FEM analysis and monolithic Newton-multigrid solver for thixo-viscoplastic flow problems

N. Begum, A. Ouazzi, S. Turek
Institute of Applied Mathematics, LS III,
TU Dortmund University, Dortmund, Germany

8th European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS Congress 2022
5-9 June 2022, Oslo, Norway

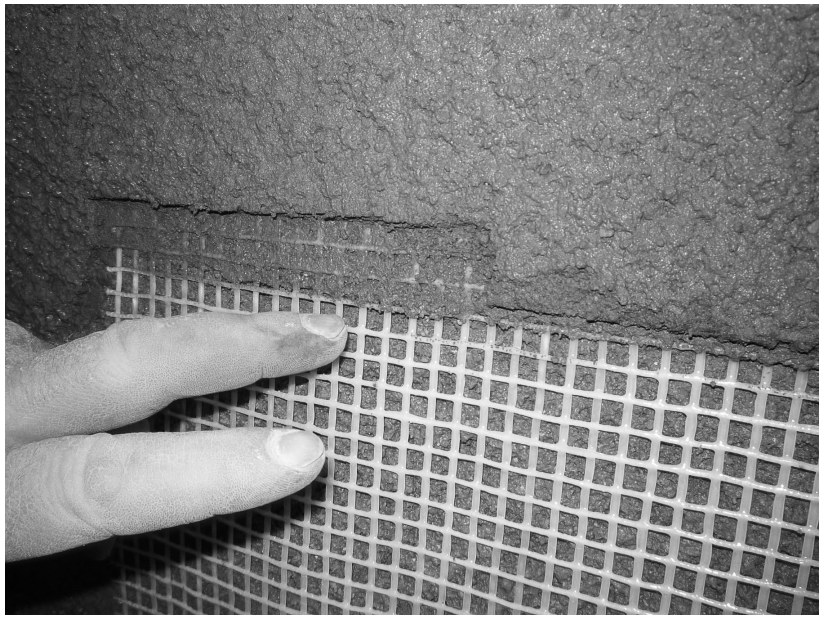
Motivation

Why "Thixotropic materials?

- **Processing of thixotropic materials relevant for industrial applications**

➡ **Lubrication, asphalt, self-compacting concrete…**

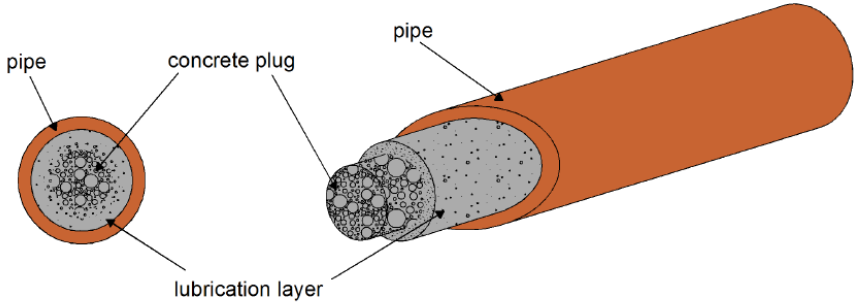
- **Physically fascinating due to improved mechanical properties**



Goal:

- **Modern CFD methods with high accuracy, robustness and efficiency for thixotropic materials**

➡ **Saving time, money and resources**



Investigation of solid/liquid and liquid/solid transitions based on micro-structure

- **Thixotropy means**
  - ➢ combination of two greek words
  - ➢Thixis: shaking/stirring
  - ➢ trepo: turning/changing

- **Thixotropy concept**
  - ➢Based on viscosity
  - ➢Flow induced by time-dependent decrease of viscosity
  - ➢The phenomena is reversible

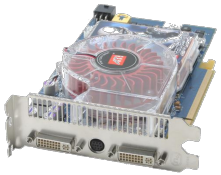
- **Rejuvenation / Breakdown**
  - ➢"Faster" flow: fluid rejuvenates
    Decreases of viscosity with acceleration of the flow

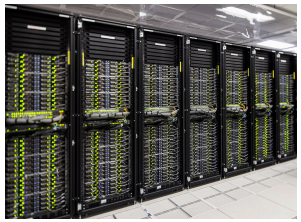
- **Aging / Buildup**
  - ➢At rest or under slow flow: fluid ages
    Increases of the viscosity in time

Realization in FeatFlow

technische universität dortmund

HPC features:
- Moderately parallel
- GPU computing
- Open source





Hardware -oriented Numerics

Numerical features:
- Higher order FEM in space & (semi-) Implicit FD/FEM in time
- Semi-(un)structured meshes with dynamic adaptive grid deformation
- Fictitious Boundary (FBM) methods
- Newton-Multigrid-type solvers

Non-Newtonian flow module:
- generalized Newtonian model (Power-law, Carreau,Houska,…)
- viscoelastic differential model (Giesekus, FENE, Oldroyd,...)

Multiphase flow module (resolved interfaces):
- ·l/l – interface capturing (Level Set)
- ·s/l – interface tracking (FBM)
- ·s/l/l – combination of l/l and s/l

Engineering aspects:
- Geometrical design
- Modulation strategy
- Optimization

Here: FEM-based tools for the accurate simulation of (thixotropic) flow problems, particularly with complex rheology

For details, please visit: www.featflow.de

Governing equations

➡**starting point: Generalized Navier-Stokes equations (+initial and boundary conditions)**

$$\rho\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u}-\nabla\cdot\boldsymbol{\sigma}+\nabla p=\rho f,$$

$$\nabla\cdot\boldsymbol{u}=0,$$

- **velocity- and pressure field $\boldsymbol{u}$ and p**
- **stress tensor $\boldsymbol{\sigma}$**
- **linear material behaviour - Newtonian fluids**

$$\boldsymbol{\sigma}=2\eta_s\boldsymbol{D}(\boldsymbol{u})$$ **: η_s is constant viscosity**

- **non-linear material behaviour- structurally viscous / viscoplastic**

$$\boldsymbol{\sigma}=2\eta_s(D_{\mathrm{II}},p,\Theta,\lambda)\boldsymbol{D}(\boldsymbol{u}),\quad D_{\mathrm{II}}=\mathrm{tr}\left(\frac{1}{2}\boldsymbol{D}(\boldsymbol{u})^2\right)$$

↓

  - **Power-law, Carreau, Bingham, Herschel-Bulkley, Houska, …**

- **structure paremeter λ**

Rheological Models

- **Archetypical thixotropic viscoplastic (TVP) models**

$$\begin{cases} \boldsymbol{\sigma} = 2\,\eta(D_{\text{II}},\lambda)\boldsymbol{D}(\boldsymbol{u}) + \sqrt{2}\,\tau(\lambda)\dfrac{\boldsymbol{D}(\boldsymbol{u})}{\sqrt{D_{\text{II}}}} & \text{if } D_{\text{II}} \neq 0 \\ \sigma_{\text{II}} \leq \tau(\lambda) & \text{if } D_{\text{II}} = 0 \end{cases}$$

- **Relations between rheological parameters and structural parameter**

	$\eta(D_{\text{II}},\lambda)$	$\tau(\lambda)$
Worrall and Tulliani[1]	$\lambda\eta_0$	τ_0
Coussot *et al.*[2]	$\lambda^a\eta_0$	–
Houska[3]	$(\eta_0+\eta_1\lambda)\,D_{\text{II}}^{\frac{(n-1)}{2}}$	$(\tau_0+\tau_1\lambda)$
Mujumbar *et al.*[4]	$(\eta_0+\eta_1\lambda)\,D_{\text{II}}^{\frac{(n-1)}{2}}$	$\lambda^{a+1}\mathrm{G}_0{\Lambda_c}^*$
Burgos *et al.*[5]	η_0	$\lambda\tau_0$
Dullaert & Mewis[6]	$\lambda\eta_0$	$\lambda\mathrm{G}_0\left(\lambda D_{\text{II}}^{\frac{1}{2}}\right)\Lambda_c^*$

*Λ_c is a constant/variable elastic strain.

Rheological Models

- **General format of evolution equation for structural parameter:**

$$\frac{\partial \lambda}{\partial t} + u \cdot \nabla \lambda = F_{buildup} - F_{breakdown}$$

- **Expressions for different thixotropic models:**

	$F_{buildup}$	$F_{breakdown}$
Worrall and Tulliani[1]	$c_1(1-\lambda)D_{II}^{\frac{1}{2}}$	$c_2\lambda D_{II}^{\frac{1}{2}}$
Coussot *et al.*[2]	c_1	$c_2\lambda D_{II}^{\frac{1}{2}}$
Houska[3]	$c_1(1-\lambda)$	$c_2\lambda^m D_{II}^{\frac{1}{2}}$
Mujumbar *et al.*[4]	$c_1(1-\lambda)$	$c_2\lambda D_{II}^{\frac{1}{2}}$
Burgos *et al.*[5]	$c_1(1-\lambda)$	$c_2\lambda D_{II}^{\frac{1}{2}}\exp(aD_{II}^{\frac{1}{2}})$
Dullaert & Mewis[6]	$(c_1 + c_3 D_{II}^{\frac{1}{2}})(1-\lambda)t^{-b}$	$c_2\lambda D_{II}^{\frac{1}{2}}t^{-b}$

Thixo-viscoplastic model

- **Viscoplastic (VP) flow**

$$\begin{cases} \boldsymbol{\sigma} = 2\,\eta_0 \boldsymbol{D}(\boldsymbol{u}) + \sqrt{2}\,\tau_0 \dfrac{\boldsymbol{D}(\boldsymbol{u})}{\sqrt{D_{\text{II}}}} & \text{if } D_{\text{II}} \neq 0 \\ \sigma_{\text{II}} \leq \tau_0 & \text{if } D_{\text{II}} = 0 \end{cases}$$

- **Thixo-viscoplastic (TVP) flow**

$$\begin{cases} \boldsymbol{\sigma} = 2\,\eta(D_{\text{II}}, \lambda) \boldsymbol{D}(\boldsymbol{u}) + \sqrt{2}\,\tau(\lambda) \dfrac{\boldsymbol{D}(\boldsymbol{u})}{\sqrt{D_{\text{II}}}} & \text{if } D_{\text{II}} \neq 0 \\ \sigma_{\text{II}} \leq \tau(\lambda) & \text{if } D_{\text{II}} = 0 \end{cases}$$

- **Micro-structure equation**

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla \right) \lambda + \mathcal{M}(D_{\text{II}}, \lambda) = f_\lambda$$

$$\mathcal{M} := \mathcal{G} - \mathcal{F}$$

Quasi-Newtonian TVP model

Viscosity model for TVP flow

Classical approximations

$$\begin{cases} I. & \dfrac{1}{\sqrt{D_{\mathrm{II},r}}} := \dfrac{1}{\sqrt{(D_{\mathrm{II}} + (k^{-1})^2)}} \\ II. & \dfrac{1}{\sqrt{D_{\mathrm{II},r}}} := \dfrac{1}{\sqrt{D_{\mathrm{II}}}} \left(1 - e^{-k\sqrt{D_{\mathrm{II}}}}\right) \end{cases}$$

Extended viscosity defined on all domain

$$\mu(D_{\mathrm{II},r}, \lambda) = \eta(D_{\mathrm{II}}, \lambda) + \tau(D_{\mathrm{II}}, \lambda) \frac{\sqrt{2}}{2} \frac{1}{\sqrt{D_{\mathrm{II},r}}}$$

Full set of equations

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) \boldsymbol{u} - \nabla \cdot \left(2\mu(\boldsymbol{D}_{\mathrm{II},r}, \lambda)\boldsymbol{D}(\boldsymbol{u})\right) + \nabla p = \boldsymbol{f_u} & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \left(\dfrac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) \lambda + \mathcal{M}(D_{\mathrm{II}}, \lambda) = f_\lambda & \text{in } \Omega \end{cases}$$

Continuous thixo-viscoplastic problem

Sobolev spaces and notations

- $H^0(\Omega) := L^2(\Omega) = \left\{ v \,|\, \int_\Omega |v|^2 d\Omega < \infty \right\}$

  $L^2_0(\Omega) = \left\{ v \in L^2(\Omega) \,|\, \int_\Omega v d\Omega = 0 \right\}, \quad \|v\|_{0,\Omega} = \left(\int_\Omega |v|^2 d\Omega \right)^{\frac{1}{2}}$

- $H^1(\Omega) = \left\{ v \in L^2(\Omega) \,|\, \nabla v \in L^2(\Omega) \right\}, \quad \|v\|_{1,\Omega} = \left(\|v\|^2_{0,\Omega} + \|\nabla v\|^2_{0,\Omega} \right)^{\frac{1}{2}}$

  $H^1_\Gamma(\Omega) = \left\{ v \in H^1(\Omega) \,|\, v_{|\Gamma} = 0, \Gamma \subset \partial\Omega \right\}$

  $H^1_0(\Omega) = \left\{ v \in H^1(\Omega) \,|\, v_{|\partial\Omega} = 0 \right\}$

- $H(\boldsymbol{u}, \Omega) = \left\{ v \in L^2(\Omega), \boldsymbol{u} \cdot \nabla v \in L^2(\Omega) \right\}, \quad \|v\|_{1,\boldsymbol{u}} = \left(\|v\|^2_{0,\Omega} + \|\boldsymbol{u} \cdot \nabla v\|^2_{0,\Omega} \right)^{\frac{1}{2}}$

  $\Gamma^- = \{ x \in \Gamma \subset \partial\Omega \,|\, \boldsymbol{u} \cdot \boldsymbol{n} < 0 \}, \quad \Gamma^+ = \{ x \in \Gamma \subset \partial\Omega \,|\, \boldsymbol{u} \cdot \boldsymbol{n} > 0 \}$

  $< v, w >_\pm = \int_{\Gamma^\pm} |\boldsymbol{u} \cdot \boldsymbol{n}| v w ds, \quad < v >_\pm = < v, v >^{\frac{1}{2}}_\pm$

- **Flow variables** $(\lambda, \boldsymbol{u}, p)$
  - **Set** $\mathbb{T} := H^1_{\Gamma^-}(\Omega), \mathbb{V} := \left[H^1_0(\Omega)\right]^2, \mathbb{Q} := L^2(\Omega), \mathbb{W} := \mathbb{T} \times \mathbb{V}$
  - **Set** $\tilde{\boldsymbol{u}} := (\lambda, \boldsymbol{u})$
  - **Find** $(\lambda, \boldsymbol{u}, p) \in \mathbb{T} \times \mathbb{V} \times \mathbb{Q}$ **s.t.**

$$\left\langle \mathcal{K}(\lambda, \boldsymbol{u}, p), (\xi, \boldsymbol{v}, q) \right\rangle = \left\langle \mathcal{L}, (\xi, \boldsymbol{v}, q) \right\rangle, \quad \forall (\xi, \boldsymbol{v}, q) \in \mathbb{T} \times \mathbb{V} \times \mathbb{Q}$$

$$\mathcal{K} = \begin{bmatrix} \mathcal{A}_{\tilde{\boldsymbol{u}}} & \mathcal{B}^{\mathrm{T}} \\ \mathcal{B} & 0 \end{bmatrix}$$

  - **Compatibily constraints**

$$\sup_{\boldsymbol{v} \in \mathbb{V}} \frac{\langle \mathcal{B}\boldsymbol{v}, q \rangle}{\|\boldsymbol{v}\|_{\mathbb{V}}} \geq \beta \|q\|_{\mathbb{Q}/Ker\mathcal{B}^{\mathrm{T}}}, \quad \forall q \in \mathbb{Q}$$

Weak Formulation

$$\Big\langle \mathcal{K}(\lambda, \boldsymbol{u}, p), (\xi, \boldsymbol{v}, q) \Big\rangle = \Big\langle \mathcal{L}, (\xi, \boldsymbol{v}, q) \Big\rangle, \quad \forall (\xi, \boldsymbol{v}, q) \in \mathbb{T} \times \mathbb{V} \times \mathbb{Q}$$

Operators

$$\Big\langle \mathcal{K}(\lambda, \boldsymbol{u}, p), (\xi, \boldsymbol{v}, q) \Big\rangle = a_\lambda(\tilde{\boldsymbol{u}})(\lambda, \xi) + a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - b(\boldsymbol{u}, q)$$

$$\mathcal{A}_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}})\tilde{\boldsymbol{u}} = \mathcal{A}_\lambda(\tilde{\boldsymbol{u}})\lambda + \mathcal{A}_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})\boldsymbol{u}$$

$$\Big\langle \mathcal{A}_\lambda(\tilde{\boldsymbol{u}})\lambda, \xi \Big\rangle = a_\lambda(\tilde{\boldsymbol{u}})(\lambda, \xi) = \Big(\mathcal{M}(D_{\mathrm{II}}, \lambda), \xi\Big) + \Big(\boldsymbol{u} \cdot \nabla\lambda, \xi\Big)$$

$$\Big\langle \mathcal{A}_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})\boldsymbol{u}, \boldsymbol{v} \Big\rangle = a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\boldsymbol{u}, \boldsymbol{v}) = \Big(2\mu(D_{\mathrm{II},r}, \lambda)\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v})\Big) + \Big(\boldsymbol{u} \cdot \nabla\boldsymbol{u}, \boldsymbol{v}\Big)$$

$$\Big\langle \mathcal{B}\boldsymbol{v},\ \mathrm{q} \Big\rangle = b(\boldsymbol{v}, q) = -\Big(\nabla \cdot \boldsymbol{v}, q\Big)$$

$$\Big\langle \mathcal{L}, (\xi, \boldsymbol{u}, q) \Big\rangle = \Big(f_\lambda, \xi\Big) + \Big(\boldsymbol{f}_{\boldsymbol{u}}, \boldsymbol{v}\Big)$$

Wellposedness of thixo-viscoplastic problem !

Coerciveness

- $a_\lambda(\tilde{\boldsymbol{v}})(\xi,\xi) \geq \alpha_0 \|\xi\|_{0,\Omega}^2 + \frac{1}{2}\left\{\langle\xi\rangle_+^2 + \langle\xi\rangle_-^2\right\} \qquad \forall\, \xi \in \mathbb{T}$
- $a_{\boldsymbol{u}}(\tilde{\boldsymbol{v}})(\boldsymbol{v},\boldsymbol{v}) \geq \alpha_1 \|\boldsymbol{v}\|_{1,\Omega}^2 \qquad \forall\, \boldsymbol{v} \in \mathbb{V}$

Continuity

- $a_\lambda(\tilde{\boldsymbol{u}})(\lambda,\xi) \leq |\boldsymbol{u}|_{0,\infty}\|\lambda\|_{0,\Omega}\|\xi\|_{1,\Omega} + K_1\|\lambda\|_{0,\Omega}\|\xi\|_{0,\Omega} + \langle\lambda,\xi\rangle_+ \qquad \forall\, \lambda,\xi \in \mathbb{T}$
  $\leq \mathcal{C}_0\|\lambda\|_{1,\Omega}\|\xi\|_{1,\Omega} \qquad \forall\, \lambda,\xi \in \mathbb{T}$
- $a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\boldsymbol{u},\boldsymbol{v}) \leq (\mu_{\mathrm{T}} + \tau_{\mathrm{T}})\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega} \qquad \forall\, \boldsymbol{u},\boldsymbol{v} \in \mathbb{V}$
  $\leq \mathcal{C}_1\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{v}\|_{1,\Omega} \qquad \forall\, \boldsymbol{u},\boldsymbol{v} \in \mathbb{V}$

Coercivity in weaker norm for the microstructure !

➢ Monotonicity

- $$a_\lambda(\tilde{\boldsymbol{u}})(\lambda, \lambda-\xi) - a_\lambda(\tilde{\boldsymbol{v}})(\xi, \lambda-\xi) \geq \Big(\mathcal{M}(D_{\mathrm{II}}(\boldsymbol{u}), \lambda) - \mathcal{M}(D_{\mathrm{II}}(\boldsymbol{v}), \xi), \lambda-\xi\Big)$$
$$\geq \alpha_0 \|\lambda-\xi\|_{0,\Omega} \qquad \forall\, \lambda, \xi \in \mathbb{T}$$

- $$a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{v}) - a_{\boldsymbol{v}}(\tilde{\boldsymbol{v}})(\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}) \geq \mu_{\mathrm{T}} |\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{v})|^2$$
$$+\frac{\sqrt{2}}{2} \frac{\tau_{\mathrm{T}}}{\sqrt{D_{\mathrm{II},r}}} \Bigg\{ \underbrace{|\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{v})|^2}_{\mathrm{I}} - \frac{\sqrt{2}}{2} \underbrace{\frac{|\boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{v})|}{\sqrt{D_{\mathrm{II},r}}} \Big(\boldsymbol{D}(\boldsymbol{v}), \boldsymbol{D}(\boldsymbol{u}) - \boldsymbol{D}(\boldsymbol{v})\Big)}_{\mathrm{II}} \Bigg\}$$
$$\geq \alpha_1 \|\boldsymbol{u}-\boldsymbol{v}\|_{1,\Omega} \quad (\mathrm{I} \geq \mathrm{II}) \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}$$

Thixoviscoplastic problem has a unique solution

Approximated TVP problem

- **Conforming approximations**

$$\mathbb{T}_h \subset \mathbb{T},\ \mathbb{V}_h \subset \mathbb{V},\ \mathbb{Q}_h \subset \mathbb{Q},\ \mathbb{W}_h := \mathbb{T}_h \times \mathbb{V}_h$$

$$\mathcal{A}_{\tilde{\boldsymbol{u}}_h} = \mathcal{A}_{\tilde{\boldsymbol{u}}},\ \mathcal{B}_h = \mathcal{B}$$

- **Discrete inf-sup condition**

$$\sup_{\boldsymbol{v}_h \in \mathbb{V}_h} \frac{\langle \mathcal{B}_h \boldsymbol{v}_h, q_h \rangle}{\|\boldsymbol{v}_h\|_{\mathbb{V}}} \geq \beta_h \left|q_h\right|_{\mathbb{Q}/Ker\mathcal{B}_h^{\mathrm{T}}}, \quad \forall q_h \in \mathbb{Q}_h$$

➢ **Find** $(\lambda_h, \boldsymbol{u}_h, p_h) \in \mathbb{T}_h \times \mathbb{V}_h \times \mathbb{Q}_h$ **s.t**

$$\left\langle \mathcal{K}(\lambda_h, \boldsymbol{u}_h, p_h), (\xi_h, \boldsymbol{v}_h, q_h) \right\rangle = \left\langle \mathcal{L}, (\xi_h, \boldsymbol{v}_h, q_h) \right\rangle, \ \forall (\xi_h, \boldsymbol{v}_h, q_h) \in \mathbb{T}_h \times \mathbb{V}_h \times \mathbb{Q}_h$$

Order of convergence !

Coercivity in a weaker norm for microstructure

$$\frac{\alpha_0}{2}\|\lambda-\lambda_h\|_{0,\Omega}^2+\frac{1}{4}\langle\lambda-\lambda_h\rangle^2\leq\frac{1}{\alpha_0}\left(1+K_1+|\boldsymbol{u}|_{0,\infty}\right)\inf_{\xi_h\in\mathbb{T}_h}\|\lambda-\xi_h\|_{1,\Omega}^2$$

$$\|\lambda-\lambda_h\|_{0,\Omega}\equiv h^r|\lambda|_{r+1,\Omega}$$

Regularization of TVP model

$$\begin{aligned}\|\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{u}_h)\|_{0,\Omega}\leq&\sqrt{2}\mu_{\mathrm{T}}^{-1}\inf_{q_h\in\mathbb{Q}_h}\|p-q_h\|_{0,\Omega}\\&+\left(3+3\tau_{\mathrm{T}}^2k^2\mu_{\mathrm{T}}^{-2}\right)^{1/2}\inf_{\boldsymbol{v}\in\mathbb{V}_h}\|\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{v}_h)\|_{0,\Omega}\end{aligned}$$

$$\begin{aligned}\|\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{u}_h)\|_{0,\Omega}\leq&\sqrt{2}\mu_{\mathrm{T}}^{-1}\inf_{q_h\in\mathbb{Q}_h}\|p-q_h\|_{0,\Omega}\\&+\sqrt{2}\inf_{\boldsymbol{v}\in\mathbb{V}_h}\|\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{v}_h)\|_{0,\Omega}\\&+\left(2\tau_{\mathrm{T}}\mu_{\mathrm{T}}^{-1}\right)^{1/2}\inf_{\boldsymbol{v}\in\mathbb{V}_h}\|\boldsymbol{D}(\boldsymbol{u})-\boldsymbol{D}(\boldsymbol{v}_h)\|_{0,\Omega}^{\frac{1}{2}}\end{aligned}$$

Coercivity in a stronger norm for the microstructure eq. !
FEM choices to counterbalance the regularisation/sub-optimality !

Numerical challenges

- **Discretization of TVP model have to address the following challenges**
  - ➢ **Stable FEM spaces**
  - ➢ **Coercivity in a weaker norm**
  - ➢ **Regularization**
  - ➢ **Convection domination**
  - ➢ **Monotonicity preserving and satisfying strongly DMP**
- **TVP Newton-multigrid Solver have to deal with**
  - ➢ **Different source of nonlinearities**
  - ➢ **Strong coupling of equations**
  - ➢ **Robustness and efficiency**

Higher order conforming FEM

- ✓ **The family of conforming FEM** $Q_r/Q_r/P_{r-1}^{\text{disc}},\ r \geq 2$ **for** $(\lambda, \boldsymbol{u}, p)$ **with stabilization**
  - ➢ **Inf-sup conditions is satisfied**
  - ➢ **Discontinuous pressure**
    - ➢ **Practical w.r.t. monolithic approach**
    - ➢ **Element-wise mass conservation**
- ✓ **Highly consistent and symmetric stabilization**

$$j_\lambda(\lambda_h, \xi_h) = \sum_{e \in \mathcal{E}_h} h^2 \gamma_\lambda \int_e [\nabla \lambda_h] : [\nabla \xi_h] \, d\Omega, \quad j_{\lambda,l} := j_\lambda \text{ for } \gamma_\lambda = \text{cst.}$$

$$j_{\boldsymbol{u}}(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{e \in \mathcal{E}_h} h^2 \gamma_{\boldsymbol{u}} \int_e [\nabla \boldsymbol{u}_h] : [\nabla \boldsymbol{v}_h] \, d\Omega, \quad j_{\boldsymbol{u},l} := j_{\boldsymbol{u}} \text{ for } \gamma_{\boldsymbol{u}} = \text{cst.}$$

  - ➢ **Coercivity in a strong norm for the microstructure eq.**
  - ➢ **Monotonocity preserving and satisfying strong DMP**
  - ➢ **Efficient and robust w.r.t. multigrid solver**

- **Coercivity in a Stronger norm for microstructure**

$$|\!|\!|\lambda|\!|\!| = \left(\alpha_0 \|\lambda\|_{0,\Omega} + \frac{1}{2}\langle\lambda\rangle^2 + j_{\lambda,l}(\lambda,\lambda)\right)^{\frac{1}{2}}$$

$$|\!|\!|\lambda - \lambda_h|\!|\!|^2 \leq \frac{1}{\alpha_0}\left(1 + K_1 + |\boldsymbol{u}|_{0,\infty}\right) \inf_{\xi_h \in \mathbb{T}_h} \|\lambda - \xi_h\|^2_{1,\Omega} + \frac{1}{2} \inf_{\xi_h \in \mathbb{T}_h} j_{\lambda,l}(\lambda_h - \xi_h, \lambda_h - \xi_h)$$

$$\|\lambda - \lambda_h\|_{0,\Omega} \equiv h^{r+\frac{1}{2}} |\lambda|_{r+1,\Omega}$$

- **Regularisation for TVP model**

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,\Omega} \equiv \sqrt{2}\mu_{\mathrm{T}}^{-1}\mathcal{O}(h^r) + \left(3 + 3\tau_{\mathrm{T}}^2\mu_{\mathrm{T}}^{-2}\right)^{1/2}\mathcal{O}(kh^r)$$

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,\Omega} \equiv \sqrt{2}\mu_{\mathrm{T}}^{-1}\mathcal{O}(h^r) + \sqrt{2}\mathcal{O}(h^r) + \left(2\tau_{\mathrm{T}}\mu_{\mathrm{T}}^{-1}\right)^{1/2}\mathcal{O}(h^{\frac{r}{2}})$$

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,\Omega} \equiv \mathcal{O}(k^{-1}) \implies h \leq \mathcal{O}(k^{-\frac{2}{r}})$$

Stabilization recover the optimal order of convergence
Higher order FEM counterbalance regularization

Generalized discrete Newton's method

Let $\{\varphi_i, i = 1, 2, \ldots, \mathbf{dim}\mathbb{W}_h\}$ and $\{\psi_i,\ i = 1, \ldots, \mathbf{dim}\mathbb{Q}_h\}$ denote the basis of spaces $\mathbb{W}_h$ and $\mathbb{Q}_h$, respectively. The Solution $\mathcal{U} = (\lambda, \boldsymbol{u}, p) = (\tilde{\boldsymbol{u}}, p) \in \mathbb{W}_h \times \mathbb{Q}_h$

$$\mathcal{U} = \sum_{i=1}^{\mathbf{dim}\mathbb{W}_h} \tilde{\boldsymbol{u}}_i \varphi_i + \sum_{i=1}^{\mathbf{dim}\mathbb{Q}_h} p_i \psi_i$$

The residuals of discrete TVP problem $\mathcal{R}(\mathcal{U}) \in \mathbb{R}^{\mathbf{dim}\mathbb{W}_h + \mathbf{dim}\mathbb{Q}_h}$

$$\mathcal{R}(\mathcal{U}) = (\mathcal{R}_\lambda(\lambda, \boldsymbol{u}), \mathcal{R}_{\boldsymbol{u}}(\lambda, \boldsymbol{u}, p), \mathcal{R}_p(\boldsymbol{u})) = (\mathcal{R}_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}}, p), \mathcal{R}_p(\boldsymbol{u}))$$

- Update of the nonlinear iteration with the correction $\delta\mathcal{U}$

$$\mathcal{U}^{l+1} = \mathcal{U}^l + \delta\mathcal{U}$$

- The linearization of the residual provides

$$\mathcal{R}(\mathcal{U}^{l+1}) = \mathcal{R}(\mathcal{U}^l + \delta\mathcal{U}) \simeq \mathcal{R}(\mathcal{U}^l) + \left(\frac{\partial \mathcal{R}(\mathcal{U}^l)}{\partial \mathcal{U}}\right) \delta\mathcal{U}$$

- The Newton's method assuming invertible Jacobian

$$\mathcal{U}^{l+1} = \mathcal{U}^l - \omega_l \left(\frac{\partial \mathcal{R}(\mathcal{U}^l)}{\partial \mathcal{U}}\right)^{-1} \mathcal{R}(\mathcal{U}^l)$$

Adaptive discrete Newton's method

The Jacobian matrix in discrete Newton is calculated using the finite difference

$$\left[\frac{\partial \mathcal{R}(\mathcal{U}^l)}{\partial \mathcal{U}}\right]_{ij} \approx \frac{\mathcal{R}_i(\mathcal{U}^l + \varepsilon_l \boldsymbol{e}_j) - \mathcal{R}_i(\mathcal{U}^l - \varepsilon_l \boldsymbol{e}_j)}{2\varepsilon_l}$$

➢ **Damping of the updated solution,** $\omega_l \in (0,1)$ **, s.t.**

$$\|\mathcal{R}(\mathcal{U}^{l+1})\| \leq \|\mathcal{R}(\mathcal{U}^l)\|$$

! Not sufficient for TVP problem

➢ **Operator-adaptive splitting**

$$\left(\frac{\partial \mathcal{R}(\mathcal{U}^l)}{\partial\, \mathcal{U}}\right) = \left(\frac{\partial \tilde{\mathcal{R}}(\mathcal{U})}{\partial \mathcal{U}}\right) + \delta_l \left(\frac{\partial \hat{\mathcal{R}}(\mathcal{U})}{\partial \mathcal{U}}\right)$$

$$\delta_{l+1} = f(r_l)\delta_l,\ r_l := \|\mathcal{R}(\mathcal{U}^l)\|/\|\mathcal{R}(\mathcal{U}^{l-1})\|,\ f(r_l) = 0.2 + 4/(0.7 + e^{1.5 r_l})$$

! Requires a priori analysis of the Jacobian

➢ **Adaptive step-length control**

$$\varepsilon_{l+1} = g(r_l)\varepsilon_l,\ g(r_l) = 1/f(r_l)$$

! Robust

Convergence analysis of the method !

Monolithic Multigrid solver

The discretization and linearization of TVP leads to systems of following linearized discrete TVP saddle point problem

$$\left(\frac{\partial \mathcal{R}(\mathcal{U})}{\partial \mathcal{U}}\right)\delta\mathcal{U} = \begin{pmatrix} \frac{\partial \mathcal{R}_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}})}{\partial \tilde{\boldsymbol{u}}} & \frac{\partial \mathcal{R}_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}})}{\partial p} \\ \frac{\partial \mathcal{R}_p(\tilde{\boldsymbol{u}})}{\partial \tilde{\boldsymbol{u}}} & 0 \end{pmatrix} \begin{pmatrix} \delta\tilde{\boldsymbol{u}} \\ \delta p \end{pmatrix} = - \begin{pmatrix} \mathcal{R}_{\tilde{\boldsymbol{u}}} \\ \mathcal{R}_p \end{pmatrix}.$$

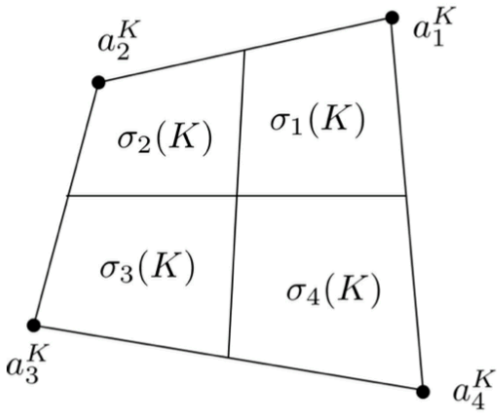
- **Coupled geometric multigrid methods CGMG**
  - **Hierarchy of multilevel triangulations**
  - **Relaxation/smoothing**
  - **Coarse-grid correction**
  - **Grid transfer operator**
  - **Local Multilevel Pressure Schur Complement**

Two-level algorithm

Linearized discrete TVP Saddle-point problem $\mathcal{J}\delta\mathcal{U} = \mathcal{R}$

- $\{\mathcal{T}_{h_k}\}$ **denote a family of hierarchy of multi-level triangulation with step-size** h_k
- **Each element on triangulation** $\mathcal{T}_{h_{k-1}}$ **is splitted into** $2^d, (d=2)$ **new elements** $\sigma_i(K),\ i = 1, \ldots, 2^d$ **to get** $\mathcal{T}_{h_k}$
- **The** k^{th} **level iteration** $MG\,(k, \delta\mathcal{U}_0, \mathcal{R})$ **of MG to solve**

$$\mathcal{J}_k\delta\mathcal{U} = \mathcal{R}$$



- **For** $k = 1$ **using the direct solver** $MG\,(1, \delta\mathcal{U}_0, \mathcal{R}) = \mathcal{J}_1^{-1}\mathcal{R}$
- **For** $k > 1$
  - **Pre-smoothing step:** $\delta\mathcal{U}_l = \delta\mathcal{U}_{l-1} + \mathcal{C}_k^{-1}\left(\mathcal{R} - \mathcal{J}_k\delta\mathcal{U}_{l-1}\right),\ 1 \le l \le \nu_1$
  - **Correction step: Let** $\tilde{\boldsymbol{\mathcal{R}}} = \boldsymbol{\mathcal{I}}_k^{k-1}\left(\mathcal{R} - \mathcal{J}_k\delta\mathcal{U}_{\nu_1}\right),\ \mathcal{Z}_i \in \mathcal{V}_{k-1},\ 0 \le i \le 2,$

$$\begin{aligned}\mathcal{Z}_0 &= 0\\ \mathcal{Z}_1 &= MG_F(k-1, \mathcal{Z}_0, \tilde{\boldsymbol{\mathcal{R}}})\\ \mathcal{Z}_2 &= MG_V(k-1, \mathcal{Z}_1, \tilde{\boldsymbol{\mathcal{R}}})\end{aligned}$$

  - **Final output:** $MG(k, \delta\mathcal{U}_0, \boldsymbol{\mathcal{R}}) = \delta\mathcal{U}_{\nu_1} + \boldsymbol{\mathcal{I}}_{k-1}^{k}\mathcal{Z}_2$
  - **Post-smoothing step:** $\delta\mathcal{U}_l = \delta\mathcal{U}_{l-1} + \mathcal{C}_k^{-1}\left(\mathcal{R} - \mathcal{J}_k\delta\mathcal{U}_{l-1}\right),\ 1 \le l \le \nu_2$

Grid Transfer Operators

Geometric transfer between FE spaces

- **The coarse-to-fine grid (prolongation):** $\mathcal{I}_{k-1}^{k} = (\mathcal{I}_{k-1,k}^{\tilde{\boldsymbol{u}}}, \mathcal{I}_{k-1,k}^{p})$
- **The fine-to coarse grid (restriction):** $\mathcal{I}_{k}^{k-1} = (\mathcal{I}_{k,k-1}^{\tilde{\boldsymbol{u}}}, \mathcal{I}_{k,k-1}^{p})$

$$\mathcal{I}_{k,k-1}^{\tilde{\boldsymbol{u}}} : \mathbb{W}_{h_k} \longrightarrow \mathbb{W}_{h_{k-1}}, \quad \mathcal{I}_{k-1,k}^{\tilde{\boldsymbol{u}}} : \mathbb{W}_{h_{k-1}} \longrightarrow \mathbb{W}_{h_k},$$
$$\mathcal{I}_{k,k-1}^{p} : \mathbb{Q}_{h_k} \longrightarrow \mathbb{Q}_{h_{k-1}}, \quad \mathcal{I}_{k-1,k}^{p} : \mathbb{Q}_{h_{k-1}} \longrightarrow \mathbb{Q}_{h_k}.$$

- **Structure and velocity, Q_2 FEM approximations:**

Nested approximation , $\mathbb{W}_{h_{k-1}} \subset \mathbb{W}_{h_k}$, Biquadratic interpolation for the prolongation and natural embedding for the restriction. For any $\tilde{\boldsymbol{u}}_{k-1} \in \mathbb{W}_{h_{k-1}}$

$$\tilde{\boldsymbol{u}}_{k-1} = \sum_{j=1}^{\mathtt{dim}\mathbb{W}_{h_{k-1}}} \tilde{\boldsymbol{u}}_{j,k-1}\varphi_j^{k-1} = \sum_{j=1}^{\mathtt{dim}\mathbb{W}_{h_k}} \tilde{\boldsymbol{u}}_{j,k}\varphi_j^{k}, \quad \tilde{\boldsymbol{u}}_{i,k} = \sum_{j=1}^{\mathtt{dim}\mathbb{W}_{h_{k-1}}} \tilde{\boldsymbol{u}}_{j,k-1}\mathcal{N}_j^k(\varphi_i^{k-1})$$

By using the dual basis $\{\mathcal{N}_i^k, i = 1, \ldots, \mathtt{dim}\mathbb{W}_{h_k}\}$, i.e. $\mathcal{N}_i^k(\varphi_j^k) = \delta_{ij}$

$$\boldsymbol{\mathcal{I}}_{k-1,k}^{\tilde{\boldsymbol{u}}} \in \mathcal{W}_k \times \mathcal{W}_{k-1}, \quad \left[\boldsymbol{\mathcal{I}}_{k-1,k}^{\tilde{\boldsymbol{u}}}\right]_{ij} = \mathcal{N}_j^k(\varphi_i^{k-1}) \qquad \boldsymbol{\mathcal{I}}_{k,k-1}^{\tilde{\boldsymbol{u}}} = \boldsymbol{M}_{k-1}^{-1}\left(\boldsymbol{\mathcal{I}}_{k-1,k}^{\tilde{\boldsymbol{u}}}\right)^T \boldsymbol{M}_k$$

Coarse grid operator

$$\mathcal{R}_{k-1} = \left(\boldsymbol{\mathcal{I}}_{k-1,k}^{\tilde{\boldsymbol{u}}}\right)^T \mathcal{R}_k \qquad \mathcal{J}_{k-1}^{\tilde{\boldsymbol{u}}} = \left(\boldsymbol{\mathcal{I}}_{k-1,k}^{\tilde{\boldsymbol{u}}}\right)^T \mathcal{J}_k^{\tilde{\boldsymbol{u}}} \boldsymbol{\mathcal{I}}_{k,k-1}^{\tilde{\boldsymbol{u}}}$$

Galerkin orthogonality is preserved and the optimal order of convergence of MG is maintained !

Grid Transfer Operators

- **Pressure, P_1^{disc} FEM approximations: Nested discontinuous approximation, $\mathbb{Q}_{h_{k-1}} \subset \mathbb{Q}_{h_k}$**

The prolongation operator for pressure $\mathcal{I}^p_{k-1,k} : \mathbb{Q}_{h_{k-1}} \longrightarrow \mathbb{Q}_{h_k}$ is defined in terms of local L^2 inner-product projection operator

$$q \in P_1^{\text{disc}}(K) \Longrightarrow q \in P_1^{\text{disc}}(\sigma_l(K)), l = 1, \cdots, 2^d)$$

$$\sum_{l=1}^{2^d} \int_{\sigma_l(K)} \mathcal{I}^p_{k-1,k} p_{k-1} q \, d\Omega = \int_K p_{k-1} q \, d\Omega, \quad \forall K \in \mathcal{T}_{h_{k-1}}$$

Its matrix representation is as follows:

$$\boldsymbol{\mathcal{I}}^p_{k-1,k} = \boldsymbol{M}_k^{-1} \boldsymbol{M}_{k-1}$$

The restriction operator for pressure $\mathcal{I}^p_{k,k-1} : \mathbb{Q}_{h_k} \longrightarrow \mathbb{Q}_{h_{k-1}}$ is the transpose counterpart of the prolongation w.r.t. L^2:

$$\forall\, K \in \mathcal{T}_{h_{k-1}},\ q \in P_1^{\text{disc}}(K)\ ;\ \int_K \mathcal{I}^p_{k,k-1} p_k q \, d\Omega = \sum_{l=1}^{2^d} \int_{\sigma_l(K)} p_k \mathcal{I}^p_{k-1,k} q \, d\Omega$$

Its matrix representation $$\boldsymbol{\mathcal{I}}^p_{k,k-1} = \boldsymbol{M}_{k-1}^{-1} \boldsymbol{M}_k$$

Indeed the transpose of prolongation operator is the restriction operator

$$\boldsymbol{\mathcal{I}}^p_{k,k-1} = \left(\boldsymbol{\mathcal{I}}^p_{k-1,k}\right)^T$$

- **Setting local subproblem,**
  - **Prolongation/Restriction:** $\mathcal{P}_K = (\mathcal{P}_K^{\tilde{\boldsymbol{u}}}, \mathcal{P}_K^p),\ \mathcal{P}_K^{\mathrm{T}} = ({\mathcal{P}_K^{\tilde{\boldsymbol{u}}}}^{\mathrm{T}}, {\mathcal{P}_K^p}^{\mathrm{T}})$

$$\mathcal{P}_K^{\tilde{\boldsymbol{u}}} : \mathbb{W}_h(K) \longrightarrow \mathbb{W}_h,\ \mathcal{P}_K^p : \mathbb{Q}_h(K) \longrightarrow \mathbb{Q}_h, \quad \forall\ K \in \mathcal{T}_h,$$

$${\mathcal{P}_K^{\tilde{\boldsymbol{u}}}}^{\mathrm{T}} : \mathbb{W}_h \longrightarrow \mathbb{W}_h(K),\ {\mathcal{P}_K^p}^{\mathrm{T}} : \mathbb{Q}_h \longrightarrow \mathbb{Q}_h(K), \quad \forall\ K \in \mathcal{T}_h,$$

- **Algebraically counterpart**
  - **Prolongation/Restriction:** $\boldsymbol{\mathcal{P}}_K = (\boldsymbol{\mathcal{P}}_K^{\tilde{\boldsymbol{u}}}, \boldsymbol{\mathcal{P}}_K^p),\ \boldsymbol{\mathcal{P}}_K^{\mathrm{T}} = ({\boldsymbol{\mathcal{P}}_K^{\tilde{\boldsymbol{u}}}}^{\mathrm{T}}, {\boldsymbol{\mathcal{P}}_K^p}^{\mathrm{T}})$

$$\mathcal{U}_K = (\tilde{\boldsymbol{u}}_K, p_K) := \boldsymbol{\mathcal{P}}_K^{\mathrm{T}} \mathcal{U} = ({\boldsymbol{\mathcal{P}}_K^{\tilde{\boldsymbol{u}}}}^{\mathrm{T}} \tilde{\boldsymbol{u}}, {\boldsymbol{\mathcal{P}}_K^p}^{\mathrm{T}} p)$$

$$\mathcal{U} = (\tilde{\boldsymbol{u}}, p) := \sum_{K \in \mathcal{T}_h} \boldsymbol{\mathcal{P}}_K \mathcal{U}_K = \sum_{K \in \mathcal{T}_h} (\boldsymbol{\mathcal{P}}_K^{\tilde{\boldsymbol{u}}} \tilde{\boldsymbol{u}}, \boldsymbol{\mathcal{P}}_K^p p)$$

- **Block Gauss-Seidel iteration reads**

$$\mathcal{U}^{k+1} = \mathcal{U}^k - \omega_k \sum_{K \in \mathcal{T}_h} \boldsymbol{\mathcal{P}}_K \left(\boldsymbol{\mathcal{P}}_K^{\mathrm{T}} \left(\frac{\partial \mathcal{R}(\mathcal{U}^k)}{\partial \mathcal{U}} \right) \boldsymbol{\mathcal{P}}_K \right)^{-1} \boldsymbol{\mathcal{P}}_K^{\mathrm{T}} \mathcal{R}(\mathcal{U}^k).$$

Coupled Monolithic Multigrid Solver !

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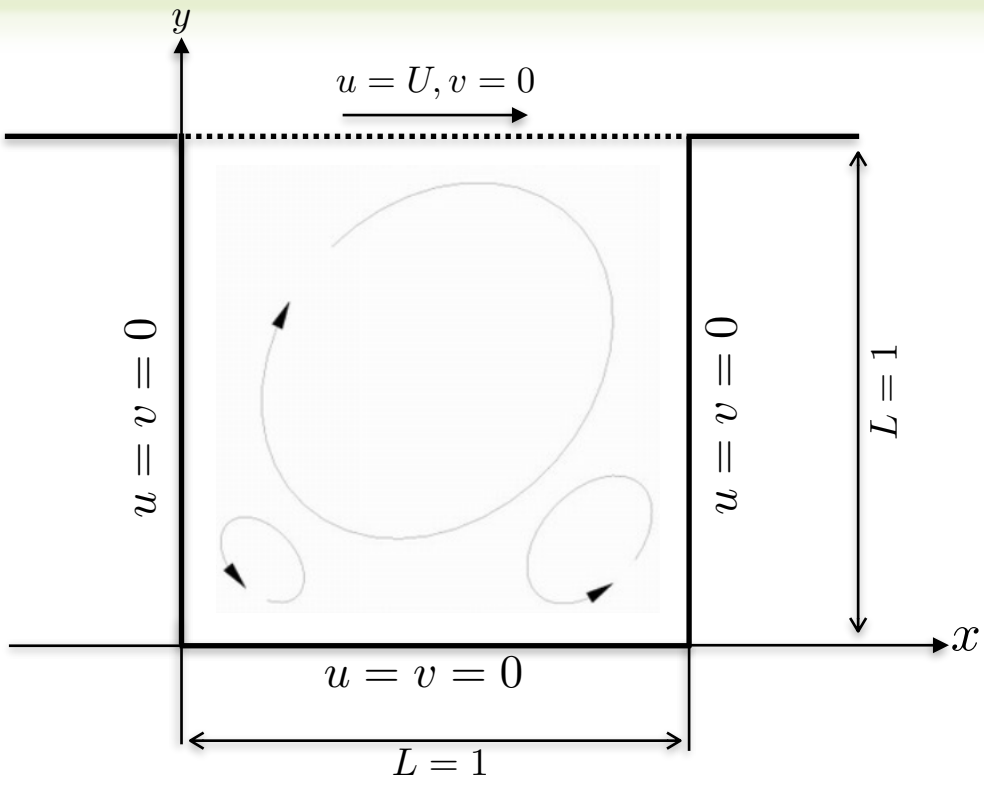
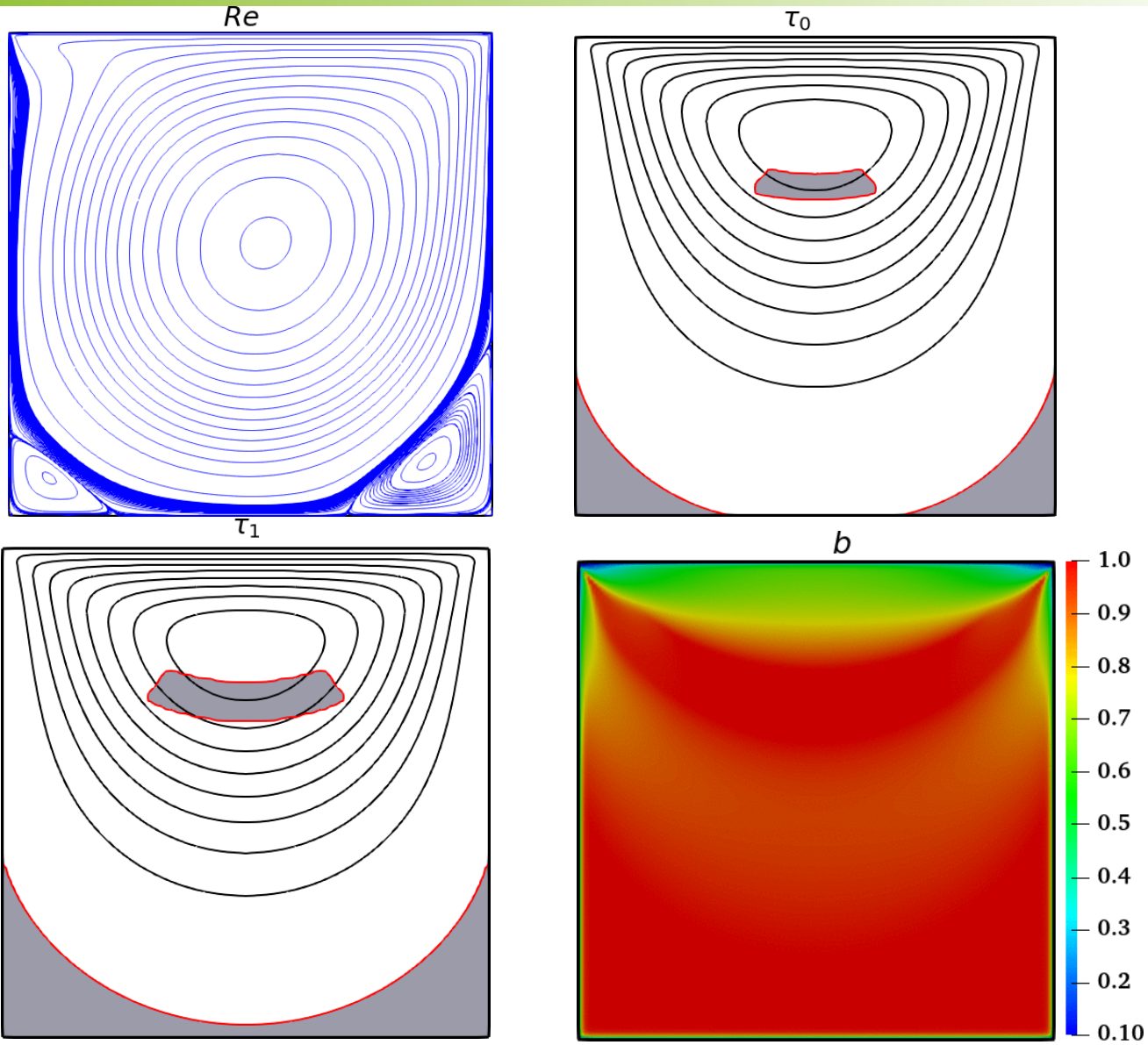
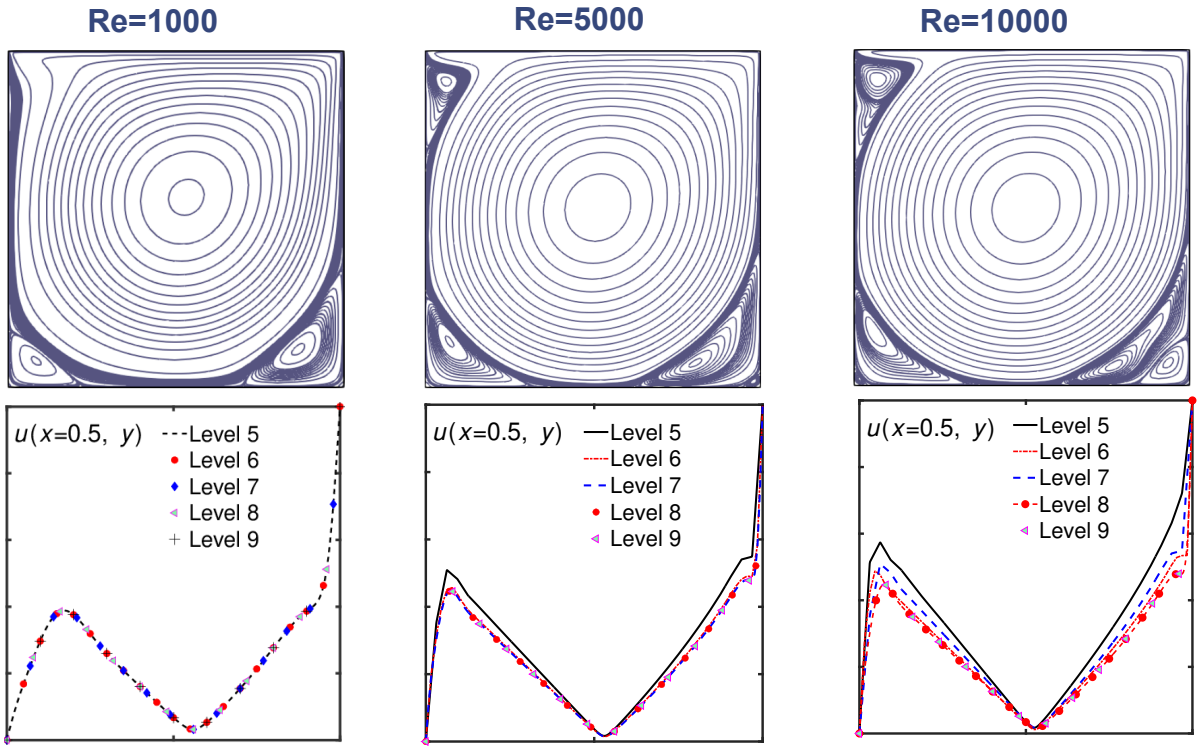
- **Starting point: consider flow in a cavity with unit height**
- **Steady, incompressible flow**
- **Constant speed at upper lid**
- **No-slip Dirichlet boundary conditions**




Re
τ_0
τ_1
b
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.10

Newtonian Lid-driven cavity flow

- **Point-wise convergence for Newtonian flow**



Re	1000		5000		10000	
Level	ψ_{max}	$\psi_{min} \times 10^3$	ψ_{max}	$\psi_{min} \times 10^3$	ψ_{max}	$\psi_{min} \times 10^3$
6	0.1190073	−1.72813	0.1249471	−3.145666	0.1586626	−5.7535749
7	0.1189360	−1.72649	0.1225439	−3.077555	0.1236127	−3.2070181
8	0.1189361	−1.72851	0.1222499	−3.072411	0.1225210	−3.1831353
9	0.1189362	−1.72963	0.1222269	−3.073524	0.1224097	−3.1910101
10	0.1189366	−1.72965	0.1222259	−3.073589	0.1223892	−3.1797390
Ref.	0.11892	−1.7292	0.12216	−3.0706		
	Brunrau	Bruneau	Kupperman	Bruneau		

Newtonian Lid-driven cavity flow

- **Global quantity and solver behaviour**

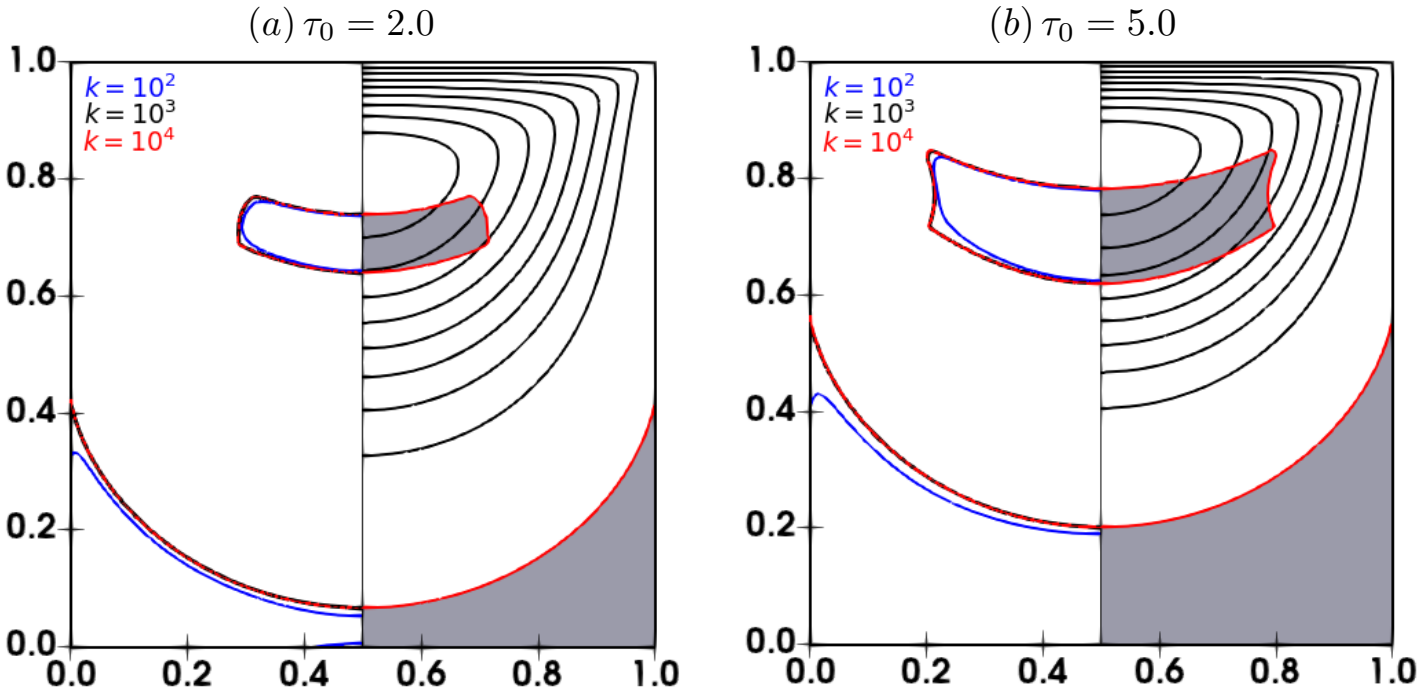
Re		1000		5000		10000	
Level	cells	Energy$\times 10^2$	N/M	Energy$\times 10^2$	N/M	Energy$\times 10^2$	N/M
5	1024	4.541506	5/1	6.082524	6/1	7.940472	7/1
6	4096	4.458877	5/1	4.955858	5/1	5.369527	6/1
7	16384	4.452357	3/1	4.768669	4/1	4.868399	5/1
8	65536	4.451904	3/1	4.744815	3/2	4.783917	4/2
9	262144	4.451846	3/1	4.742921	3/1	4.773500	3/2
10	1048576	4.451834	2/1	4.742815	3/1	4.772692	3/1
Ref. values $\approx$:		4.45		4.74		4.77	

- ✓ **Mesh convergence of the solutions irrespective of Re number**
- ✓ **Efficient non-linear solver**
- ✓ **Mesh independent linear solver**

Accurate, robust and efficient Monolithic Multigrid Solver

Viscoplastic flow in Lid-driven cavity

- **Boundary limit for rigid-zone w.r.t regularization k**



- **Accurate track of interface requires**
  - ✓ **larger k solutions**
  - ✓ **finer mesh refinement**
- **Existence of pair (k, L) beyond which no further improvement in solutions is expected**

Viscoplastic flow in Lid-driven cavity

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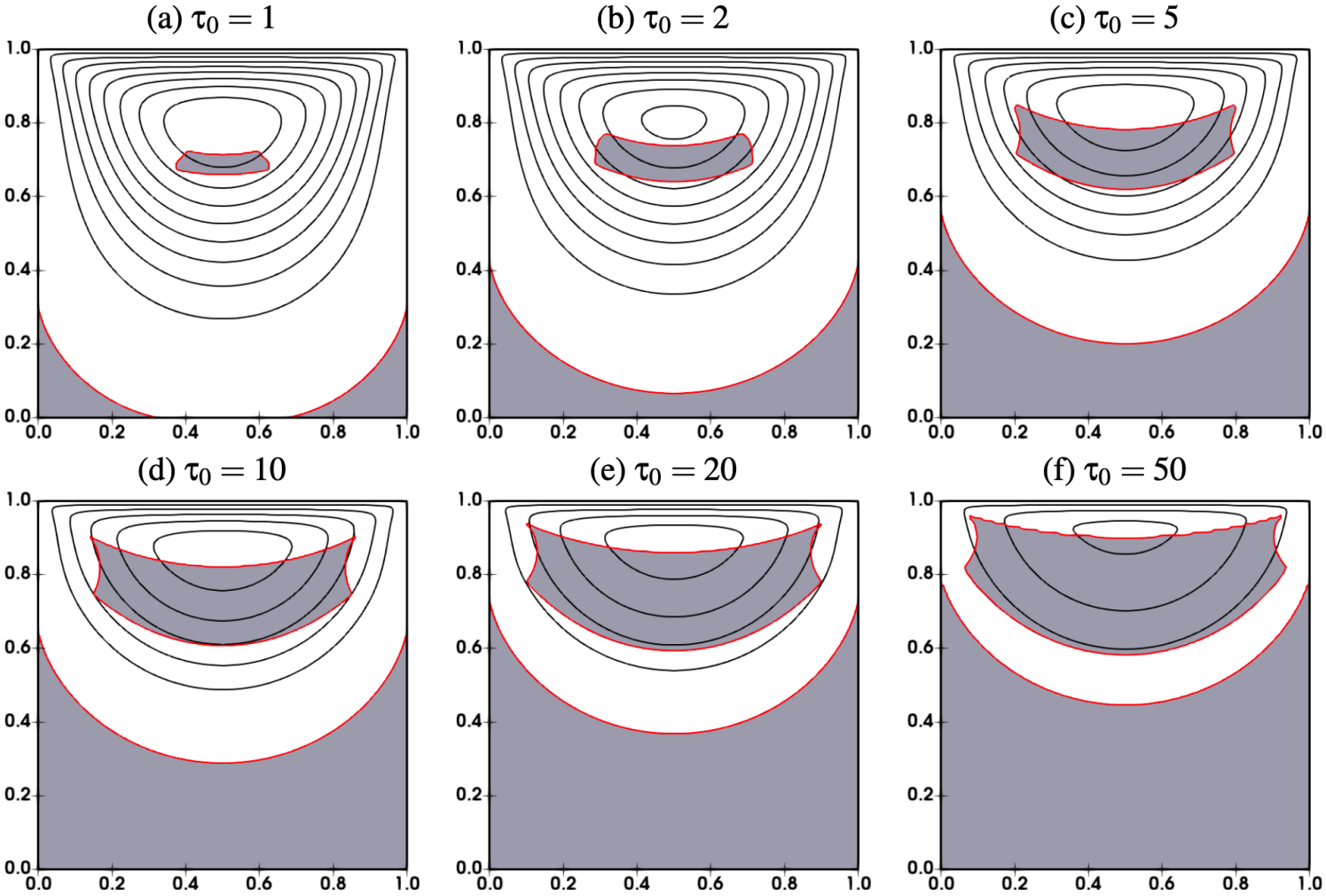
- **progressive growth of unyielded zones for non-thixotropic (Bingham Plastic) flow**



✓ **Unyielded zones's shape and extent is in agreement with Ref. Results**

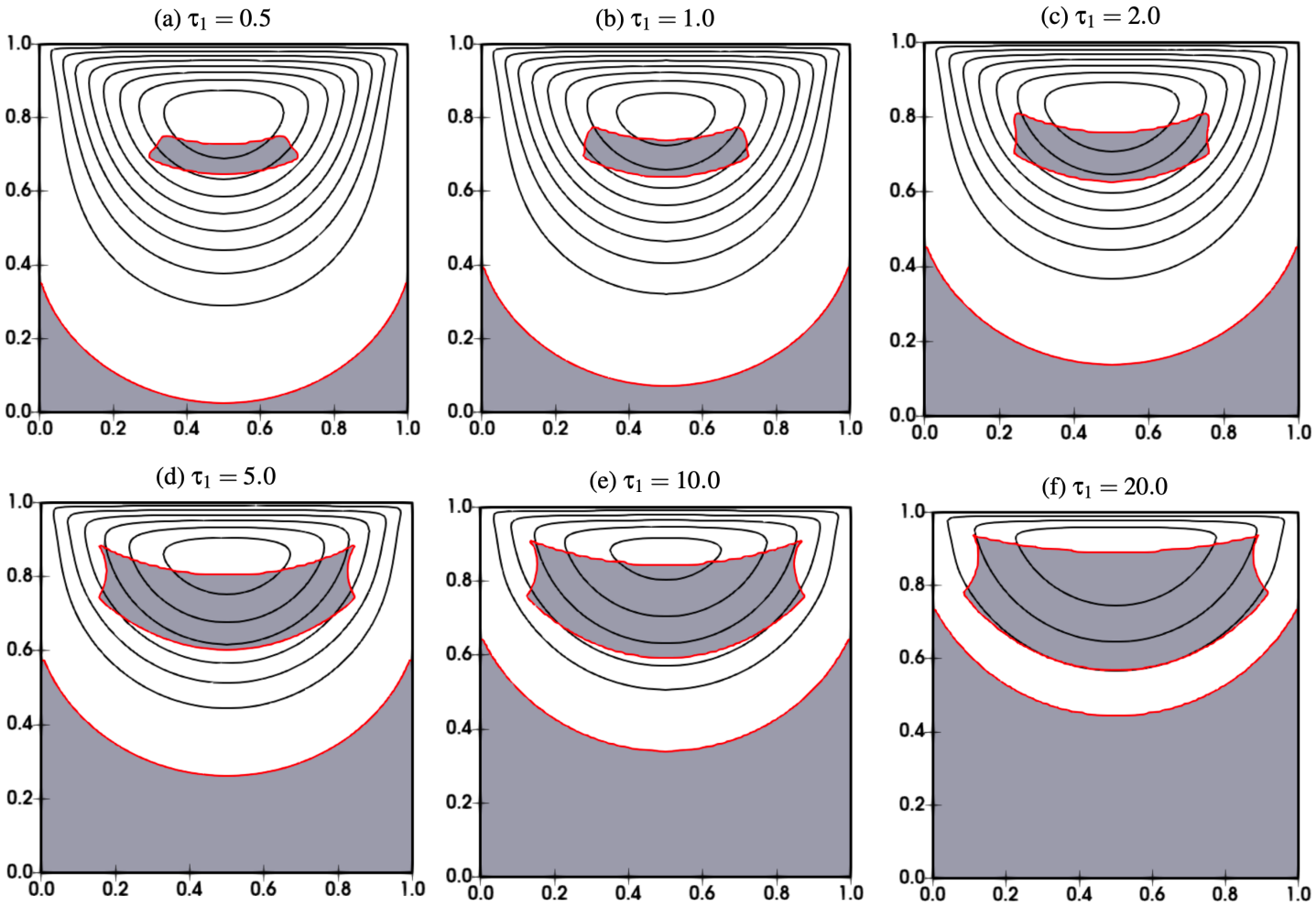
- **Solver behaviour w.r.t Regulariztaion and mesh refinement**

$k\backslash L$	5	6	7	5	6	7	5	6	7
	$\tau_0 = 1.0$			$\tau_0 = 2.0$			$\tau_0 = 5.0$		
1×10^1	3/1	3/1	3/1	3/1	3/1	3/1	4/1	4/1	4/1
5×10^1	2/1	2/1	2/1	2/1	2/1	2/1	3/1	3/1	3/1
1×10^2	3/1	3/1	3/1	3/1	3/1	3/1	4/1	4/1	4/1
5×10^2	3/1	2/1	2/1	3/1	2/1	3/1	3/2	3/2	3/1
1×10^3	2/2	3/2	3/1	3/1	3/1	4/1	4/1	5/2	5/2
5×10^3	2/1	2/1	4/1	3/1	3/2	6/2	4/1	8/2	6/1
1×10^4	2/1	2/2	5/1	3/1	3/1	6/1	4/1	5/4	6/3
	$\tau_0 = 10.0$			$\tau_0 = 20.0$			$\tau_0 = 50.0$		
1×10^1	5/1	5/1	5/1	6/1	6/1	6/1	5/1	7/1	7/1
5×10^1	4/1	3/1	3/1	4/1	4/1	3/2	5/4	4/2	4/2
1×10^2	5/2	4/1	4/1	5/2	5/2	5/1	6/5	5/4	5/1
5×10^2	5/3	3/2	3/1	4/4	3/4	4/3	5/4	4/2	4/3
1×10^3	5/2	7/4	9/1	5/5	7/2	8/1	5/5	9/2	9/2
5×10^3	5/1	7/3	8/2	6/3	6/4	6/4	6/4	7/2	8/2
1×10^4	6/1	7/2	8/3	6/3	5/5	7/3	6/3	7/3	8/2

- ✓ **Efficient non-linear solver**
- ✓ **Mesh independent linear solver**
- ✓ **Solutions are obtained with continuation strategy w.r.t. k**
- ➡ **Integration of continuation strategy w.r.t. k in the solver**

Thixo-viscoplastic flow in Lid-driven cavity

- **Impact of thixotropic yield stress on morphology of unyielded zones in TVP flow**



✓ **Main rheological characteristics of materials with yield stress is preserved**

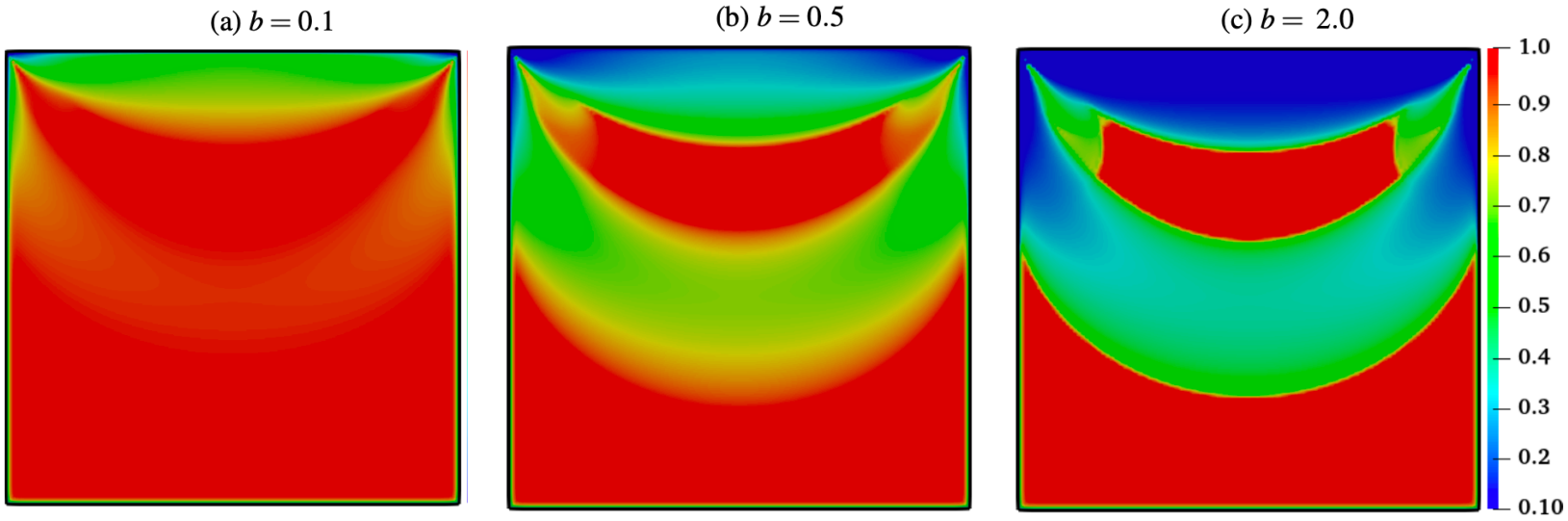
- **Solver behaviour w.r.t Regulariztaion and mesh refinement**

$k\backslash L$	5	6	7	5	6	7	5	6	7
	$\tau_1 = 0.5$			$\tau_1 = 1.0$			$\tau_1 = 2.0$		
1×10^1	5/2	5/3	6/2	5/2	5/2	9/1	5/2	5/2	9/1
5×10^1	4/2	4/2	4/2	3/2	3/3	7/1	3/2	3/3	8/1
1×10^2	4/1	4/2	5/1	4/1	4/2	7/1	4/2	4/2	8/1
5×10^2	4/1	4/1	5/1	3/1	4/1	6/1	4/2	4/2	8/1
1×10^3	4/1	4/1	4/1	4/2	4/2	8/1	4/4	6/1	7/1
5×10^3	4/1	4/1	3/2	7/1	9/1	5/1	6/1	9/1	8/1
1×10^4	4/1	4/2	4/2	5/1	7/1	4/1	7/1	10/1	8/2
	$\tau_1 = 5.0$			$\tau_1 = 10.0$			$\tau_1 = 20.0$		
1×10^1	6/2	6/2	10/1	11/1	8/2	11/1	10/1	9/2	11/1
5×10^1	4/2	3/2	11/1	11/1	4/2	7/1	12/1	5/3	9/1
1×10^2	4/2	5/2	11/1	10/1	5/3	8/1	12/1	6/3	10/1
5×10^2	5/2	4/2	10/1	9/1	5/3	5/1	8/1	5/5	11/1
1×10^3	5/2	9/1	10/1	10/1	9/1	7/1	8/2	9/1	9/2
5×10^3	5/1	5/1	5/1	8/1	8/2	6/1	8/1	7/1	11/1
1×10^4	5/1	5/2	5/1	8/3	7/1	5/1	8/2	7/1	9/1

- ✓ **Robust non-linear solver**
- ✓ **Mesh independent linear solver**
- ✓ **Solutions are obtained via continuation strategy w.r.t. k**
- ➡ **Integration of continuation strategy w.r.t. k in the solver**

Thixo-viscoplastic flow in Lid-driven cavity

- **Material micro-structural level w.r.t. breakdown parameter**



- **Interplay of yield stress and thixotropy**

  - ✓ **Structuring level is predicting shape and extent of rigid zones**
  - ✓ **Induction of more breakdown layers**
  - ✓ **Shear localization**
  - ✓ **Shear band**

Thixo-viscoplastic flow in Couette

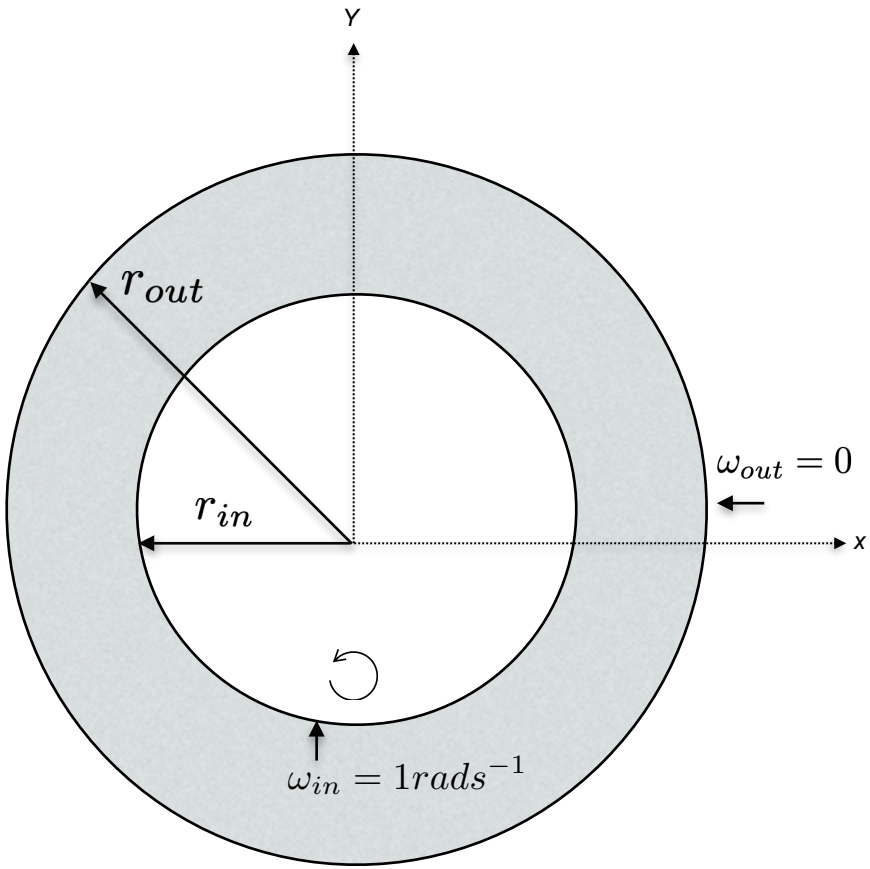
Continuous axial-Flow Couette device:

The material is sheared in the annulus between the interior and exterior cylinder shells of radii r_{in} and r_{out} respectively.

- **Concentric cylinders**
- **Rotating inner cylinder with $\omega_{in} = 1 rads^{-1}$**
- **Stationary outer cylinder**
- **Vertical flow super-imposed in radial direction**

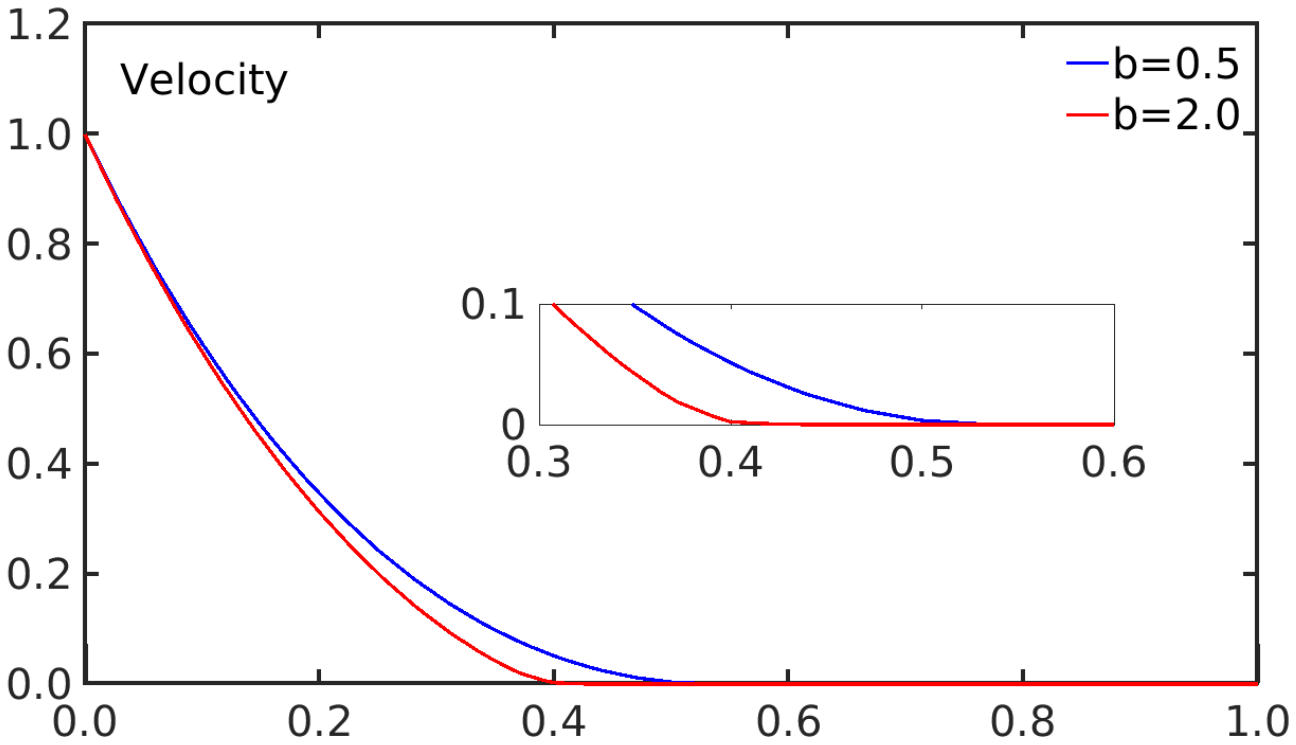
Investigations of thixo-viscoplastic phenomena

- **Shear localization**
- **Shear banding**
- **Consistent transition points between velocity and structure**
- **Discontinuous jump to infinity for viscosity**



Thixo-viscoplastic flow in Couette

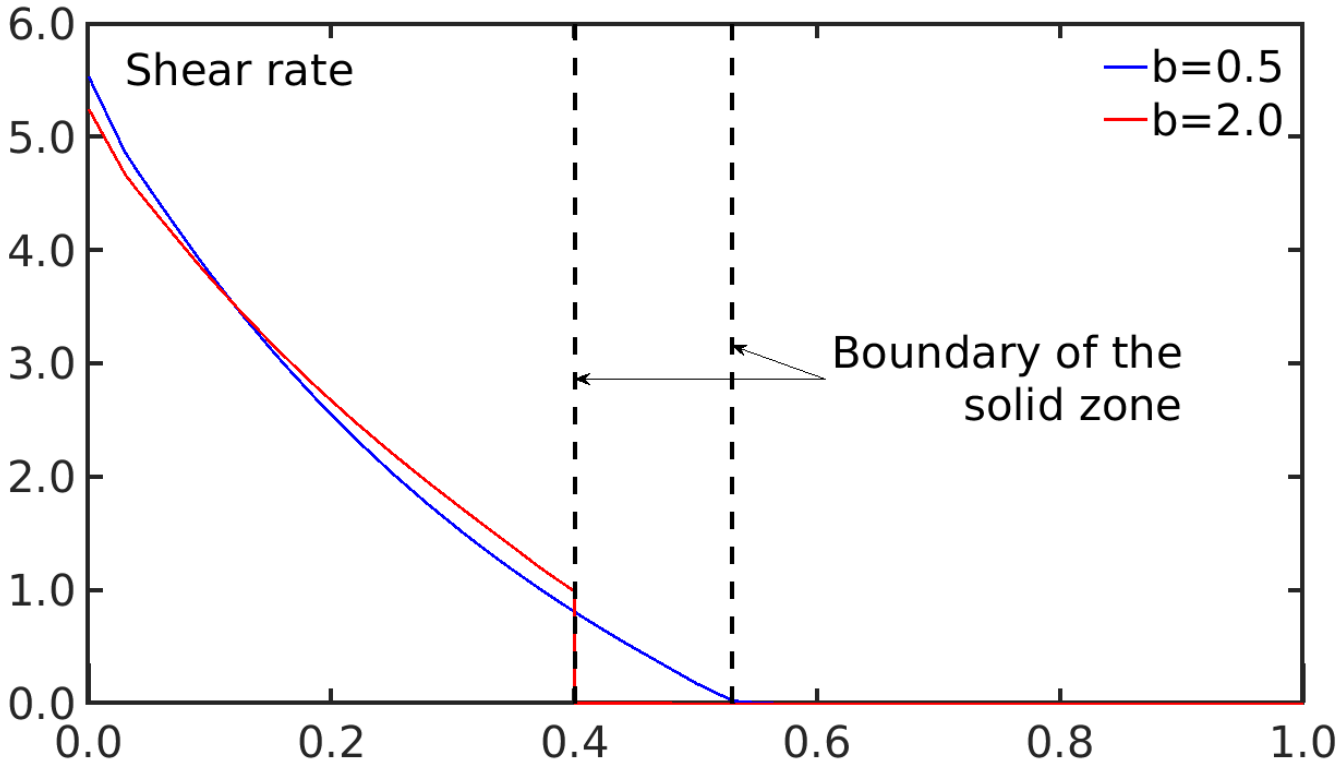
- **Velocity profile at cut-line positions $c; c \in [0, 2\pi]$ in a Couette device w.r.t breakdown parameter**



- ✓ **Localization**
- ✓ **Shear banding**

Thixo-viscoplastic flow in Couette

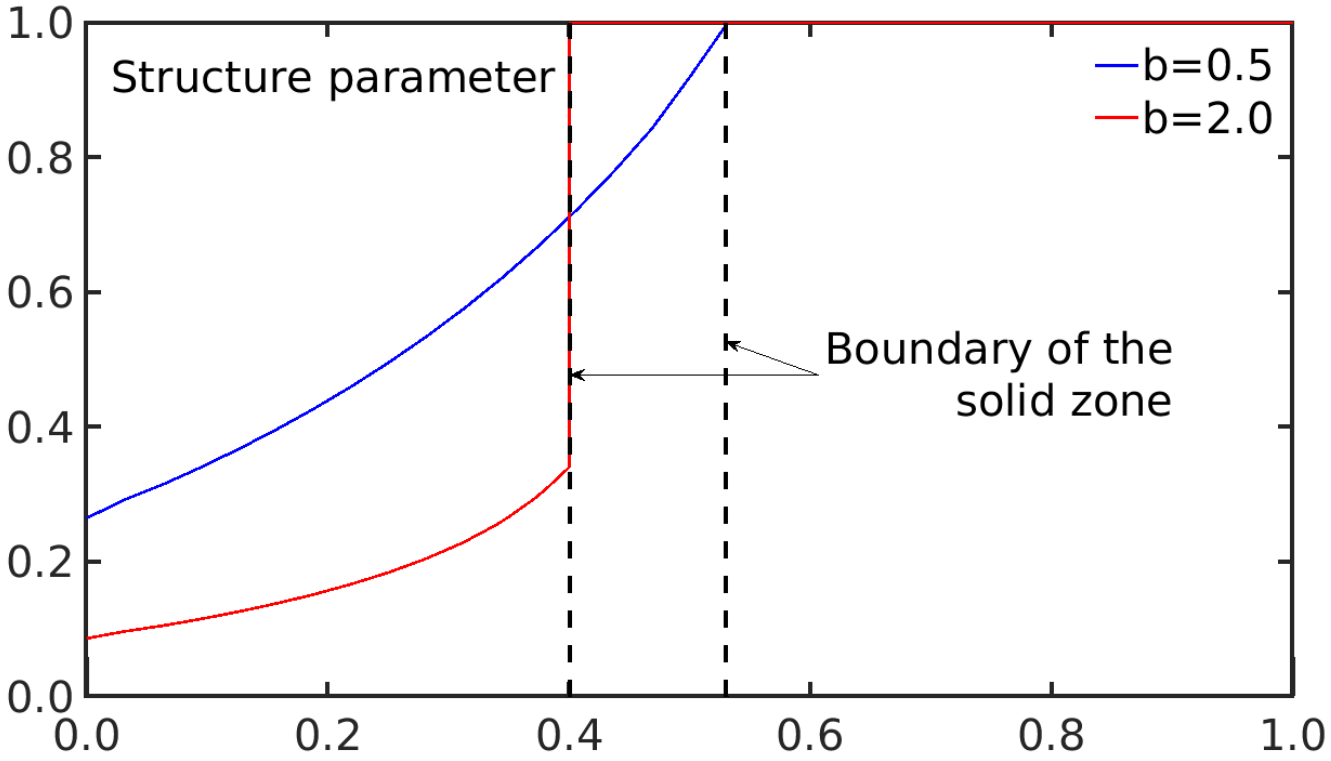
- **Shear rate at cut-line positions $c; c \in [0, 2\pi]$ in a Couette device w.r.t breakdown parameter**



- ✓ **Smooth and sharp transition are possible**
- ✓ **Transition point matches with the velocity**

Thixo-viscoplastic flow in Couette

- **Structure parameter at cut-line positions $c; c \in [0, 2\pi]$ in a couette w.r.t. breakdown parameter**



✓ **Transition point matches with the velocity**
✓ **Structuring level is predicting shape and extent of rigid zones**

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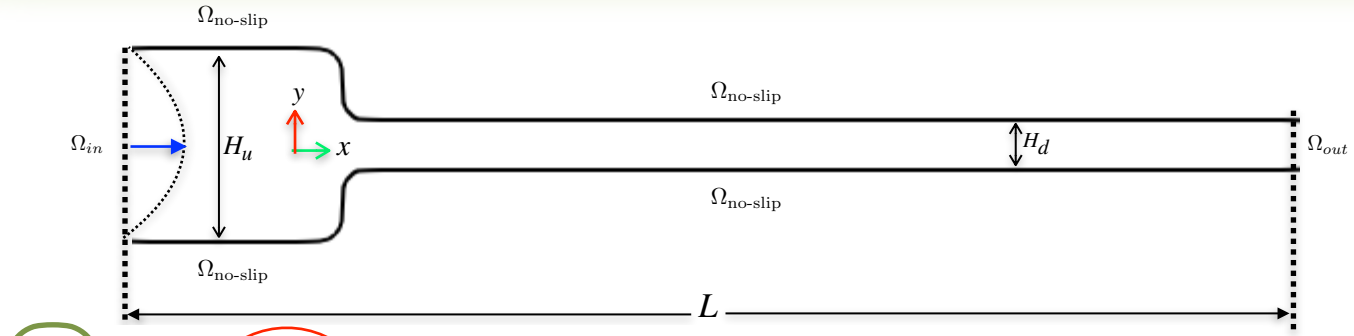
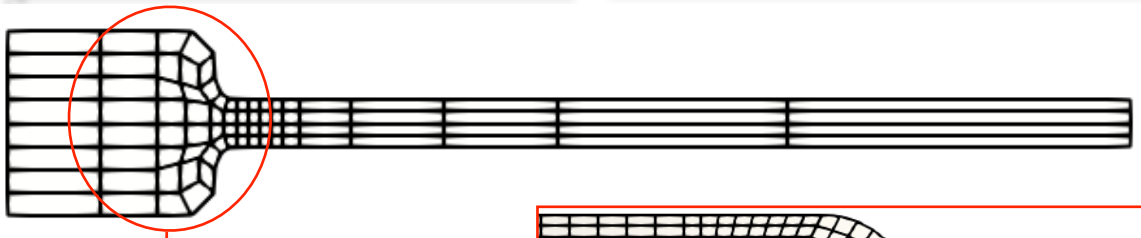
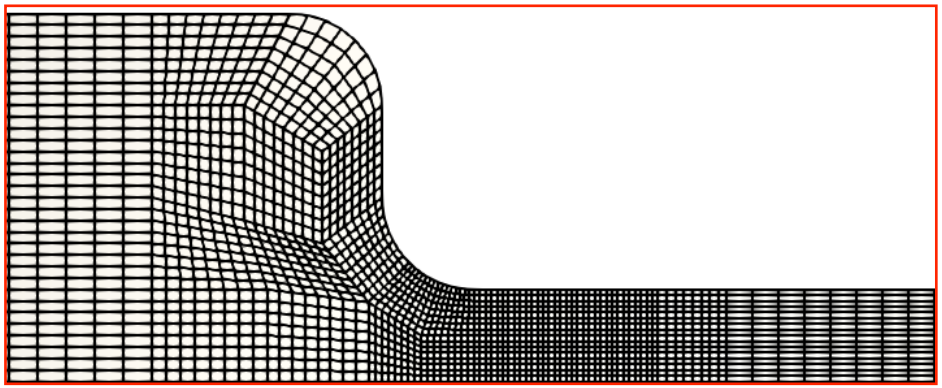
- **2D-FEM simulation results for thixo-viscoplastic flow- validation of 1D tool**
- **Specifying the "unidirectional profiles as boundary Data" in 2D for contraction**



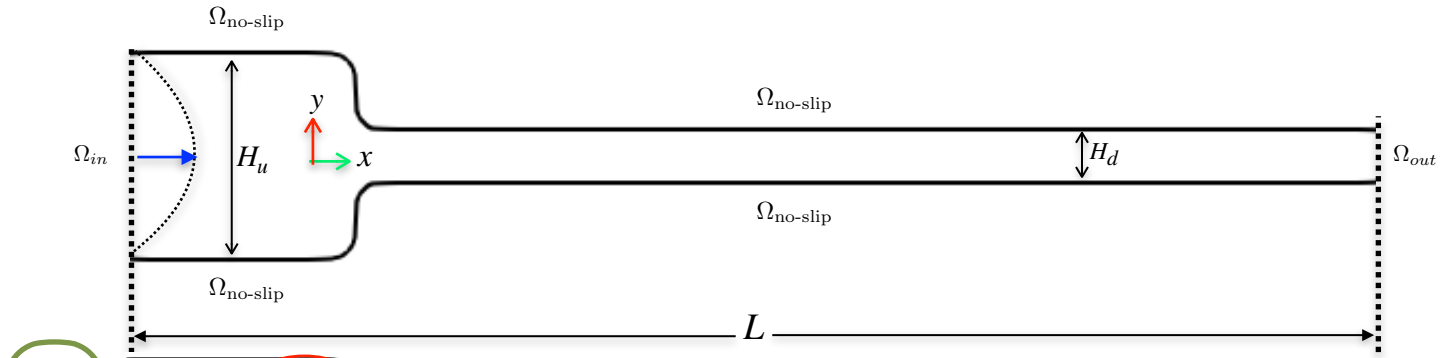
Coarse Mesh

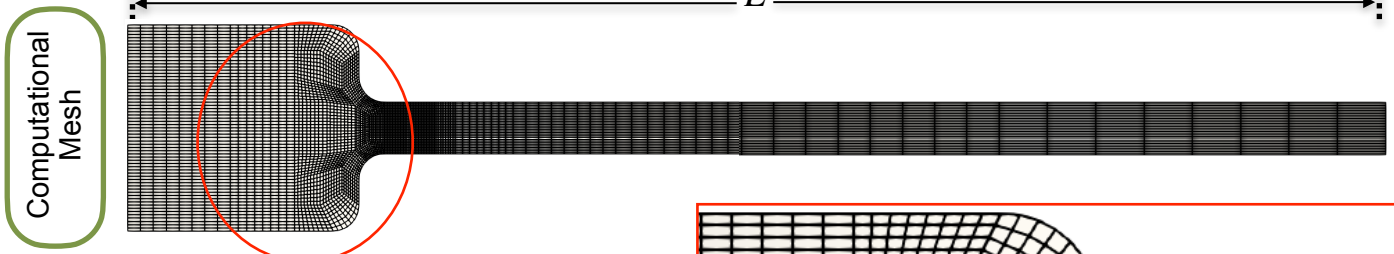


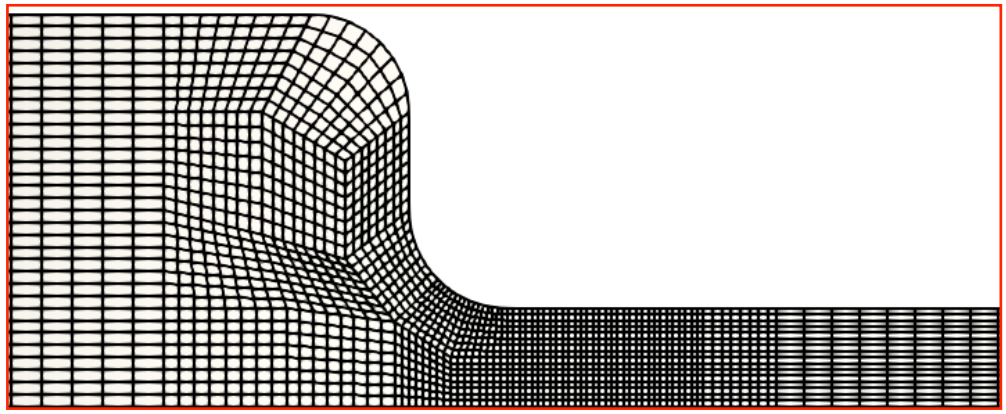


Thixo-viscoplastic flow in curved contractions

- **2D-FEM simulation results for thixotrpic flow- validation of 1D tool**
- **Specifying the "1D-profiles as boundary Data" in 2D simulations for contraction domain**







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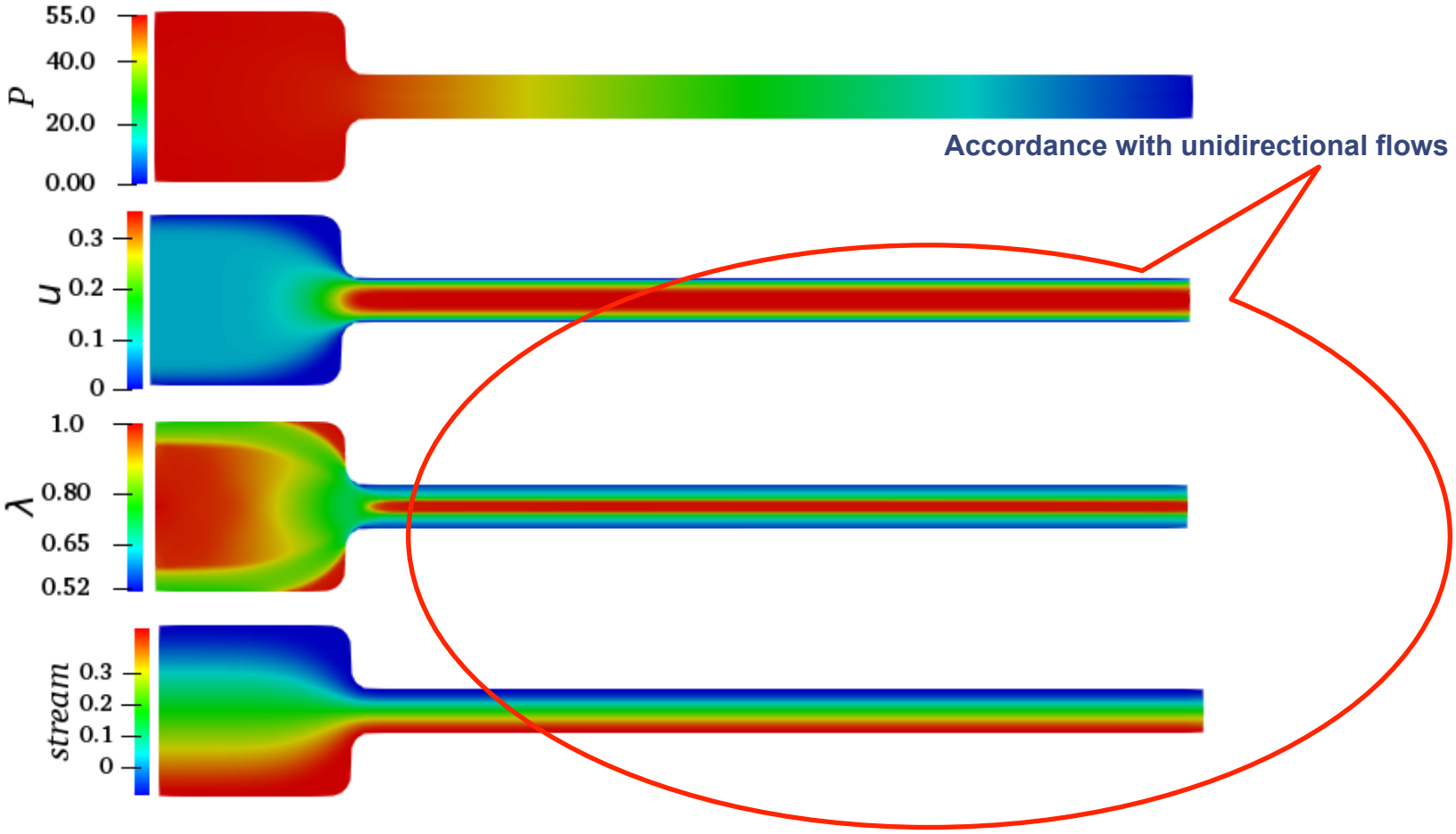
Contraction flow (smoothed)



- ✓ **As predicted (u, λ, p) solutions**
- ✓ **Structuring level is predicting shape and extent of rigid zones**

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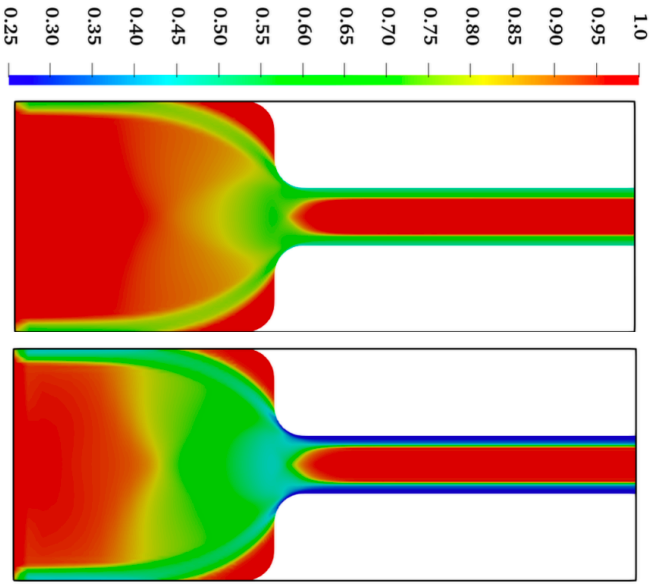
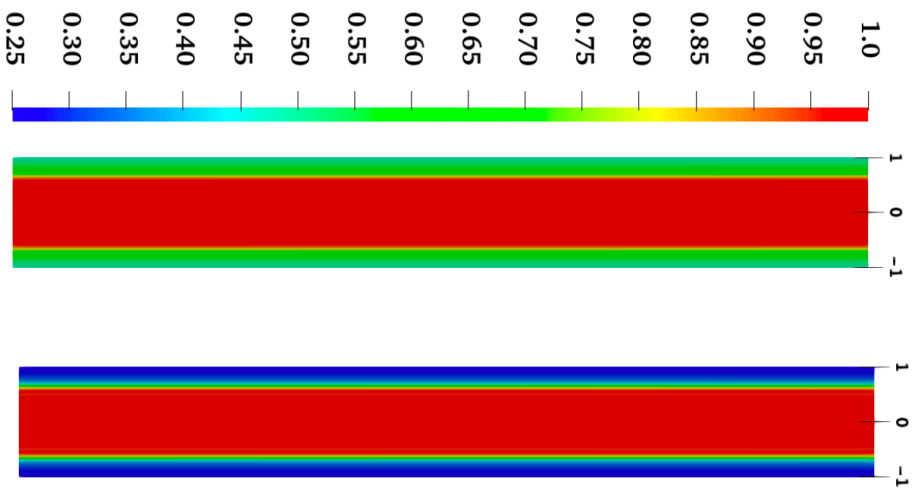
- **Material micro-structural level w.r.t. breakdown**

(i) Upstream channel & Entrance zone



(ii) Downstream channel



- **Inherent thixotropy speed-up the breakdown**
  - ✓ **Appearance of more breakdown layers**
  - ✓ **Applications: restart pressure in pipelines should not be over-estimated**

Summary

We analysed FEM for regularized TVP problem

- ✓ **Wellposedness**
- ✓ **Convergence analysis**

Analyzed the accuracy, robustness, and efficiency of the TVP solver using

- ✓ **Higher order finite element method**
- ✓ **Monolithic Newton-multigrid**
  - **Adaptive discrete Newton's method with global convergent property**
  - **Geometric multigrid with local MPSC**

Simulations for TVP materials for Benchmarks in Lid-driven cavity, Couette devices and 4:1 contraction configuration

FEM analysis and monolithic Newton-multigrid solver for thixo-viscoplastic flow problems

N. Begum, A. Ouazzi, S. Turek
Institute of Applied Mathematics, LS III,
TU Dortmund University, Dortmund, Germany

8th European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS Congress 2022
5-9 June 2022, Oslo, Norway

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