithm, iteration $k+1$

1. Fluid: Solve for flow variables $\mathbf{u}_{k+1}, p_{k+1}$ on $\Omega_{f,k}$ with boundary condition $\mathbf{u}_{k+1} = \dot{\mathbf{x}}_k$ on Γ_k.
2. Structure: Solve for the structure position $\mathbf{x}_{k+1}$ with load $\mathbf{f}_{\Gamma,k+1}$ on Γ_k and obtain Γ_{k+1}, which defines $\Omega_{f,k+1}$.
3. Check: if the stopping criterion

$$\frac{||\mathbf{x}_{k+1} - \mathbf{x}_k||}{||\mathbf{x}_k||} < tol$$

is satisfied set $\mathbf{u}^{n+1} = \mathbf{u}_{k+1}$, $p^{n+1} = p_{k+1}$, $\mathbf{x}^{n+1} = \mathbf{x}_{k+1}$, $\Gamma^{n+1} = \Gamma_{k+1}$, and $\Omega_f^{n+1} = \Omega_{f,k+1}$; otherwise we go back to step 1.

To speed up the convergence, we use an Aitken acceleration technique[4].

[4] Küttler, Wall, *Fixed-point fluid-structure interaction solvers with dynamic relaxation*, CMECH (2008)

- For the time discretization we use BDF1 or BDF2.
- Inertial term in the momentum equation is treated implicitly by Picard iteration.
- For the space discretization we use inf-sup stable Taylor-Hood FE pair $\mathbb{P}_2 - \mathbb{P}_1$.
- We allow for discontinuities of the pressure across Γ_k, since accurately resolving the pressure discontinuity across Γ_k is needed for the correct evaluation of the hydrodynamic force.
- The Subspace Projection Method[5] is used to enforce the continuity of the velocity across Γ_k.
- The linear systems are solved by a direct solver (UMFPACK).

[5] Bäumler, Bänsch, *A subspace projection method for the implementation of interface conditions in a single-drop flow problem*, JCP (2013)

- For the time discretization we use a generalized Crank-Nicolson scheme[6].
- For the space discretization we use a third order Hermite finite element method[7].
- After time discretization, at every time step we have to solve a quasi-static problem which is equivalent to minimization problem:

$$\mathbf{x}_{k+1} = \arg\min_{\mathbf{y}\in K} J(\mathbf{y}), \quad \text{with } K = \left\{ \mathbf{y} \in (H^2(0,L))^2, |\mathbf{y}'| = 1, B.C. \right\},$$

where the total energy of the beam can be written as:

$$J(\mathbf{y}) = \frac{1}{2}\int_0^L \frac{\rho_s}{\Delta t^2} |\mathbf{y}|^2 ds + \frac{1}{2}\int_0^L EI\alpha \left|\mathbf{y}''\right|^2 ds - \int_0^L \tilde{\mathbf{f}}_{k+1} \cdot \mathbf{y} ds.$$

[6] Glowinski, Le Tallec, *Augmented Lagrangian and operator-splitting methods in nonlinear mechanics*, SIAM (1988)

[7] Glowinski, Le Tallec, *Large Displacement Calculations of Flexible Pipelines by Finite Element and Nonlinear Programming Methods*, SIAM J. Sci. Stat. Comput (1980)

- To treat the inextensibility condition $|\mathbf{x}'| = 1$, we use an augmented Lagrangian Method[8]. for the equivalent minimization problem

$$\{\mathbf{x}_{k+1}, \mathbf{x}'_{k+1}\} = \underset{\{\mathbf{y},\mathbf{q}\}\in W}{\arg\min}\, J(\mathbf{y}), \quad \text{with } W = \{\mathbf{y} \in \mathcal{V},\ \mathbf{q} \in \mathcal{Q},\ \mathbf{y}' - \mathbf{q} = \mathbf{0}\},$$

where

$$\mathcal{V} = \left\{\mathbf{y} \in (H^2(0,L))^2,\ B.C.\right\},$$
$$\mathcal{Q} = \left\{\mathbf{y} \in (L^2(0,L))^2, |\mathbf{y}| = 1 \text{ a.e. on } (0,L)\right\}.$$

- To solve the saddle-point problem associated with the augmented Lagrangian functional, we employ ALG2[9], which is a ‘disguised’ Douglas-Rachford operator-splitting scheme.

[8] Fortin, Glowinski, *The augmented Lagrangian method*, North Holland (1983)

[9] Glowinski, Le Tallec, *Augmented Lagrangian and operator-splitting methods in nonlinear mechanics*, SIAM (1988)

The computation of the hydrodynamic force $\mathbf{f}_\Gamma$ is crucial for the numerical stability and accuracy of the solver[10].

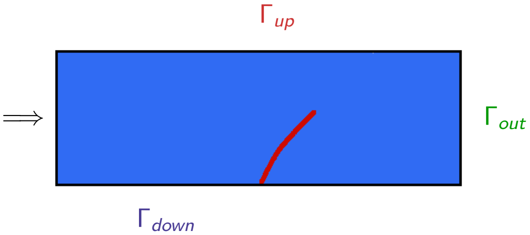
The load exerted by the fluid onto the structure can be computed as the variational residual $\mathcal{R}$ of the momentum conservation equation for the fluid tested with test functions $\mathbf{v}$ that do not vanish at $\Gamma(t)$:

$$\int_{\Gamma(t)} \mathbf{f}_\Gamma \cdot \mathbf{v} \, \mathrm{d}\Gamma = -\int_{\Gamma(t)} \boldsymbol{\sigma}^1 \mathbf{n}^1 \cdot \mathbf{v} \, \mathrm{d}\Gamma - \int_{\Gamma(t)} \boldsymbol{\sigma}^2 \mathbf{n}^2 \cdot \mathbf{v} \, \mathrm{d}\Gamma$$
$$= \mathcal{R}(\Omega_f^1(t); \mathbf{u}, p, \mathbf{v}) + \mathcal{R}(\Omega_f^2(t); \mathbf{u}, p, \mathbf{v}).$$

Since $\Gamma_h^{f,n+1}$ and $\Gamma_h^{s,n+1}$ are aligned but do not coincide and the fluid and structure discretizations are based on different elements, the discrete power exchanged at the interface is not exactly balanced. However, with the numerical results we show that the mismatch is small.

[10] Farhat, Lesoinne, Le Tallec, *Load and motion transfer algorithms for fluid/structure interaction problems with non-matching discrete interfaces: Momentum and energy conservation, optimal discretization and application to aeroelasticity*, CMAME (1998)

The leaflet is clamped at the midpoint of the base and it is 0.5 cm long. We set $\rho_f = 1$ g/cm^3 and μ varies to achieve $Re = 100$ in each test.



- The inlet condition changes depending on the test.
- No slip condition in imposed on Γ_{down}
- Symmetry condition is imposed on Γ_{up}
- Homogeneous Neumann condition is enforced on Γ_{out}

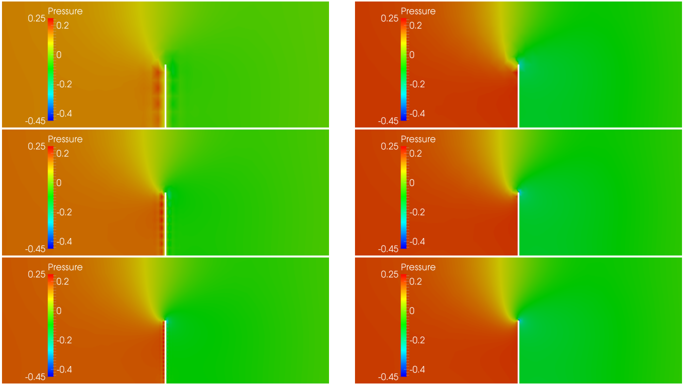
We take: $U = 1$ cm/s , $\rho_s = 10^6$ g/cm, $EI = 0.01$ g/(cm s^2), $h_s = 1/44$, $h_f = \sqrt{2}/8 \cdot 2^{-l}$ with $l = 1$ (*coarse*), 2 (*medium*), 3 (*fine*).

continuous pressure | *discontinuous pressure*

coarse

medium

fine



Test 1: standard ALE vs extended ALE

We set $\rho_s = 5$ g/cm, $EI = 0.05$ g/(cm s^2), $h_s = 1/44$, $h_f = \sqrt{2}/8 \cdot 2^{-l}$ with $l = 1, 2$.

At Γ_{in}, we impose a time dependent Poiseuille profile, with maximum velocity:

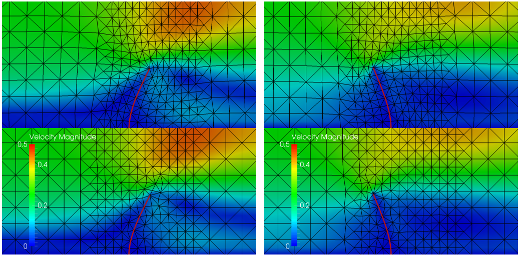
$$U(t) = \frac{1}{4}\left(1 - \cos\left(\frac{\pi}{2}t\right)\right).$$

PLAY

$t = 7\ s$ | $t = 9\ s$

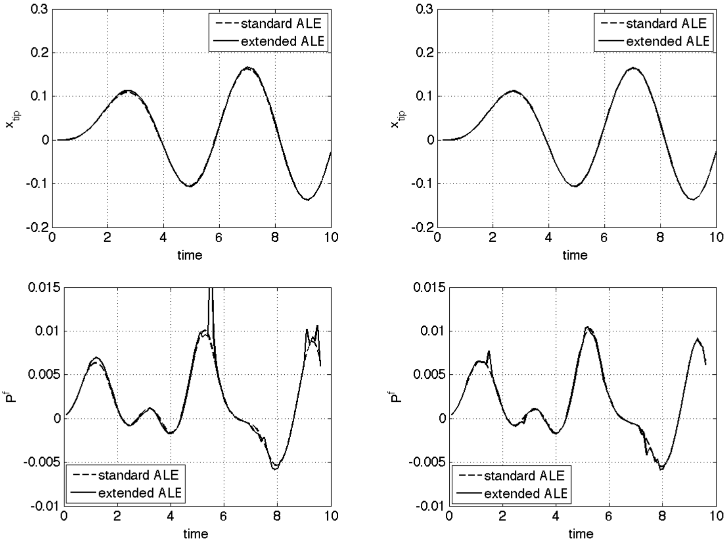
st. ALE

ex. ALE

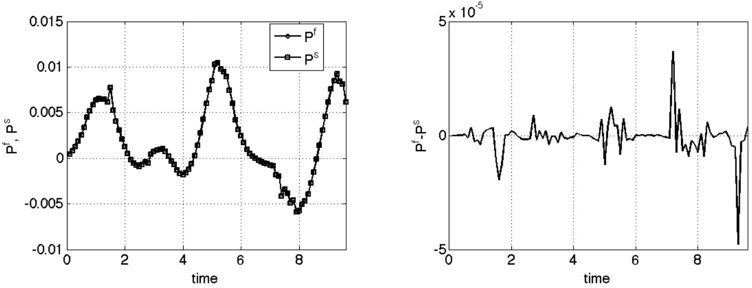


Test 1: tip movement and power exchanged at Γ

coarse mesh | *medium mesh*



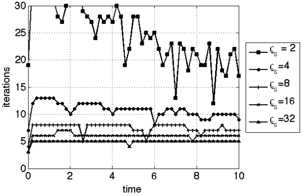
Now we take the medium mesh and check the unbalance in the power exchange at the interface.



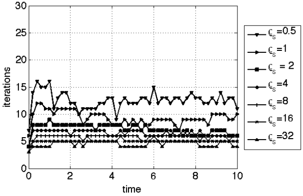
Notice that difference between the two powers exchanged at the interface is of the order of 10^{-5}, which is 0.1% of the power value.

We set $EI = 0.05$ g/(cm s^2) and let the structure density vary: $\rho_s = 32, 16, 8, 4, 2, 1, 0.5$ g/cm, with $\rho_f = 1$ g/cm^3.

no relaxation



relaxation with Aitken's acceleration



- The number of DN iterations increases as ρ_s decreases.
- The DN algorithm with no relaxation ceases to converge[11] when $\rho_s \leq \rho_f$.
- Aitken's acceleration method allows a reduction in the number of DN iterations[12].

[11] Causin, Gerbeau, Nobile, *Added-mass effect in the design of part. algorithms for fluid–structure prob.*, CMAME (2005)

[12] Küttler, Wall, *Fixed-point fluid-structure interaction solvers with dynamic relaxation*, CMECH (2008)

We set $\rho_s = 5$ g/cm, $EI = 0.05$ g/(cm s^2), $h_s = 1/44$, $h_f = \sqrt{2}/16$.

At Γ_{in}, we impose a time dependent Poiseuille profile, with maximum velocity:

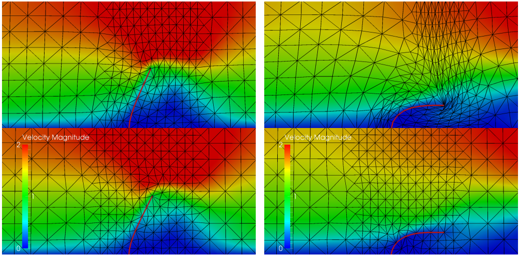
$$U(t) = \left(1 - \cos\left(\frac{\pi}{2} t\right)\right).$$

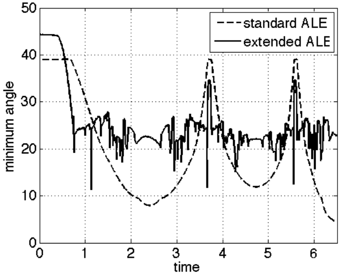
PLAY

$t = 5.8\ s$ | $t = 6.5\ s$

st. ALE

ex. ALE





- The minimum angle in the meshes given by the standard ALE method occasionally drops below 10 degrees. In particular, it is equal to 4 degrees at $t = 6.5$ s, shortly before the simulation breaks down.
- The minimum angle for the meshes given by the extended ALE method oscillates around 23 degrees most of the time.

- We proposed an extended ALE method for the simulation of fluid-structure interaction problems with large structural displacements.
- Our extended ALE method relies on mesh optimization technique with an additional constraint to enforce the alignment of the interface with the edges of the resulting triangulation.
- We applied it to the interaction of an incompressible fluid with an inextensible beam.
- We showed that when the structural displacement is mild the results given by our extended ALE method are in excellent agreement with the results given by a standard ALE method.
- We showed that when the structural displacement is large the quality of the mesh given by the extended ALE method is still high.