The CG1-DG2 method for conservation laws

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CG1-DG2 Method - Motivation

hp-adaptivity for transport problems:

1. non-smooth solution:
   continuous linear elements + FCT (flux corrected transport) $\Rightarrow$ only h-adaptivity
2. smooth solution:
   continuous higher order elements without stabilization

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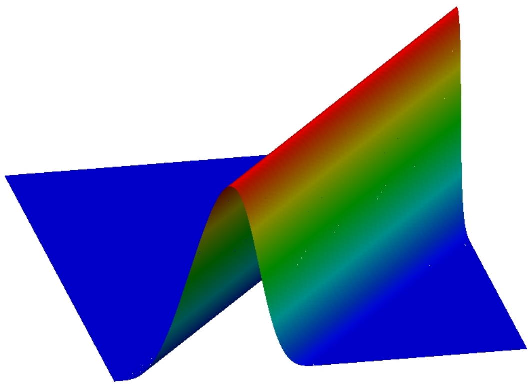
Consider the stationary problem

$$u + \nabla \cdot (\mathbf{v}u) = f \qquad \text{in } \Omega = (0,1) \times (0,1)$$

$$\mathbf{v}(x, y) = (1, 1).$$



CG1-DG2 Method - Motivation

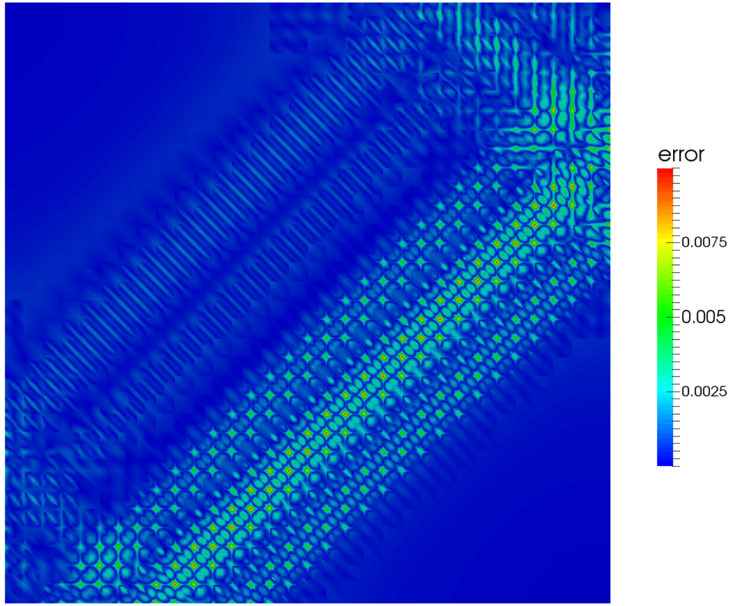


Figure : Error for continuous Galerkin with quadratic elements (CG2)

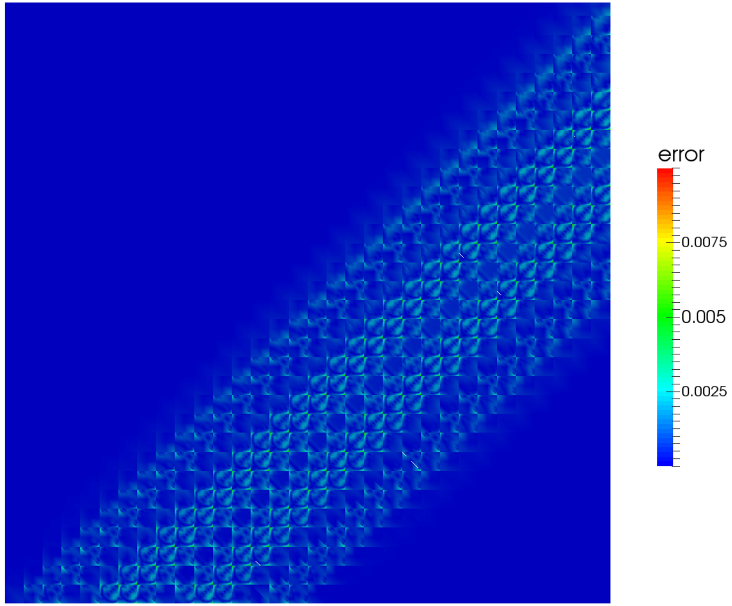


Figure : Error for discontinuous Galerkin with quadratic elements (DG2)

Idea

A new method which fulfills

1. Stability
2. Optimal convergence rate
3. Lower computational cost than
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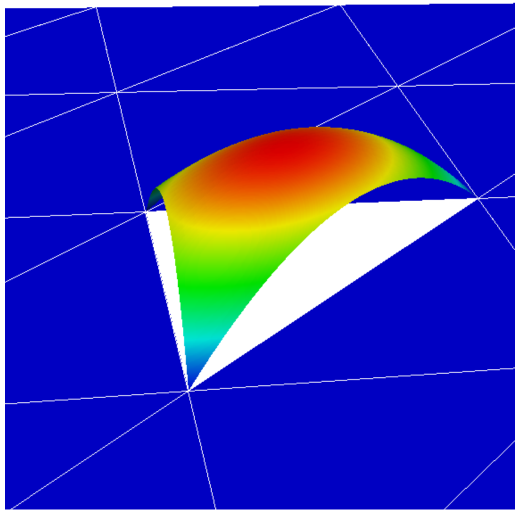
The CG1-DG2 method:
Enriching the space of continuous linear finite elements with discontinuous quadratic basis functions:

CG1-DG2 functions are **continuous at the vertices** but may have **jumps across edges**.



CG1-DG2 space for triangles

On each triangle K of a shape-regular triangulation $\mathcal{K}_h$ of Ω we have:

1. linear basis function φ_K^i, $i = 1, 2, 3$
2. quadratic basis function $\psi_K^i := \varphi_K^i \varphi_K^{i+1 \bmod 3}$, $i = 1, 2, 3$

The space of discontinuous quadratic basis functions is defined

$$D_h := \{v_h \in L^2(\Omega) : v_h|_K \in D_K\} \quad \text{with } D_K := \{\psi_K^i : \ 1 \leq i \leq 3\}.$$

The CG1-DG2 space is defined by:

$$V_h^{1,2} := V_h^1 \oplus D_h.$$

where V_h^1 is the space of linear finite elements.

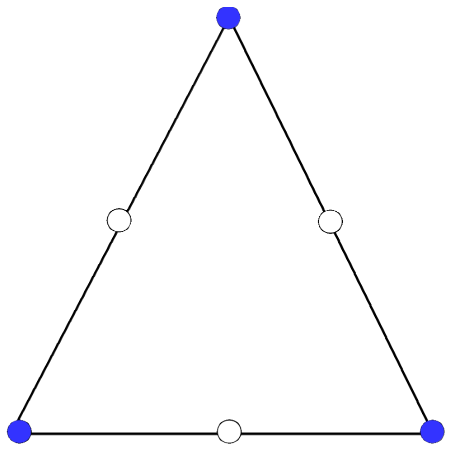


Figure : blue: continuous linear basis functions, white: discontinuous quadratic basis functions

Some analysis for transport equations

Model problem in 2d:

$$\begin{aligned} \sigma u + \operatorname{div}(\mathbf{v}u) &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega^- \end{aligned}$$

We will assume that

1. $\sigma + \frac{1}{2}\operatorname{div}\mathbf{v} \geq \sigma_0 > 0$,
2. $\sup_{\mathbf{x}\in\Omega} |\sigma(\mathbf{x}) + \operatorname{div}\mathbf{v}(\mathbf{x})| =: \sigma_1 < \infty$.
3. $\sigma \in L^\infty(\Omega)$ and $v_1, v_2 \in W^{1,\infty}(\Omega), \mathbf{v} = (v_1, v_2)^T$.

Weak Formulation

On each element $K \in \mathcal{K}_h$: Multiplying with test function φ and integration by parts on each element K.
Summation over all elements K:

$$\underbrace{\int_{\mathcal{K}_h} -(\mathbf{v}u)\cdot\nabla\varphi + \sigma u\varphi \,\mathrm{d}\mathbf{x} + \int_{\mathcal{S}_h^{\mathrm{int}}} v_n\hat{u}[\varphi]\,\mathrm{d}\mathbf{s} + \int_{\mathcal{S}_h^{\partial,+}} v_n u\varphi\,\mathrm{d}\mathbf{s}}_{a(u,\varphi)} = \underbrace{\int_{\mathcal{K}_h} f\varphi\,\mathrm{d}\mathbf{x} - \int_{\mathcal{S}_h^{\partial,-}} v_n g\varphi\,\mathrm{d}\mathbf{s}}_{f(\varphi)}\,.$$

where $\hat{u}$ is the upwind value, $v_n = \mathbf{v}\cdot\mathbf{n}$, $\mathcal{S}^{\mathrm{int}}$ interior sides, $\mathcal{S}^{\partial}$ boundary sides.

Stability[1]

Lemma

There exists a mesh-independent constant $\gamma > 0$ such that

$$\sup_{v_h \in V_h^{1,2} \setminus \{0\}} \frac{a(u_h, v_h)}{|||v_h|||} \geq \gamma\, |||u_h|||.$$

$$\|u_h\|_{\mathrm{DG}} := \sqrt{\sigma_0 \|u_h\|^2 + \frac{1}{2}\int_{\partial\Omega} |v_n| u_h^2 + \frac{1}{2}\int_{\mathcal{S}_h} |v_n| [u_h]^2},$$

$$|||u_h|||_{\mathbf{v}} := \sqrt{\sum_{K \in \mathcal{K}_h} \delta_K \|\mathbf{v} \cdot \nabla u_h\|_K^2}, \quad \delta_K = \frac{h_K}{\|\mathbf{v}\|_{\infty,K}},$$

$$|||u_h||| := \sqrt{\|u_h\|_{\mathrm{DG}} + |||u_h|||_{\mathbf{v}}^2}.$$

[1] R. Becker, M. Bittl, and D. Kuzmin. "Analysis of a combined CG1-DG2 method for the transport equation". In: *SINUM, submitted* (september 2013).

A priori estimate[2]

Theorem

Let $h := \max_{K\in\mathcal{K}_h} h_K$. *Let* $0 \leq k \leq 2$ *and* $u \in H^{k+1}(\Omega)$.
Then we have the a priori error estimate

$$|||u - u_h||| \leq C\, h^{k+1/2}\, \|u\|_{H^{k+1}(\Omega)}$$

[2] R. Becker, M. Bittl, and D. Kuzmin. "Analysis of a combined CG1-DG2 method for the transport equation". In: *SINUM, submitted* (september 2013).

Numerical examples: Stationary convection problem

$$u + \nabla \cdot (\mathbf{v}u) = f \quad \text{in } \Omega = (0,1) \times (0,1)$$

$$\mathbf{v}(x,y) = (1,1).$$

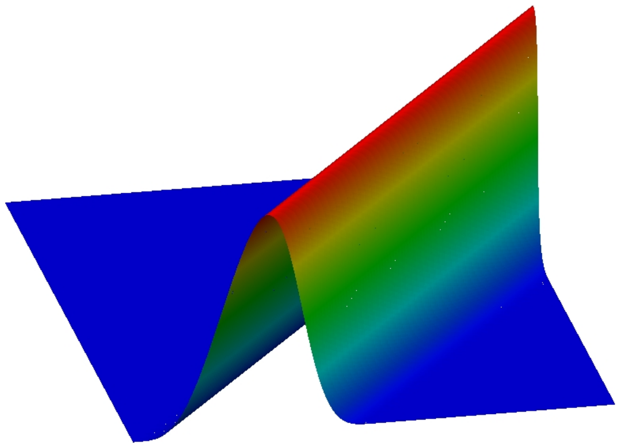


Figure : exact solution

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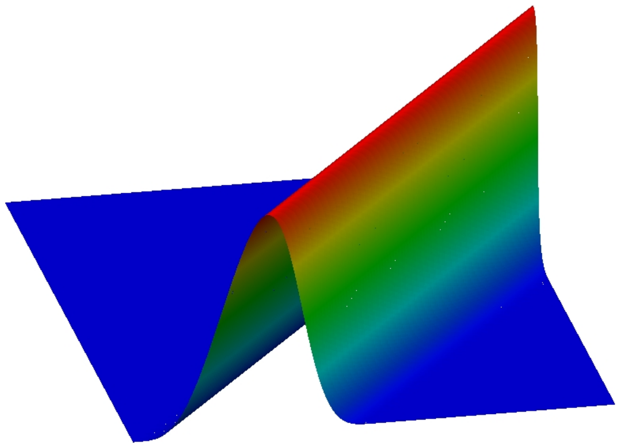

Figure : exact solution

h	CG2 $\lvert\lvert\lvert u - u_h \rvert\rvert\rvert$	CG1-DG2 $\lvert\lvert\lvert u - u_h \rvert\rvert\rvert$	DG2 $\lvert\lvert\lvert u - u_h \rvert\rvert\rvert$
1/32	4.45e-02	4.11e-02	1.51e-02
1/64	1.52e-02	9.74e-03	2.67e-03
1/128	5.34e-03	1.95e-03	4.72e-04
1/256	1.88e-03	3.60e-04	8.33e-05
EOC	1.50	2.44	2.50

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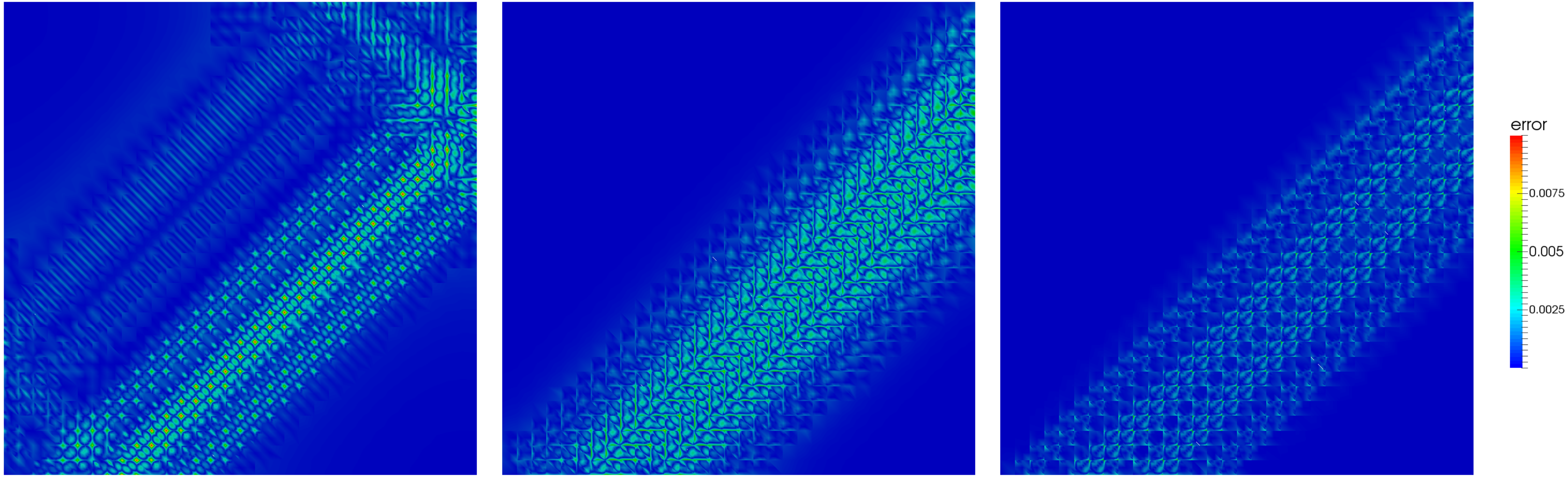


(a) Error for CG2 (b) Error for CG1-DG2 (c) Error for DG2

Stationary convection problem: DOFs

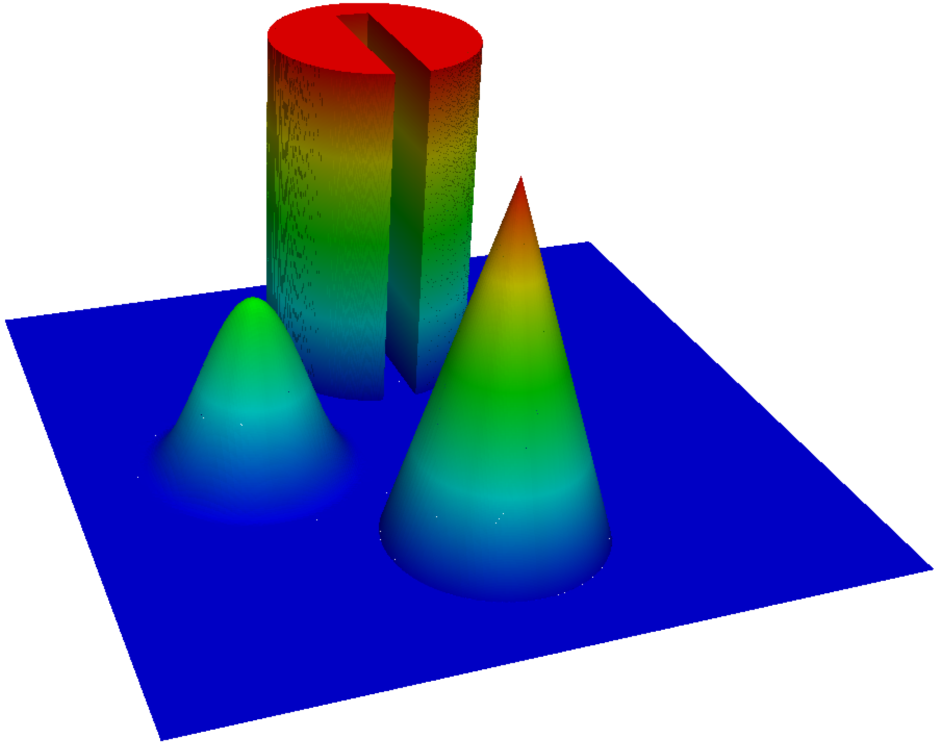
h	CG1	CG2	CG1-DG2	DG1	DG2
1/32	1089	4225	7233	6144	12288
1/64	4225	16641	28801	24576	49152
1/128	16641	66049	114945	98304	196608
1/256	66049	263169	459265	393216	786432

Table : DOFs on triangular meshes for the unit square

Solid body rotation

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) = 0 \quad \text{in } \Omega = (0,1) \times (0,1)$$

$$\mathbf{v}(x,y) = (0.5 - y, x - 0.5)$$



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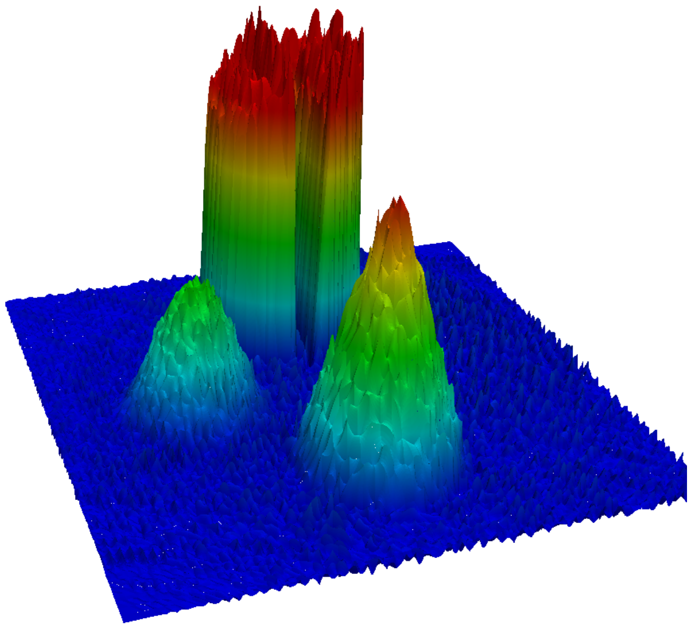
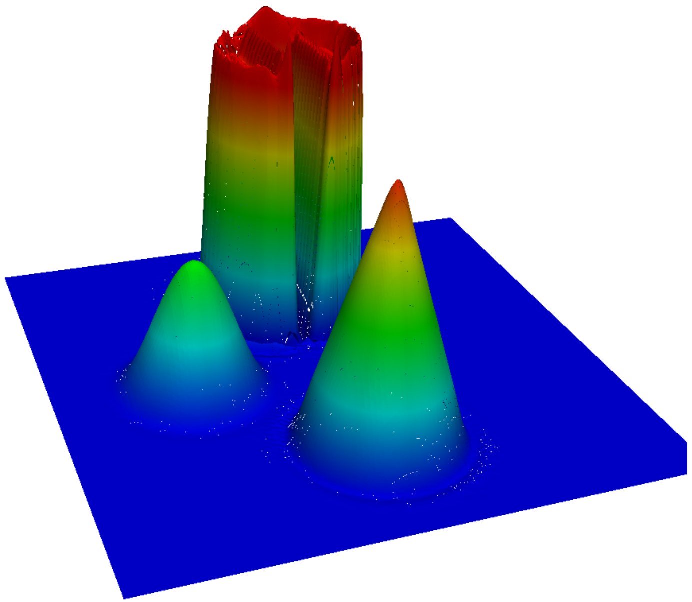
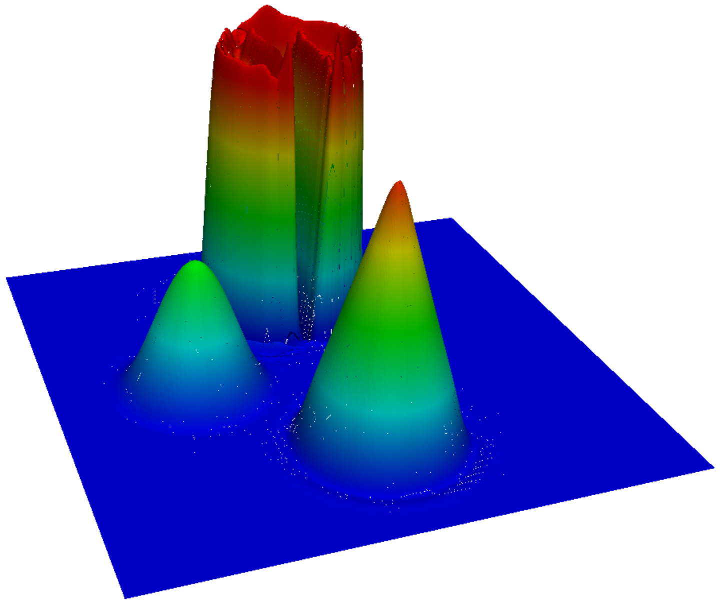


(a) CG2

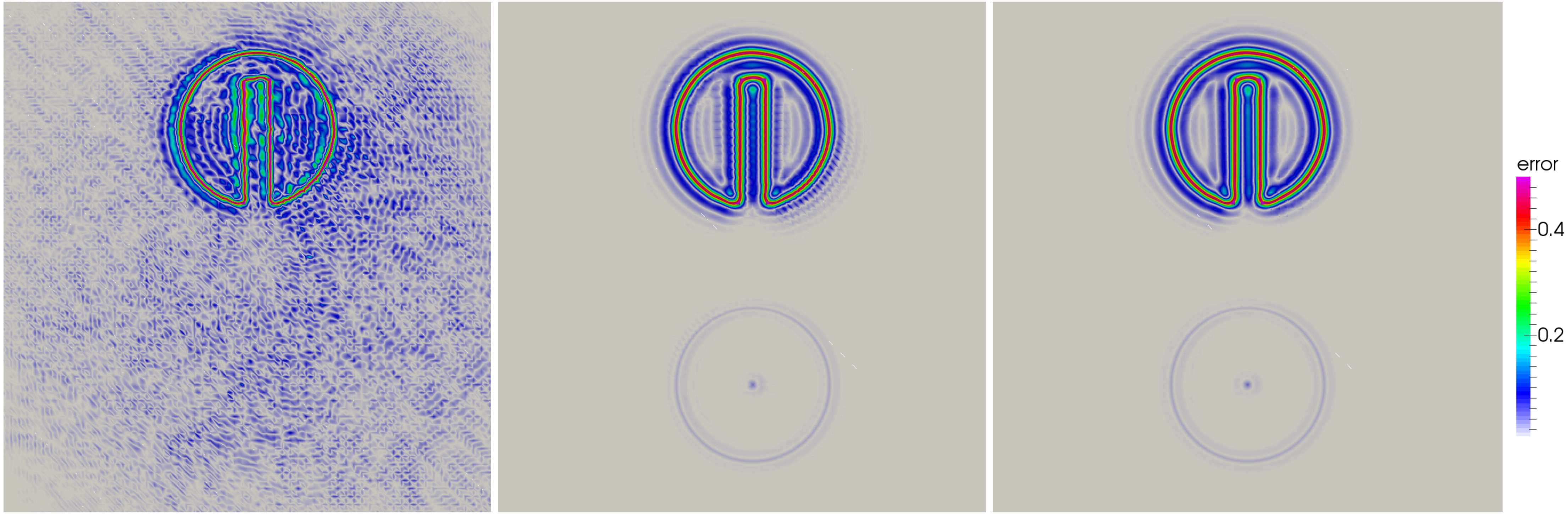


(b) CG1-DG2



(c) DG2

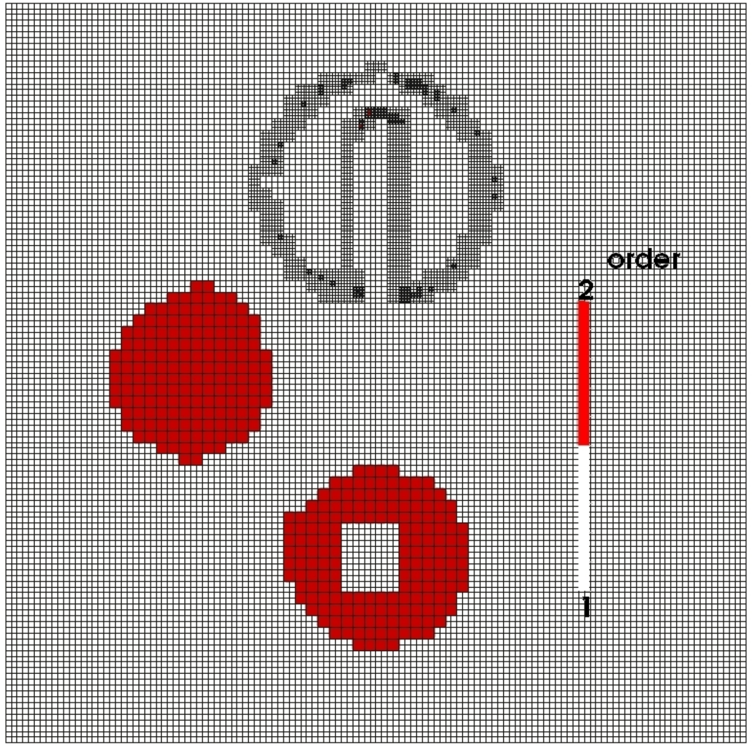
Solid body rotation - error

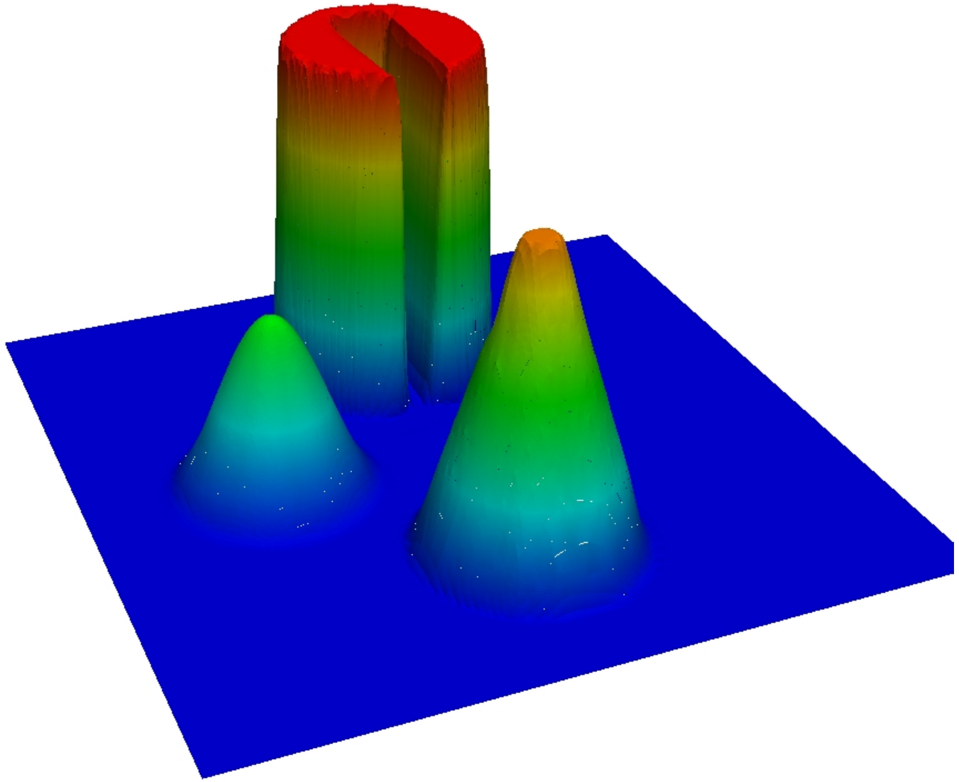


(a) Error for CG2 (b) Error for CG1-DG2 (c) Error for DG2

Solid body rotation - hp-adaptivity

CG1+FCT (non-smooth) and CG1-DG2 (smooth)





Summary & Outlook

We introduced the CG1-DG2 method:

1. Stability
2. A priori error estimate
3. Numerical studies

In the future:

- Euler equations
- Inviscid 2-fluid model

Thank you for your attention!