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Monolithic Newton-multigrid FEM for the simulation of thixotropic flow problems

N. Begum, A. Ouazzi, S. Turek
Institute of Applied Mathematics, LS III,
TU Dortmund University, Dortmund, Germany

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GESELLSCHAFT für ANGEWANDTE MATHEMATIK und MECHANIK
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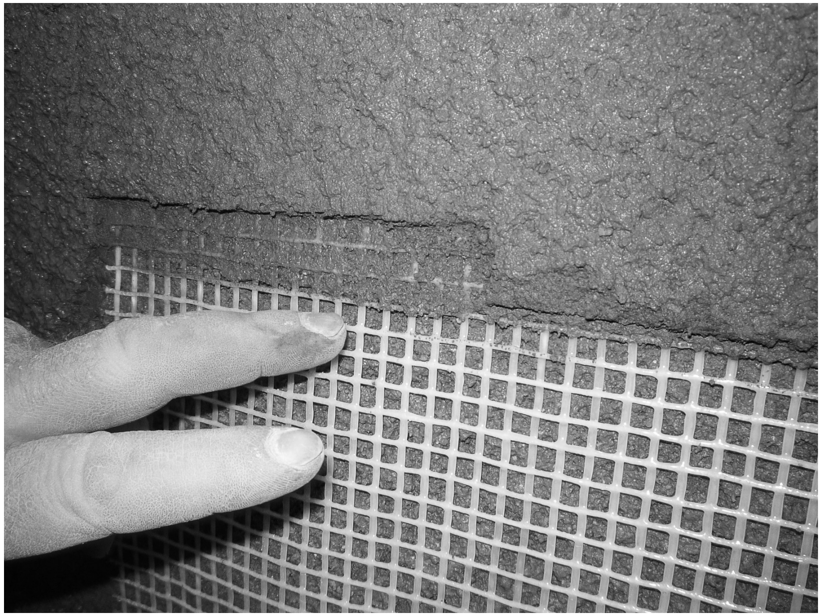
Why "Thixotropic materials?

- **Processing of thixotropic materials relevant for industrial applications**

➡ **Lubrication, asphalt, self-compacting concrete…**

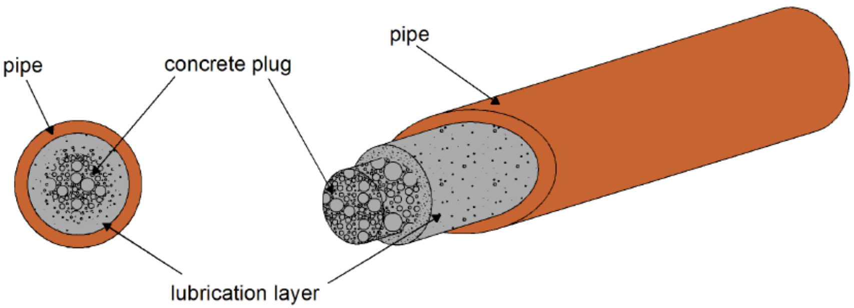
- **Physically fascinating due to improved mechanical properties**



Goal:

- **Modern CFD methods with high accuracy, robustness and efficiency for thixotropic materials**

➡ **Saving time, money and resources**



Investigation of solid/liquid and liquid/solid transitions based on micro-structure

Introduction

- **Thixotropy means**
  - ➢ **combination of two greek words**
  - ➢ **Thixis: shaking/stirring**
  - ➢ **trepo: turning/changing**

- **Thixotropy concept**
  - ➢ **Based on viscosity**
  - ➢ **Flow induced by time-dependent decrease of viscosity**
  - ➢ **The phenomena is reversible**

- **Rejuvenation / Breakdown**
  - ➢ **"Faster" flow: fluid rejuvenates**
    Decreases of viscosity with acceleration of the flow

- **Aging / Build-up**
  - ➢ **At rest or under slow flow: fluid ages**
    Increases of the viscosity in time

Realization in FeatFlow

HPC features:
- Moderately parallel
- GPU computing
- Open source


Hardware -oriented Numerics

Numerical features:
- Higher order FEM in space & (semi-) Implicit FD/FEM in time
- Semi-(un)structured meshes with dynamic adaptive grid deformation
- Fictitious Boundary (FBM) methods
- Newton-Multigrid-type solvers

Non-Newtonian flow module:
•generalized Newtonian model (Power-law, Carreau,Houska,…)
•viscoelastic differential model (Giesekus, FENE, Oldroyd,...)

Multiphase flow module (resolved interfaces):
•l/l – interface capturing (Level Set)
•s/l – interface tracking (FBM)
•s/l/l – combination of l/l and s/l

Engineering aspects:
•Geometrical design
•Modulation strategy
•Optimization

Here: FEM-based tools for the accurate simulation of (thixotropic) flow problems, particularly with complex rheology

For details, please visit: www.featflow.de

Governing equations

➡starting point: Generalized Navier-Stokes equations (+initial and boundary conditions)

$$\rho\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u}-\nabla\cdot\boldsymbol{\sigma}+\nabla p=\rho f,$$

$$\nabla\cdot\boldsymbol{u}=0,$$

- velocity- and pressure field $\boldsymbol{u}$ and p
- stress tensor $\boldsymbol{\sigma}$
- linear material behaviour - Newtonian fluids

$$\boldsymbol{\sigma}=2\eta_s\boldsymbol{D}(\boldsymbol{u})$$: η_s is constant viscosity

- non-linear material behaviour- structurally viscous / viscoplastic

$$\boldsymbol{\sigma}=2\eta_s(D_{\mathrm{II}},p,\Theta,\lambda)\boldsymbol{D}(\boldsymbol{u}),\quad D_{\mathrm{II}}=\mathrm{tr}\left(\frac{1}{2}\boldsymbol{D}(\boldsymbol{u})^2\right)$$

↓

- Power-law, Carreau, Bingham, Herschel-Bulkley, Houska, …

- structure paremeter λ

Rheological Models

Archetypical yield-stress driven thixotropic models

$$\begin{cases} \boldsymbol{\sigma} = 2\,\eta(D_{\mathrm{II}},\lambda)\boldsymbol{D}(\boldsymbol{u}) + \sqrt{2}\,\tau(\lambda)\dfrac{\boldsymbol{D}(\boldsymbol{u})}{\sqrt{D_{\mathrm{II}}}} & \text{if } D_{\mathrm{II}} \neq 0 \\ \sigma_{\mathrm{II}} \leq \tau(\lambda) & \text{if } D_{\mathrm{II}} = 0 \end{cases}$$

Relations between rheological parameters and structural parameter

	$\eta(D_{\mathrm{II}},\lambda)$	$\tau(\lambda)$
Worrall and Tulliani [1]	$\lambda\eta_0$	τ_0
Coussot *et al.* [2]	$\lambda^a\eta_0$	–
Houska [3]	$(\eta_0+\eta_1\lambda)\,D_{\mathrm{II}}^{\frac{(n-1)}{2}}$	$(\tau_0+\tau_1\lambda)$
Mujumbar *et al.* [4]	$(\eta_0+\eta_1\lambda)\,D_{\mathrm{II}}^{\frac{(n-1)}{2}}$	$\lambda^{a+1}\mathrm{G}_0\Lambda_c{}^*$
Burgos *et al.* [5]	η_0	$\lambda\tau_0$
Dullaert & Mewis [6]	$\lambda\eta_0$	$\lambda\mathrm{G}_0\left(\lambda D_{\mathrm{II}}^{\frac{1}{2}}\right)\Lambda_c^*$

$^*\Lambda_c$ is a constant/variable elastic strain.

General format of evolution equation for structural parameter:

$$\frac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda = F_{buildup} - F_{breakdown}$$

Expressions for different thixotropic models:

	$F_{buildup}$	$F_{breakdown}$
Worrall and Tulliani[1]	$c_1(1-\lambda)D_{\mathrm{II}}^{\frac{1}{2}}$	$c_2\lambda D_{\mathrm{II}}^{\frac{1}{2}}$
Coussot *et al.*[2]	c_1	$c_2\lambda D_{\mathrm{II}}^{\frac{1}{2}}$
Houska[3]	$c_1(1-\lambda)$	$c_2\lambda^m D_{\mathrm{II}}^{\frac{1}{2}}$
Mujumbar *et al.*[4]	$c_1(1-\lambda)$	$c_2\lambda D_{\mathrm{II}}^{\frac{1}{2}}$
Burgos *et al.*[5]	$c_1(1-\lambda)$	$c_2\lambda D_{\mathrm{II}}^{\frac{1}{2}} \exp(aD_{\mathrm{II}}^{\frac{1}{2}})$
Dullaert & Mewis[6]	$(c_1 + c_3 D_{\mathrm{II}}^{\frac{1}{2}})(1-\lambda)t^{-b}$	$c_2\lambda D_{\mathrm{II}}^{\frac{1}{2}} t^{-b}$

Quasi-Newtonian thixotropic model

- **Viscoplastic model**

$$\begin{cases} \eta_s(D_{\mathrm{II}}) = \eta_0 + \dfrac{\sqrt{2}}{2}\tau_0 D_{\mathrm{II}}^{-\frac{1}{2}} & \text{if } D_{\mathrm{II}} \neq 0 \\ \|\boldsymbol{\sigma}\| \leq \tau_0 & \text{else} \end{cases}$$

$$(\tau_0: \ \text{yield stress})$$

- **Thixotropic model**

$$\begin{cases} \eta_s(D_{\mathrm{II}}, \lambda) = \eta(\lambda) + \dfrac{\sqrt{2}}{2}\tau(\lambda) D_{\mathrm{II}}^{-\frac{1}{2}} & \text{if } D_{\mathrm{II}} \neq 0 \\ \|\boldsymbol{\sigma}\| \leq \tau(\lambda) & \text{else} \end{cases}$$

$$(\lambda: \ \text{structure parameter})$$

➡ **Structure evolution equation**

$$\frac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda = a(1-\lambda) - b\lambda D_{\mathrm{II}}^{\frac{1}{2}}$$

$$(a, b \text{ are structure parameters})$$

- **Viscosity model for thixotropic flow i.e. extended viscosity defined on all domains s.t.**

$$\begin{cases} \eta_s(D_{\mathrm{II}}, \lambda) = \eta(\lambda) + \dfrac{\sqrt{2}}{2}\tau(\lambda) D_{\mathrm{II}}^{-\frac{1}{2}} & \text{if } D_{\mathrm{II}} \neq 0 \\ \|\boldsymbol{\sigma}\| \leq \tau(\lambda) & \text{else} \end{cases}$$

- **Examples:**

$$\begin{cases} I. \quad \eta_s(D_{\mathrm{II}}, \lambda) = \eta(\lambda) + \dfrac{\sqrt{2}}{2}\tau(\lambda)\dfrac{1}{\sqrt{(D_{\mathrm{II}} + (k^{-1})^2)}} \\ II. \quad \eta_s(D_{\mathrm{II}}, \lambda) = \eta(\lambda) + \dfrac{\sqrt{2}}{2}\tau(\lambda)\dfrac{1}{D_{\mathrm{II}}^{\frac{1}{2}}}(1 - e^{-kD_{\mathrm{II}}^{\frac{1}{2}}}) \end{cases}$$

$$(k: \text{ regularization parameter})$$

- **Full set of equations**

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + u \cdot \nabla\right) u - \nabla \cdot \Big(2\eta_s(D_{\mathrm{II}}, \lambda) D(u)\Big) + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ \dfrac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda - a(1-\lambda) + b\lambda D_{\mathrm{II}}^{\frac{1}{2}} = 0 & \text{in } \Omega \end{cases}$$

Newton's method

Let $\mathcal{U} = (\lambda, \boldsymbol{u}, p)$**, and** $\mathcal{R}_{\mathcal{U}}(\mathcal{U})$ **be the continuous or the discrete corresponding system's residum.**

- **Update of the nonlinear iteration with the correction** $\delta\mathcal{U}$ **i.e.**

$$\mathcal{U}^N = \mathcal{U} + \delta\mathcal{U}$$

- **The linearization of the residual provides**

$$\begin{aligned}\mathcal{R}_{\mathcal{U}}\left(\mathcal{U}^N\right) &= \mathcal{R}_{\mathcal{U}}\left(\mathcal{U} + \delta\mathcal{U}\right) \\ &= \mathcal{R}_{\mathcal{U}}\left(\mathcal{U}\right) + \mathcal{J}\left(\mathcal{U}\right) \cdot \delta\mathcal{U}\end{aligned}$$

- **The Newton's method assuming invertible Jacobian**

$$\mathcal{U}^N = \mathcal{U} - \mathcal{J}^{-1}(\mathcal{U})\,\mathcal{R}_{\mathcal{U}}(\mathcal{U})$$

Adaptive Newton's method

Jacobian calculations $$\mathcal{J}\left(\mathcal{U}\right)=\left(\frac{\partial\mathcal{R}_{\mathcal{U}}\left(\mathcal{U}\right)}{\partial\mathcal{U}}\right)$$

- **Continuous Adaptive Newton based on a priori study of Jacobian's properties and decompositions**

$$\mathcal{J}\left(\mathcal{U}\right)=\left(\frac{\partial\hat{\mathcal{R}}_{\mathcal{U}}\left(\mathcal{U}\right)}{\partial\mathcal{U}}\right)+\delta\left(\frac{\partial\tilde{\mathcal{R}}_{\mathcal{U}}\left(\mathcal{U}\right)}{\partial\mathcal{U}}\right)$$

- **Discrete Adaptive Newton based on the rate of residum's convergence**

$$\left(\frac{\partial\mathcal{R}}{\partial\mathcal{U}}\right)_{ij}\approx\left(\frac{\mathcal{R}_i\left(\mathcal{U}+\epsilon e_j\right)-\mathcal{R}_i\left(\mathcal{U}-\epsilon e_j\right)}{2\epsilon}\right)$$

- **Flow variables** $(\lambda, \boldsymbol{u}, p)$

  - **Set** $\mathbb{T} := L^2(\Omega), \mathbb{V} := \left[H_0^1(\Omega)\right]^2, \mathbb{Q} := L_0^2(\Omega)$
  - **Set** $\tilde{\boldsymbol{u}} := (\lambda, \boldsymbol{u})$
  - **Find** $(\lambda, \boldsymbol{u}, p) \in \left(\mathbb{T} \cap H^1(\Omega)\right) \times \mathbb{V} \times \mathbb{Q}$ **s.t.**

$$\left\langle \mathcal{K}(\lambda, \boldsymbol{u}, p), (\xi, \boldsymbol{v}, q) \right\rangle = \left\langle \mathcal{L}, (\xi, \boldsymbol{v}, q) \right\rangle, \quad \forall (\xi, \boldsymbol{v}, q) \in \mathbb{T} \times \mathbb{V} \times \mathbb{Q}$$

$$\mathcal{K} = \begin{bmatrix} \mathcal{A}_{\tilde{\boldsymbol{u}}} & \mathcal{B}^{\mathrm{T}} \\ \mathcal{B} & 0 \end{bmatrix}$$

  - **Compatibily constraints**

$$\sup_{\boldsymbol{v} \in \mathbb{V}} \frac{\langle \mathcal{B}\boldsymbol{v}, q \rangle}{\|\boldsymbol{v}\|_{\mathbb{V}}} \geq \beta \|q\|_{\mathbb{Q}/Ker\mathcal{B}^{\mathrm{T}}}, \quad \forall q \in \mathbb{Q}$$

Mathematical & Numerical challenges

- **Discretizations have to handle the following challenges**
  - **Stable FEM spaces**
  - **Non-symmetric, non-coercive and ill-posedness**
  - **Convection and positivity preserving**
  - **Locally adatpted meshes for steep gradients**

- **Solvers have to deal with**
  - **Different source of nonlinearities**
  - **Strong coupling of equations**
  - **Robustness and efficiency**

Approximated problem

- **Conforming approximations**

$$\mathbb{T}_h \subset \mathbb{T}, \quad \mathbb{V}_h \subset \mathbb{V}, \quad \mathbb{Q}_h \subset \mathbb{Q}$$
$$\mathcal{A}_{\tilde{\boldsymbol{u}}h} = \mathcal{A}_{\tilde{\boldsymbol{u}}}, \ \mathcal{B}_h = \mathcal{B}$$

- **Discrete inf-sup condition**

$$\sup_{\boldsymbol{v}_h \in \mathbb{V}_h} \frac{\langle \mathcal{B}_h \boldsymbol{v}_h, q_h \rangle}{\|\boldsymbol{v}_h\|_{\tilde{\mathbb{V}}}} \geq \beta_h \|q_h\|_{\mathbb{Q}/Ker\mathcal{B}_h^{\mathrm{T}}}, \quad \forall q_h \in \mathbb{Q}_h$$

Robust conforming FEM

The family of conforming FEM $Q_r/Q_r/P_{r-1}^{\text{disc}},\ r \geq 2$ for $(\lambda, \boldsymbol{u}, p)$ with stabilization

$$J_\lambda(\lambda_h, \xi_h) = \gamma_\lambda \sum_{e \in \mathcal{E}_h} h \int_e [\nabla \lambda_h] : [\nabla \xi_h] \, d\Omega$$

- **Inf-sup conditions is satisfied**
- **Discontinuous pressure**
  - **Good for the solver**
  - **Element-wise mass conservation**
- **Discrete problem is well-posed**
- **Highly consistent and symmetric stabilization**
- **Robust solver w.r.t. the monolitic approach**
- **Efficient solver w.r.t. multigird solver**

Monolitic-multigrid linear solver

- **Standard geometric multigrid solver for linearized system**
- **Full Q_r and P_{r-1}^{disc} restriction and prolongation**
- **Local Multilevel Pressure Schur Complement via Vanka-like smoother**

$$\begin{pmatrix} \lambda^{l+1} \\ u^{l+1} \\ p^{l+1} \end{pmatrix} = \begin{pmatrix} \lambda^{l} \\ u^{l} \\ p^{l} \end{pmatrix} + \omega^l \sum_{T \in \mathcal{T}_h} \left(\Big(\mathcal{K}_h + \mathcal{J}_\lambda \Big)_{|T} \right)^{-1} \begin{pmatrix} \mathcal{R}_{\lambda^l} \\ \mathcal{R}_{u^l} \\ \mathcal{R}_{p^l} \end{pmatrix}_{|T}$$

Coupled Monolithic Multigrid Solver !

Bingham flow in unit channel

- **"Validation": manufactured exact solution for Bingham flow in a unit channel domain**

  ➡ **Dirichlet boundary conditions**

  ➡ **analytical solution for velocity $u_{exact} = (u,0)^T$, such that u**

$$
u = \begin{cases} \frac{1}{8}\left((1-2\tau_0)^2 - (1-2\tau_0-2y)^2\right) & \text{if } y \in [0, \frac{1}{2}-\tau_0) \\ \frac{1}{8}(1-2\tau_0)^2 & \text{if } y \in [\frac{1}{2}-\tau_0, \frac{1}{2}+\tau_0] \\ \frac{1}{8}\left((1-2\tau_0)^2 - (2y-2\tau_0-1)^2\right) & \text{if } y \in (\frac{1}{2}+\tau_0, 1] \end{cases}
$$

  ➡ **solution show that the plug zone, $y \in [\frac{1}{2} - \tau_0, \frac{1}{2} + \tau_0]$, is a moving rigid body with a constant velocity.**

Bingham flow in unit channel

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Table 1: **Newton Iterations:** Corresponding results of No. of non-linear iterations w.r.t. regularization parameter k and refinement level L for Bingham viscoplastic flow for different values of non-thixotropic yield stress parameter τ_0.

$k \backslash L$	2	3	4	5	6	2	3	4	5	6
	For $\tau_0 = 0.25$					For $\tau_0 = 0.375$				
10^1	3	3	4	3	4	3	3	3	3	3
10^2	15	15	14	12	12	11	11	11	11	11
10^3	16	17	18	16	16	13	17	17	16	19
10^4	22	22	24	24	25	15	23	25	26	26
10^5	28	22	27	29	32	11	25	29	32	35
10^6	34	22	30	50	52	9	25	32	51	53

Table 2: **L^2-error of velocity $\|u - u_{ex}\|_2$:** Velocity errors in L_2 for Bingham viscoplastic flow for different values of non-thixotropic yield stress.

$k \backslash L$	2	3	4	5	6
	$\tau_0 = 0.25$				
10^1	$3.007423E-3$	$2.986835E-3$	$2.985134E-3$	$2.985008E-3$	$2.985049E-3$
10^2	$8.070790E-4$	$7.910534E-4$	$7.771159E-4$	$7.766747E-4$	$7.766779E-4$
10^3	$8.854564E-5$	$8.713851E-5$	$8.999407E-5$	$8.962862E-5$	$8.876830E-5$
10^4	$9.265935E-6$	$8.958942E-6$	$8.871151E-6$	$8.863787E-6$	$8.919511E-6$
10^5	$9.311507E-7$	$8.991498E-7$	$8.892617E-7$	$8.868517E-7$	$8.860986E-7$
10^6	$9.316114E-8$	$8.994771E-8$	$8.894896E-8$	$8.869936E-8$	$8.863445E-8$
	$\tau_0 = 0.375$				
10^1	$1.598519E-3$	$1.553552E-3$	$1.552506E-3$	$1.552413E-3$	$1.552401E-3$
10^2	$1.138180E-3$	$8.094995E-4$	$7.961622E-4$	$7.961382E-4$	$7.961298E-4$
10^3	$8.799405E-4$	$9.477742E-5$	$1.009830E-4$	$9.963894E-5$	$9.836866E-5$
10^4	$8.887246E-4$	$9.998811E-6$	$9.857640E-6$	$9.893807E-6$	$1.004589E-5$
10^5	$8.893694E-4$	$1.007135E-6$	$9.912362E-7$	$9.901165E-7$	$9.913865E-7$
10^6	$8.893644E-4$	$1.007866E-7$	$9.918013E-8$	$9.906410E-8$	$9.910985E-8$

Thixotropic flow in channel

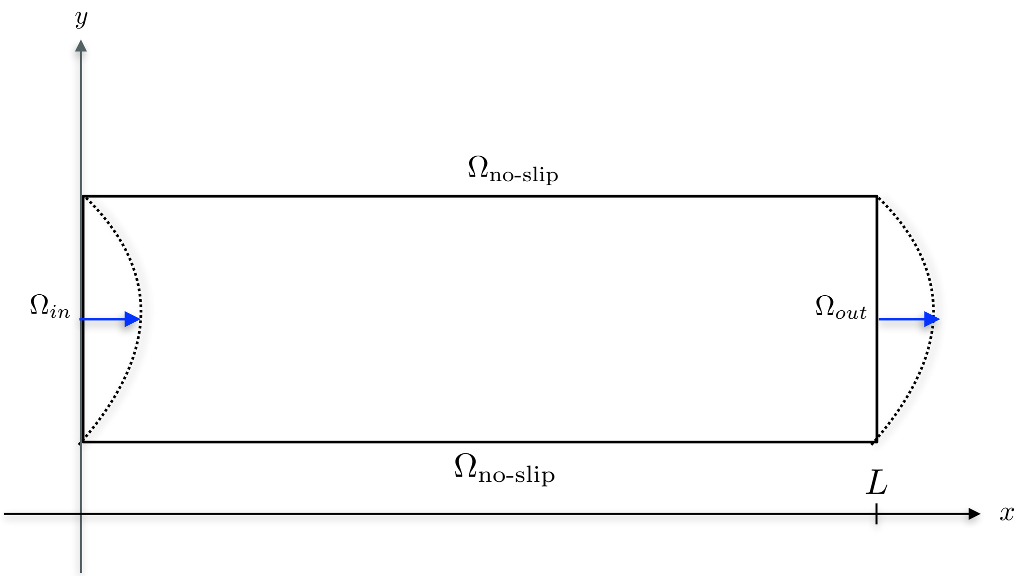
- **Starting point : consider the flow in infinitely long channel**

➡ **Flow variables varies only w.r.t 'y' i.e., no change in x-direction**

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u(y) \\ 0 \end{pmatrix}, \quad \mathbf{D}(\mathbf{u}) = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{pmatrix}, \quad \frac{\partial u}{\partial x} = \frac{\partial \lambda}{\partial x} = 0$$

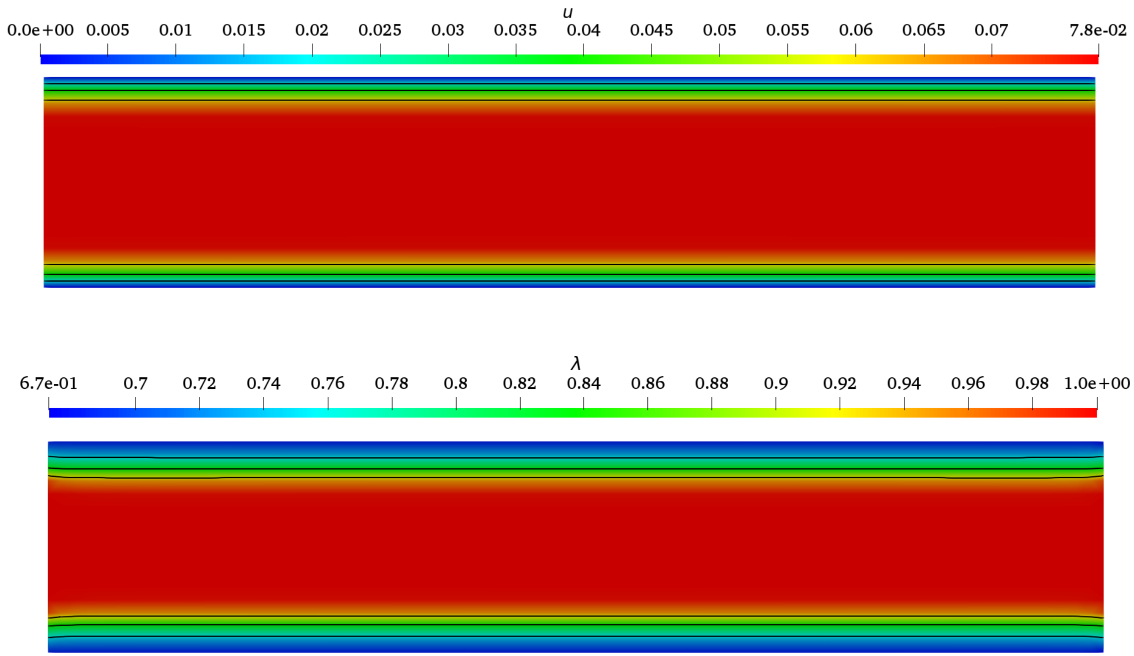
➡ **Reduction of complete flow model in 1D problem**

- **Fast , efficient, high resolutions calculations possible for fully developed flow**



Thixotropic flow in channel

- **Prototypical simulation results for thixotrpic flow- validation of 1D tool**
- **Specifying the "1D-profiles as boundary Data" in 2D simulations of channel flow**
- **Parameters:** $\eta_0 = 1.0, \tau_1 = 0.25, \eta_1 = \tau_0 = 0.0, a = b = 1.0, k = 10^4$



✓ **Horizontal isolines w.r.t. each flow variable**
✓ **Same flow profiles over y at each x-position**

Thixotropic flow in channel

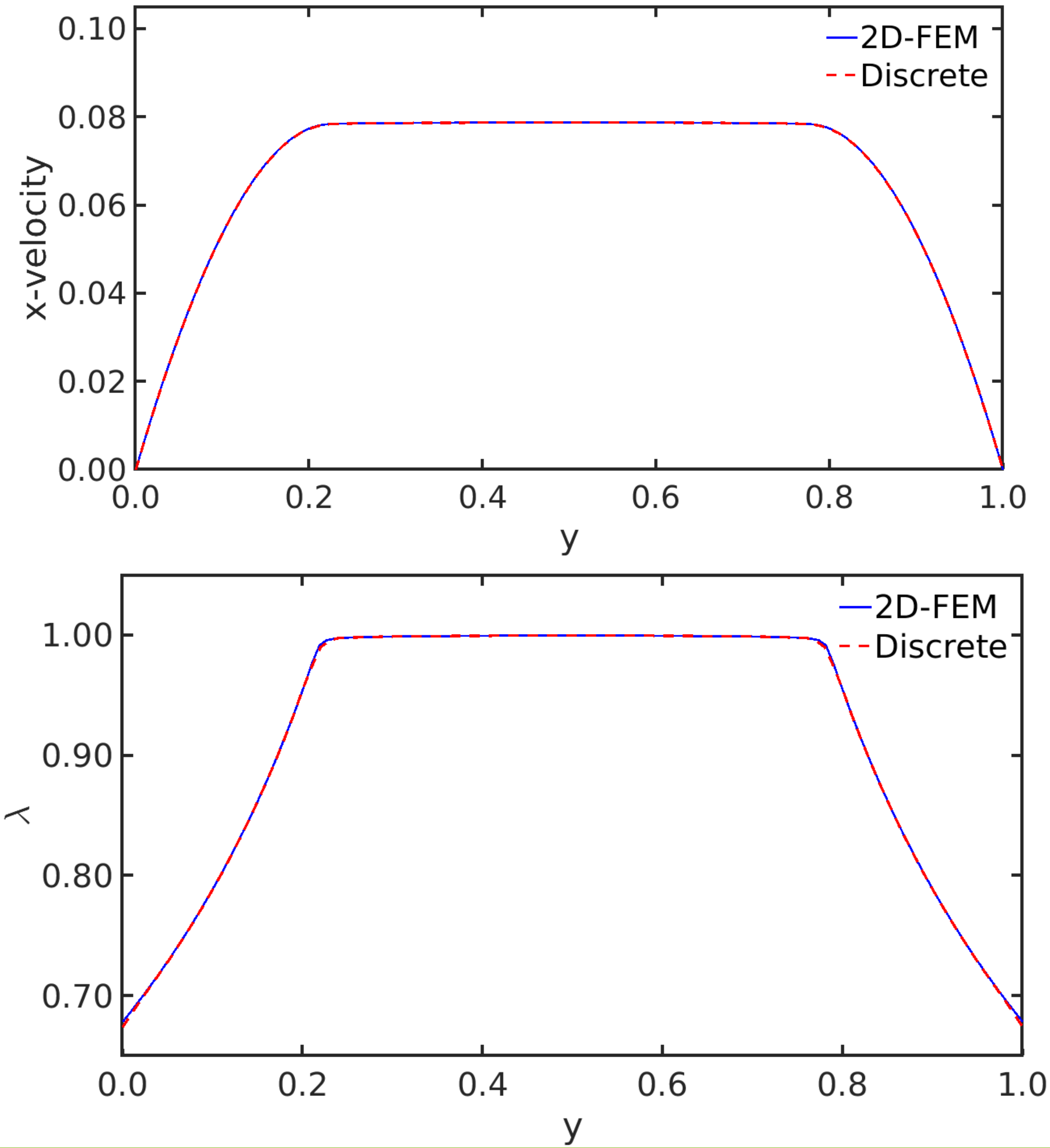
- **Comparison of velocity and structure parameter profiles at middle x-position for thixotropic material**

✓ **1D and 2D-FEM simulations leads to same results**

✓ **Detailed validation for varying the parameters $(k, \eta(\lambda), \tau(\lambda), \ldots)$**

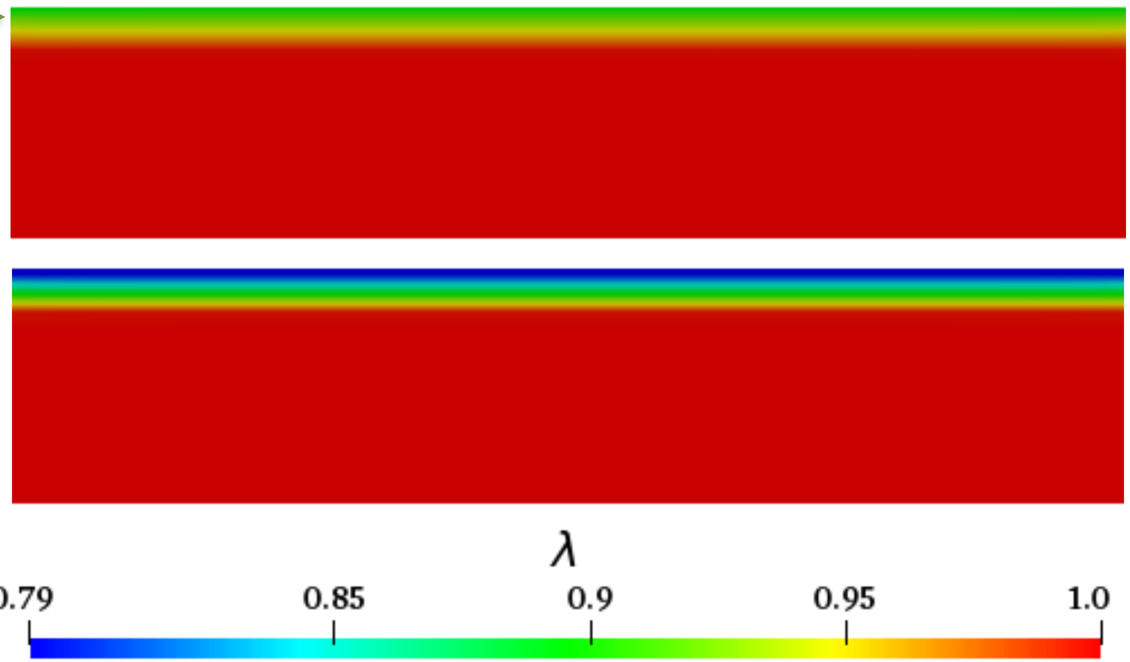
✓ **Validated to recover the analytical Bingham profiles in unit channel**

➡ **In principle, this approach can be analogously applied to any thixotropic elasto-viscoplastic model**



- Impact of thixotropic viscosity and yield-stress on structural parameter

  - Without thixotropy
  - With thixotropy



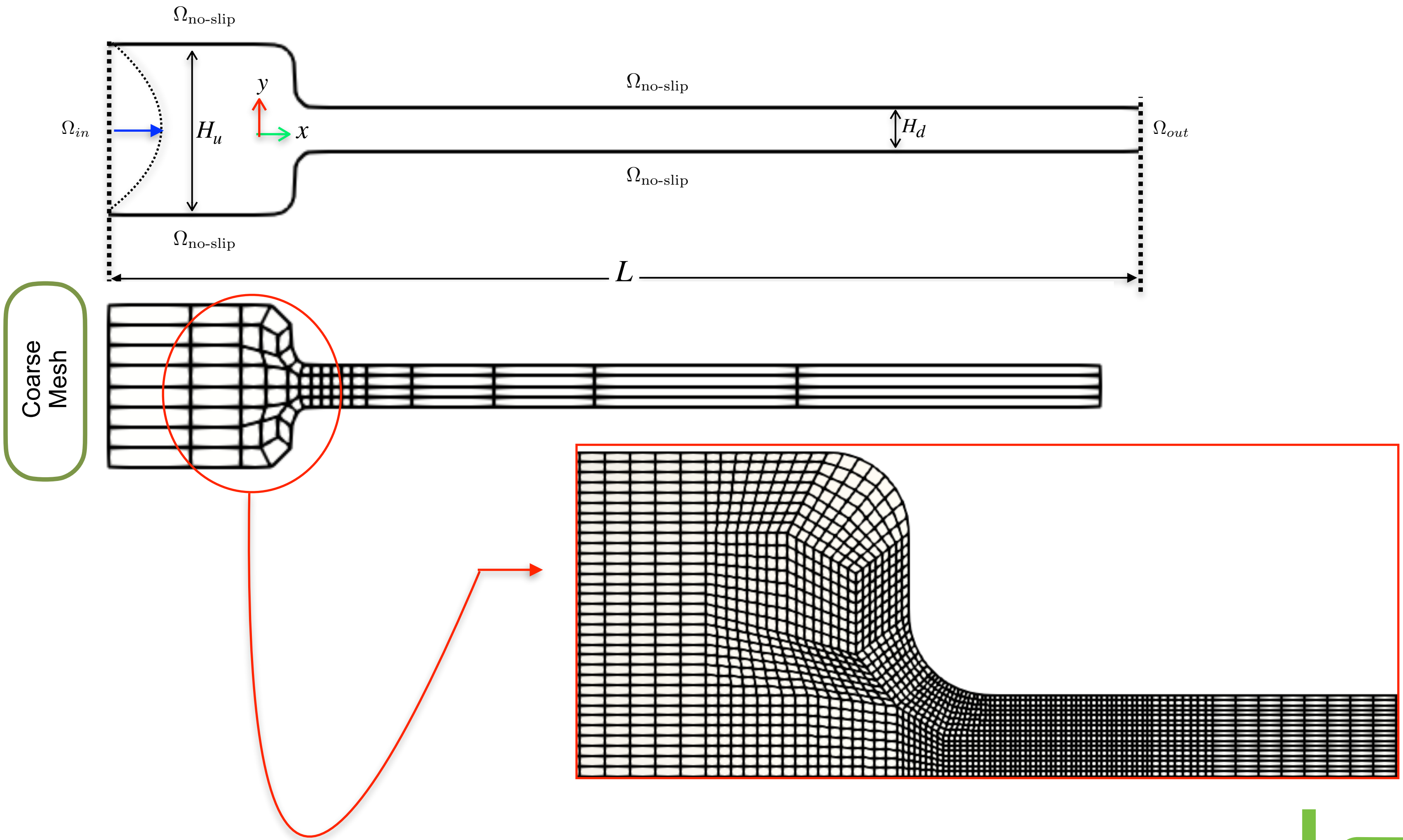
- Inherent thixotropy speed-up the breakdown

  ✓ Appearance of breakdown layer

  ✓ Restart pipelines pressure should not be over-estimated

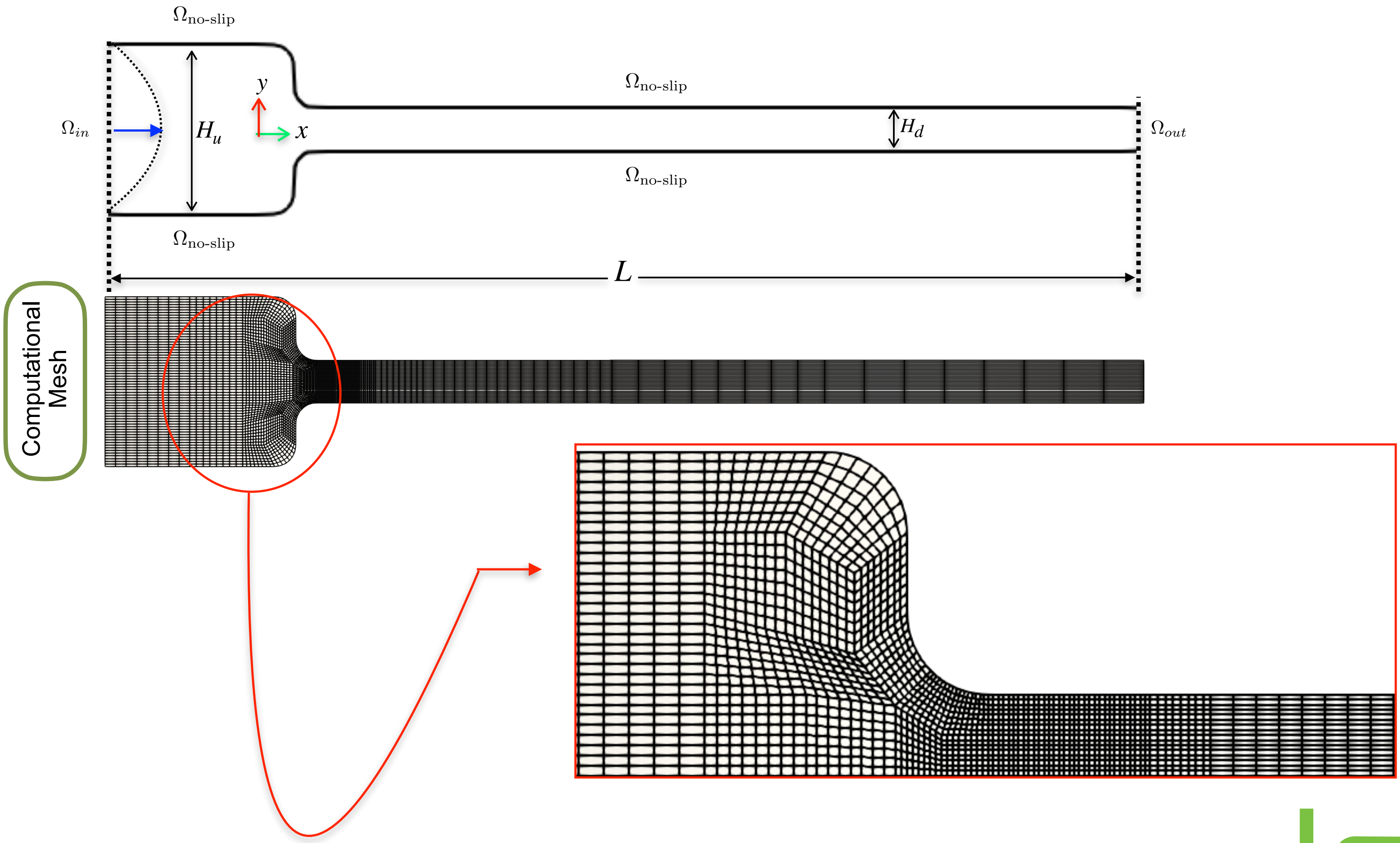
Thixotropic flow in curved contraction domain

- **2D-FEM simulation results for thixotrpic flow- validation of 1D tool**
- **Specifying the “1D-profiles as boundary Data” in 2D simulations for contraction domain**

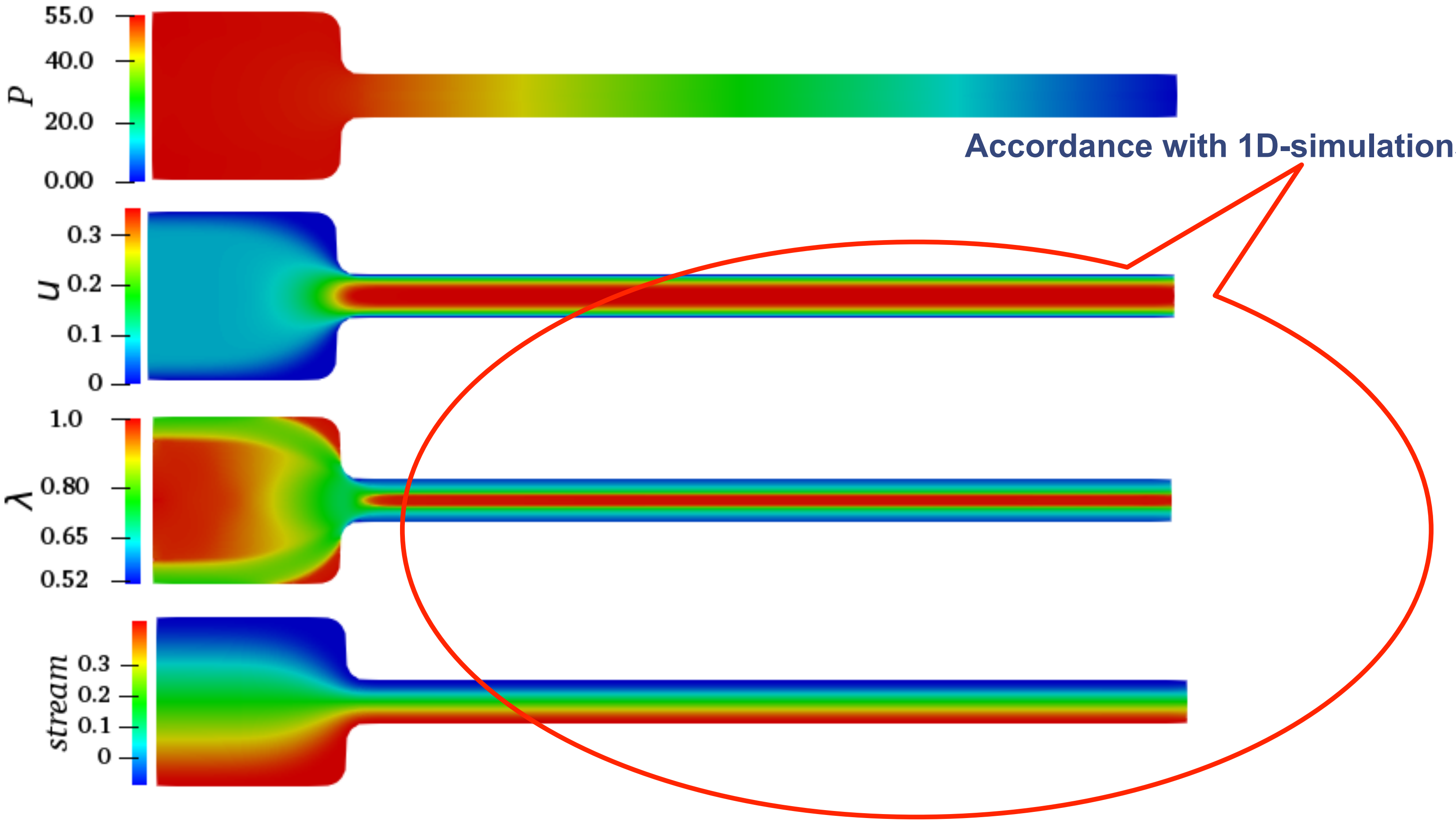


Thixotropic flow in curved contraction domain

- **2D-FEM simulation results for thixotrpic flow- validation of 1D tool**
- **Specifying the "1D-profiles as boundary Data" in 2D simulations for contraction domain**

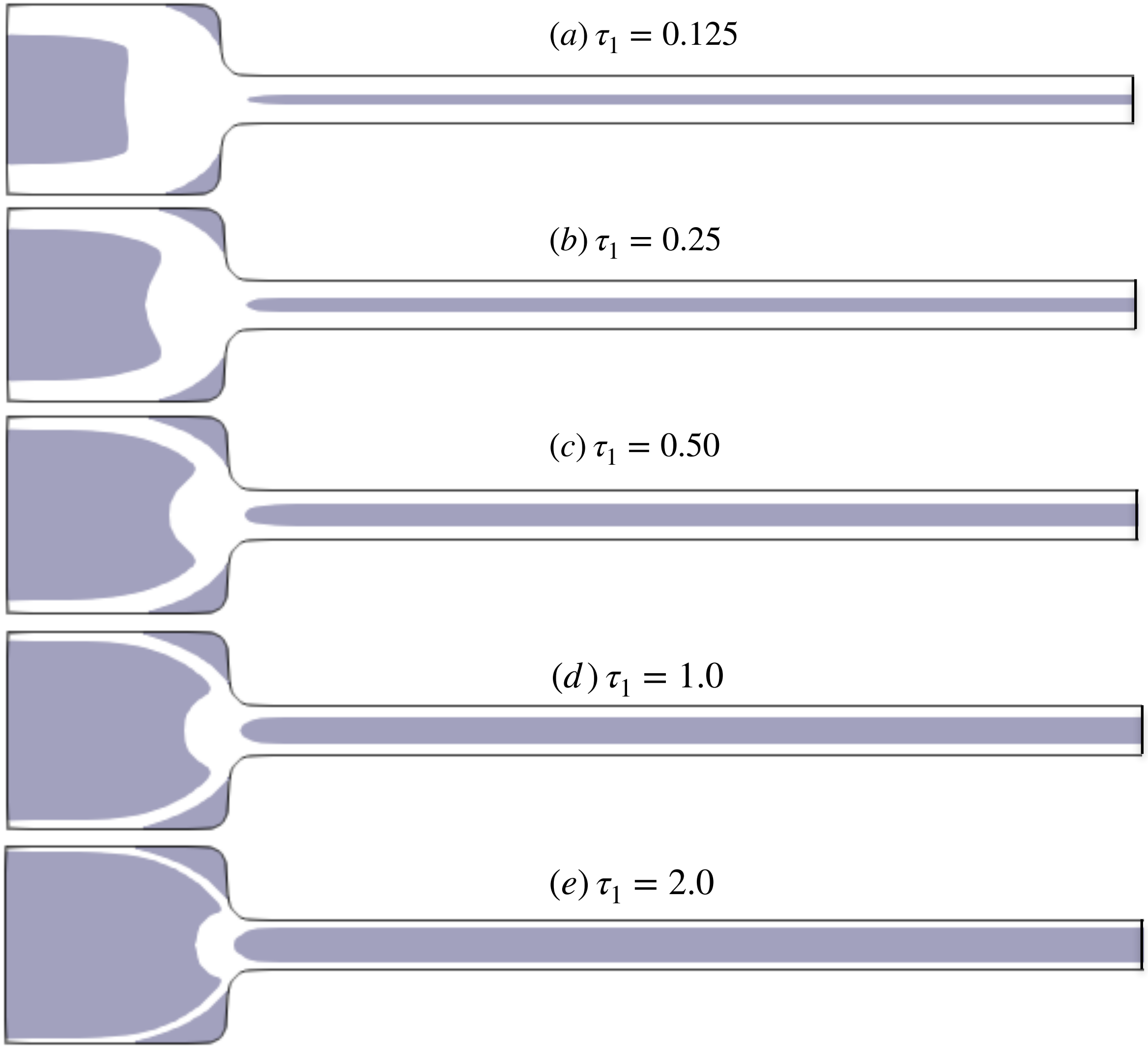


Contraction flow (smoothed)

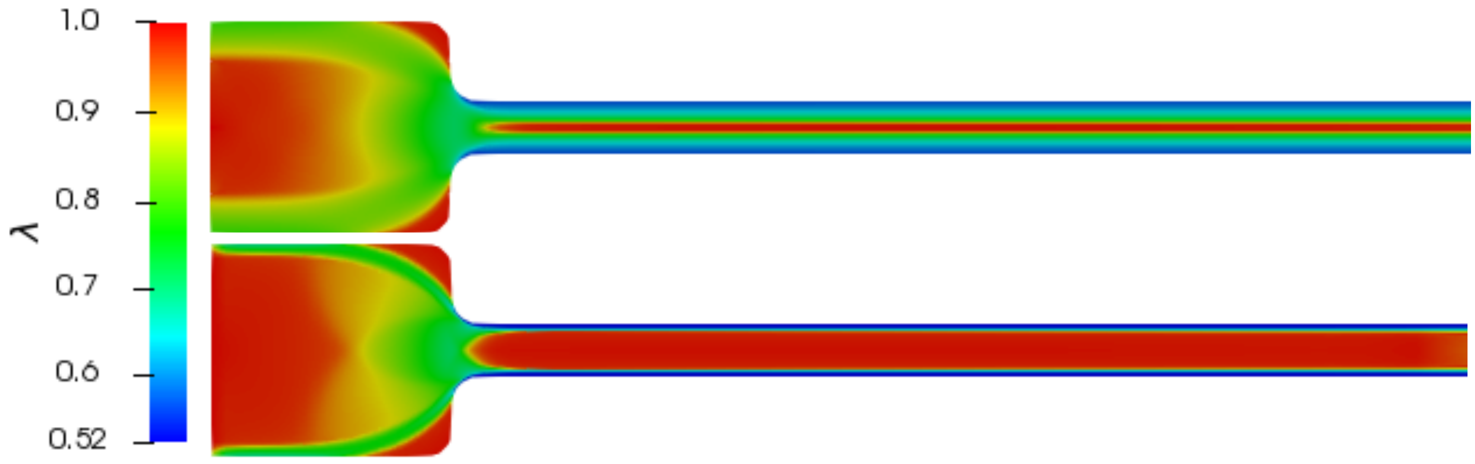


- ✓ **As predicted** (u, λ, p) **solutions**
- ✓ **Structuring level is predicting shape and extent of rigid zones**

- **progressive growth of unyielded zones (shaded)**



- **Material structural behaviour w.r.t. yield stress**



To Do: "Further investigation" regarding material structuring and prediction of plug-zones in the reservoir with sharp/abrupt contraction.

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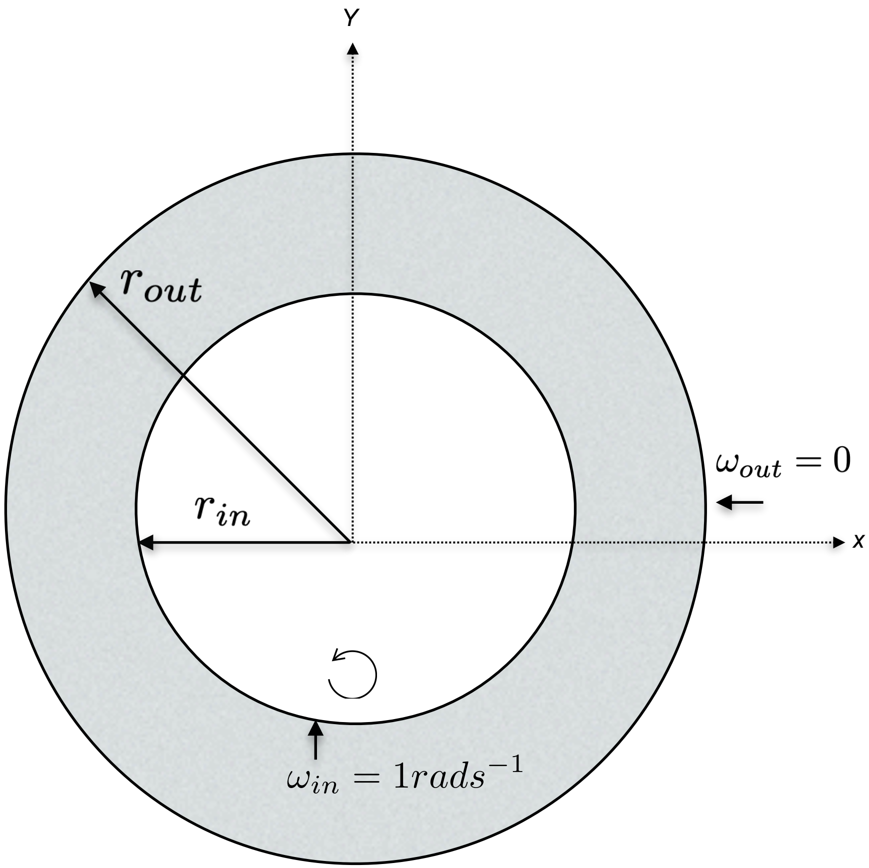
Continuous axial-Flow Couette device:

The material is sheared in the annulus between the interior and exterior cylinder shells of radii r_{in} and r_{out} respectively.

- Concentric cylinders
- Rotating inner cylinder with $\omega_{in} = 1rads^{-1}$
- Stationary outer cylinder
- Vertical flow super-imposed in radial direction
- Torque is computed via:

$$M = -\oint_S (X - X_0) T_{ij} \vec{n} dS$$

where S is the surface of inner cylinder, and X_0 the center of concentric couette device.



Thixotropic flow in Couette

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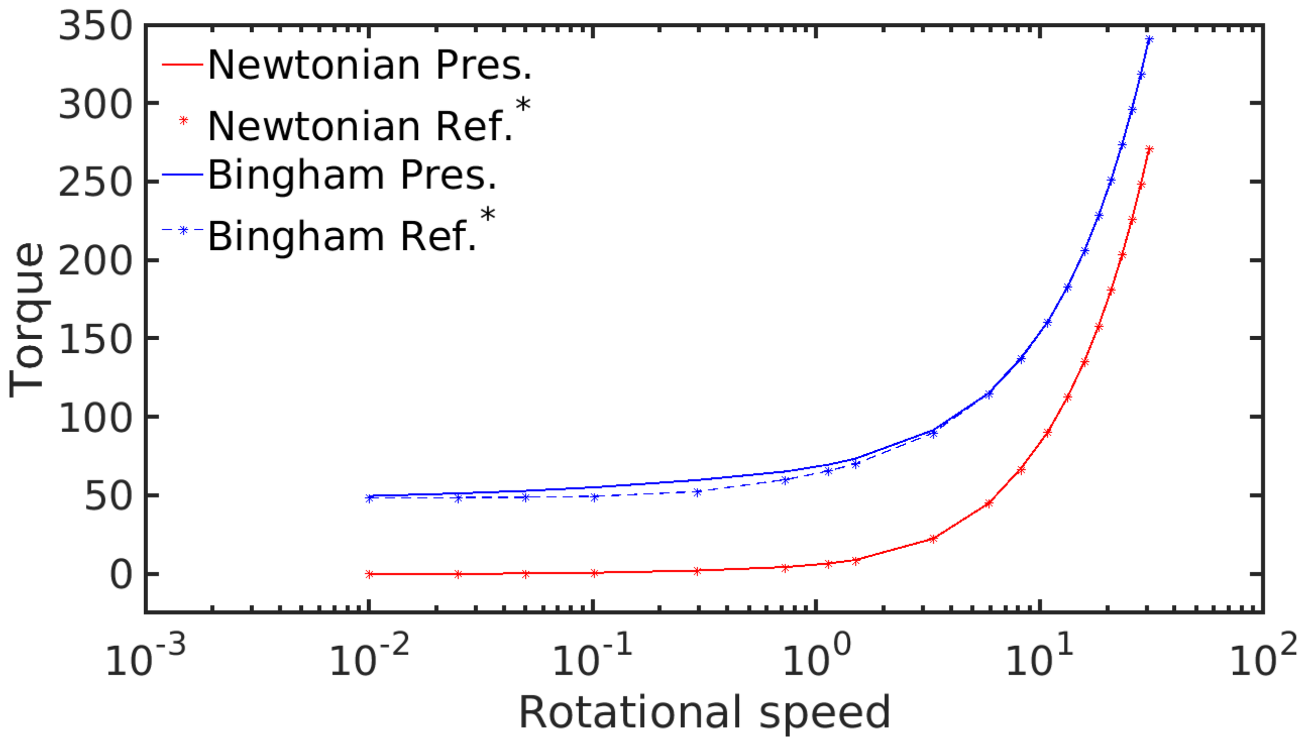
- **"Validation":Results for classical Newtonian and Bingham model are recovered**



Material 1. Newtonian
Material 2. Viscoplastic described by Bingham-Papanastasiou's model
Material 3. Thixotropic described by Houska's Model

Material	η_0	η_1	τ_0	τ_1
Newtonian	1.0	0.0	0.0	0.0
Viscoplastic	1.0	0.0	0.48	0.0
Thixotropic	1.0	0.0	0.0	0.48

*M. Kheiripour Langroudi, S. Turek, A. Ouazzi, G.I. Tardos, An investigation of frictional and collisional powder flows using a unified constitutive equation,Powder Technology, 197, 2010, 91-101.

Thixotropic flow in Couette

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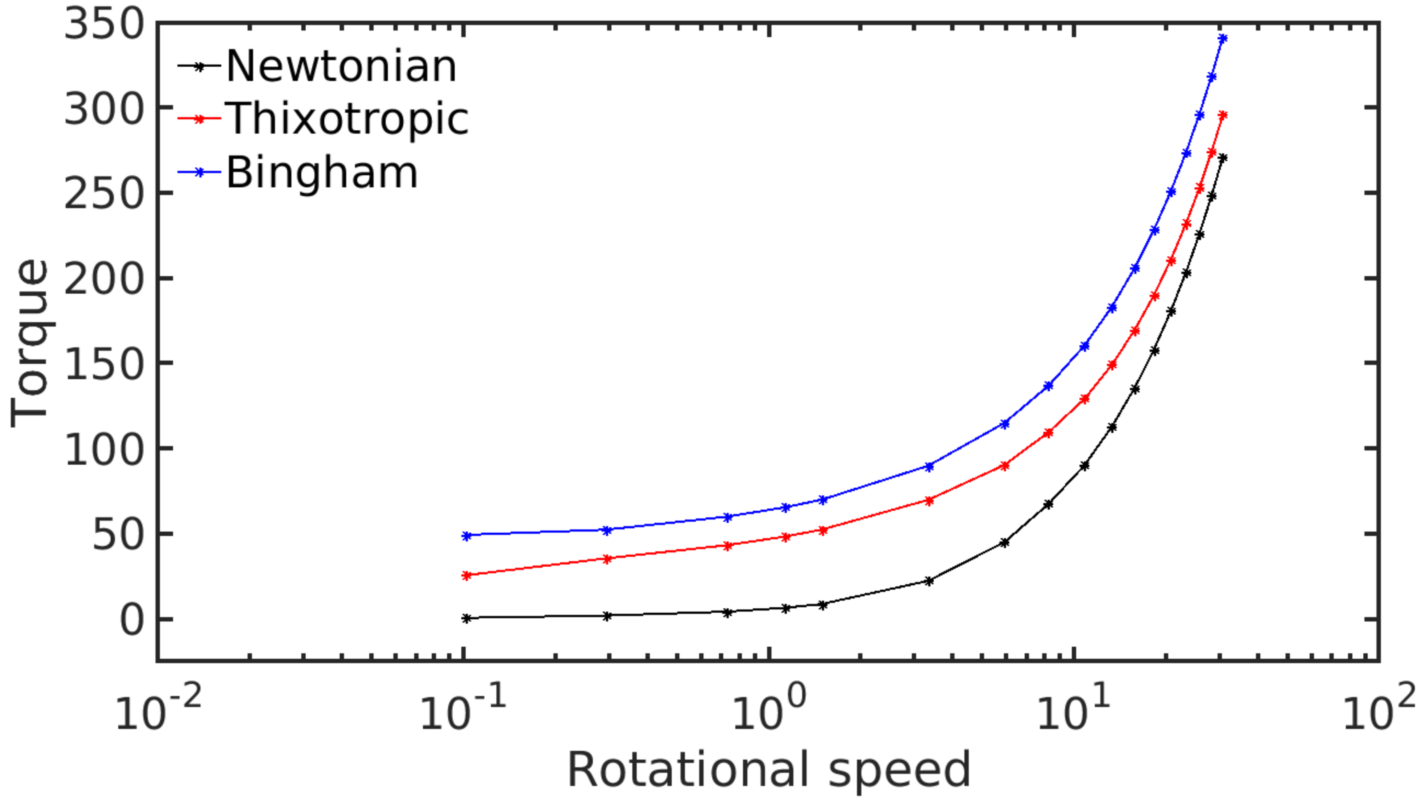
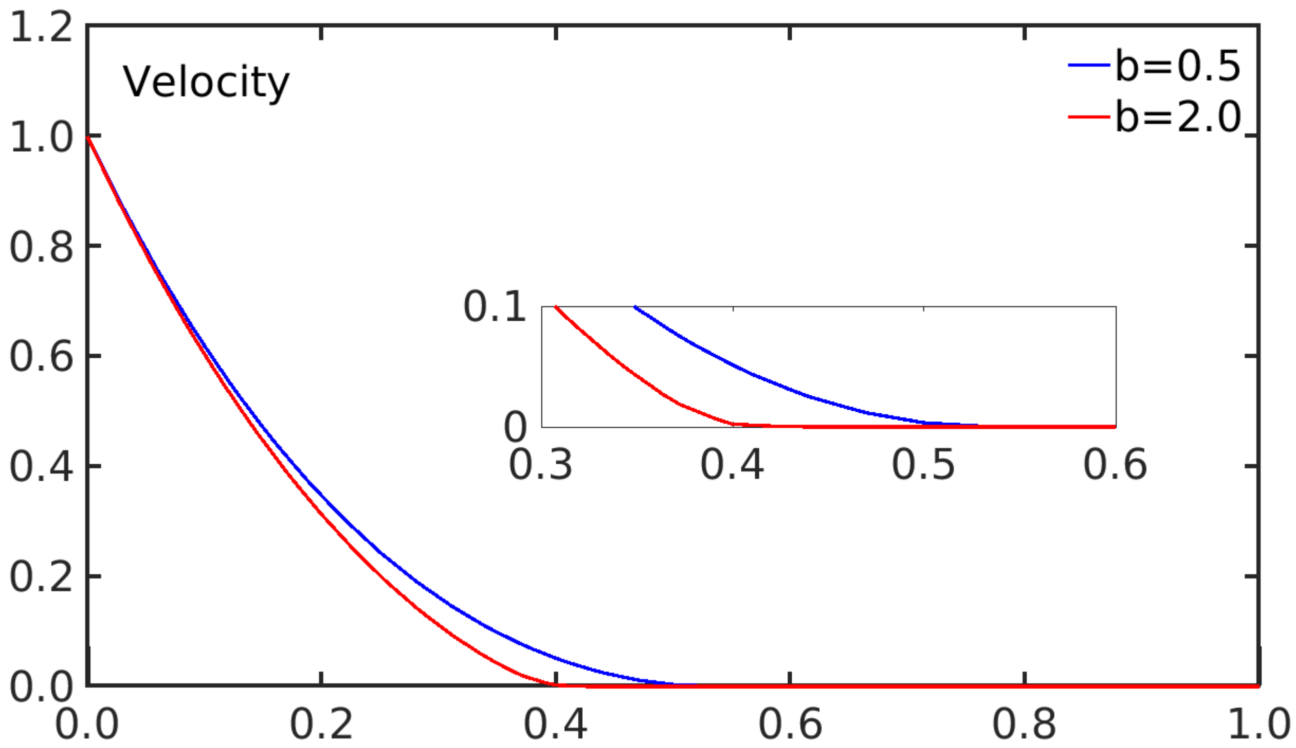
- **Torque results for thixotropic material**



- **Thixotropic material also yields a constant torque at small shear rates but then behaves like a Newtonian/Bingham and, data becomes indistinguishable where rotational speed is** $\geq 10 rads^{-1}$
- **"Observation": inherent thixotropy of Houska's model controls the transitions between solid/liquid and liquid/solid.**

Thixotropic flow in Couette

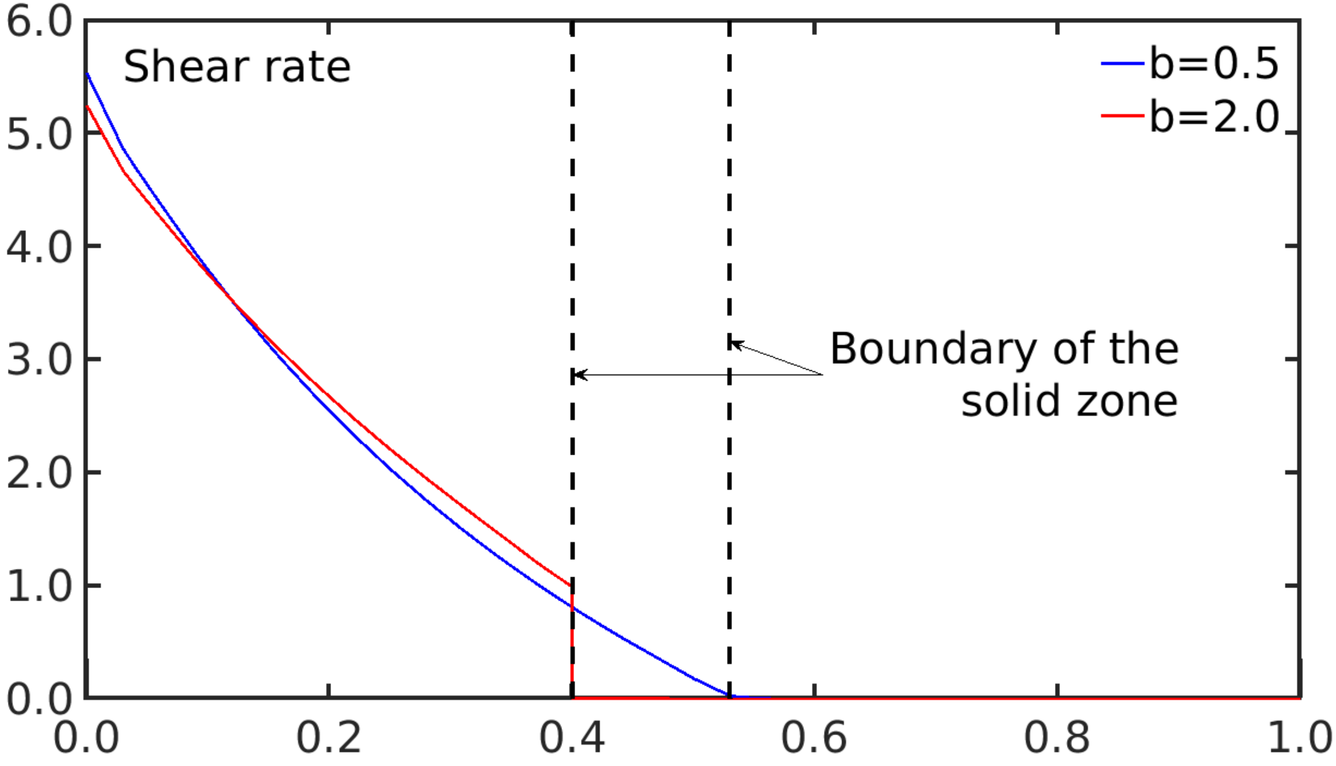
- **Velocity profile at cut-line positions** $c; c \in [0, 2\pi]$ **in a Couette device w.r.t breakdown parameter**



✓ **Localization**
✓ **Shear banding**

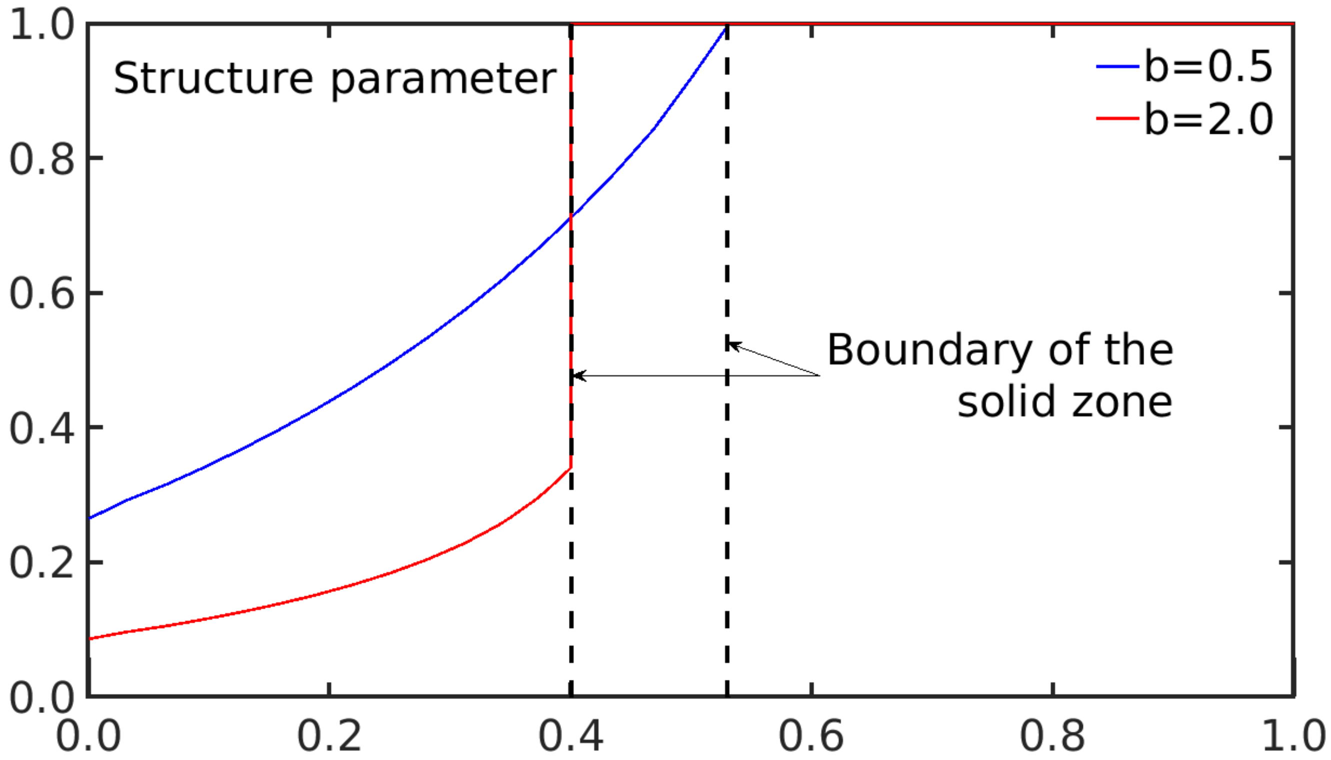
Thixotropic flow in Couette

- **Shear rate at cut-line positions** $c; c \in [0, 2\pi]$ **in a Couette device w.r.t breakdown parameter**



✓ **Smooth and sharp transition are possible**
✓ **Transition point matches with the velocity**

- **Structure parameter at cut-line positions $c; c \in [0, 2\pi]$ in a couette w.r.t. breakdown parameter**



✓ **Transition point matches with the velocity**
✓ **Structuring level is predicting shape and extent of rigid zones**

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Summary

A quasi-Newtonian model for simulating thixotropic materials via time and shear rate dependent extended viscosity is analysed and validated for different prototypical flow simulations

- ✓ **exact solution in simple channel**
- ✓ **complex flow geometry with curved contraction and in Couette devices**

using robust numerical algorithms

- ✓ **Monolithic Finite element method**
- ✓ **Generalized Newton's method w.r.t. singularities with global convergent property**
- ✓ **Fast Multigrid solver with local MPSC smoother**