On Solving Frictional Contact Problems Part I: Abstract Framework and the Static Case.

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This paper presents mixed variational formulation and its discretization with finite elements of higher-order for Signorini's problem with Tresca's friction. To guarantee the unique existence of the discrete saddle point of the mixed method, a discrete inf-sup condition is proven. Moreover, a solution scheme based on the dual formulation of the mixed method is proposed. Numerical results confirm the theoretical findings.

1 Introduction

This paper deals with finite element methods of higher-order for Signorini's problem with Tresca's friction, which plays an important role in mechanical engineering [14, 15, 24]. The discretization approach is based on a mixed variational formulation. For lower-order finite elements, this approach was introduced by Haslinger et al. in [16, 18, 21]. In this paper, we extend it to higher-order finite elements. The approach relies on a saddle point formulation where the geometrical contact condition and the frictional condition are captured by Lagrange multipliers. The constraints for the Lagrange multipliers are sign conditions and box constraints and are, therefore, simpler than the original contact conditions. However, the Lagrange multipliers are additional variables which also have to be discretized. In mixed variational formulations, unique

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existence of the discrete saddle point usually follows from an inf-sup condition associated to the discretization spaces. Its verification is often a crucial point. For lower-order finite elements, the inf-sup condition is proven in the above mentioned references. In this work, we prove the inf-sup condition for higher-order finite elements for Signorini's problem with Tresca's friction. We use approximation results for the p-method of finite elements, and some inverse estimates for higher-order polynomials, [1,11]. The key is to use a discretization of the Lagrange multipliers on boundary meshes with a larger mesh size than that of the primal variable and, moreover, different polynomial degrees for the primal variable and Lagrange multipliers.
In general, higher-order discretization schemes for contact problems are rarely studied in literature, especially for mixed variational formulation. For discretization techniques based on a primal, non-mixed formulation, we refer to [26,27].

This paper is organized as follows: To motivate the subject, to show the analytical background behind and, in particular, to introduce the discrete inf-sup condition, we briefly summarize the main arguments of convex analysis for the derivation of a mixed variational formulation in Section 2. If necessary, some of the proofs are given in the appendix. In Section 3, we apply the abstract framework to obtain a mixed variational formulation for Signorini's problem with Tresca's friction and to introduce a higher-order finite element discretization. In Section 4, we consider some simplifications of Signorini's problem and also assert them to the abstract framework of Section 2. The main part of this work, the derivation of the inf-sup condition for higher-order finite elements, is proposed in Section 5.
The second focus of this work is to present a solution scheme to solve the discrete mixed variational formulation. The scheme is based on a dual variational formulation leading to a minimization problem in terms of the Lagrange multipliers. It follows the same line as in the approach presented in [17,19,20]. In Section 6, we extend it to the higher-order approach. Furthermore, we discuss an extension of the solution scheme to time-dependent problems in Section 7. Numerical results confirming the theoretical findings are presented in Section 8.

2 General remarks on mixed variational formulations

Frictional contact problems can be captured by the minimization problem

$$(H+j)(u) = \min_{v\in K}(H+j)(v). \tag{1}$$

Here, K is a subset of a reflexive Banach space V and $H, j : V \to \mathbb{R}$. The special choice for V, H and j in the context of contact problems with friction will become clear in Section 3, below. The following results are well-known, their proofs can be found, for instance, in [5,10,24].

Theorem 1 *Let K be convex.*

(i) If K is closed and $H+j$ is weakly lower semicontinuous and coercive, then there exists a minimizer $u\in K$ of (1).
(ii) If $H+j$ is strictly convex, (1) *admits at most one minimizer.*

(iii) Let H be Fréchet-differentiable in $u \in K$ with the Fréchet-derivative $H' : V \to V'$. If u is a minimizer of (1) *and j is convex, then*

$$\langle H'(u), v-u\rangle + j(v) - j(u) \geq 0 \tag{2}$$

for all $v \in K$. If H is convex and (2) *holds, then u is a minimizer of* (1).

To derive a mixed variational formulation, we resolve the condition $v \in K$ and the functional j by using Lagrange multipliers. To this end, let $\Phi_i : V \times \Lambda_i \to \mathbb{R}$, $i = 0, 1$, fulfill

$$\sup_{\mu_0 \in \Lambda_0} \Phi_0(v, \mu_0) = \begin{cases} 0, & v \in K \\ \infty, & v \notin K \end{cases} \tag{3}$$

and

$$j(v) = \sup_{\mu_1 \in \Lambda_1} \Phi_1(v, \mu_1) \tag{4}$$

for all $v \in V$ with $\Lambda_i \subset U_i'$ and reflexive Banach spaces U_i'. Obviously, it holds

$$(H + j)(u) = \inf_{v \in V} \sup_{\mu_0 \in \Lambda_0, \mu_1 \in \Lambda_1} \mathscr{L}(v, \mu_0, \mu_1)$$

with the Lagrange functional $\mathscr{L}(v, \mu_0, \mu_1) := H(v) + \Phi_0(v, \mu_0) + \Phi_1(v, \mu_1)$. Therefore, u is a minimizer of (1), whenever the triple $(u, \lambda_0, \lambda_1) \in V \times \Lambda_0 \times \Lambda_1$ is a saddle point,

$$\mathscr{L}(u, \lambda_0, \lambda_1) = \inf_{v \in V} \sup_{\mu_0 \in \Lambda_0, \mu_1 \in \Lambda_1} \mathscr{L}(v, \mu_0, \mu_1). \tag{5}$$

Defining $\Phi_{i,\mu_i}(v) := \Phi_i(v, \mu_i)$ and $\Phi_{i,v}(\mu_i) := \Phi_i(v, \mu_i)$ and applying Theorem 1, we immediately obtain

Theorem 2 *Let K, Λ_0 and Λ_1 be convex. Moreover, let H, Φ_{0,λ_0}, Φ_{1,λ_1} be Fréchet-differentiable in $u \in V$ and $\Phi_{0,u}$, $\Phi_{1,u}$ in $\lambda_0 \in \Lambda_0$ and $\lambda_1 \in \Lambda_1$.*

(i) If $(u, \lambda_0, \lambda_1)$ is a saddle point, then

$$\begin{aligned} &(H' + \Phi'_{0,\lambda_0} + \Phi'_{1,\lambda_1})(u) = 0, \\ &\langle \Phi'_{0,u}(\lambda_0), \mu_0 - \lambda_0\rangle + \langle \Phi'_{1,u}(\lambda_1), \mu_1 - \lambda_1\rangle \leq 0 \end{aligned} \tag{6}$$

for all $(\mu_0, \mu_1) \in \Lambda_0 \times \Lambda_1$.

(ii) If H, Φ_{0,λ_0}, Φ_{1,λ_1}, $-\Phi_{0,u}$, and $-\Phi_{1,u}$ are convex and (6) *holds, then $(u, \lambda_0, \lambda_1)$ is a saddle point.*

The existence of a saddle point is stated in the following theorem,

Theorem 3 *Let Λ_0 and Λ_1 be closed and convex. Furthermore, let the following conditions hold:*

(i) $-\Phi_{0,v}$ and $-\Phi_{1,v}$ are convex and weakly lower semicontinuous for all $v \in V$,
(ii) H, Φ_{0,μ_0} and Φ_{1,μ_1} are convex and weakly lower semicontinuous for all $(\mu_0, \mu_1) \in \Lambda_0 \times \Lambda_1$,
(iii) There exists a $(\hat{\mu}_0, \hat{\mu}_1) \in \Lambda_0 \times \Lambda_1$, so that $H + \Phi_{0,\hat{\mu}_0} + \Phi_{1,\hat{\mu}_1}$ is coercive.
(iv) $\Lambda_0 \times \Lambda_1$ is bounded or $(\mu_0, \mu_1) \mapsto \sup_{v \in V} -\mathscr{L}(v, \mu_0, \mu_1)$ is coercive.

Then, there exists a saddle point $(u,\lambda_0,\lambda_1)\in V\times\Lambda_0\times\Lambda_1$ *of* (5).

See Remark IV.2.1 and Prop IV.2.3 in [10] for a proof. A simple criterion for condition (3) is given by the following assertion.

Lemma 1 *Let* Λ_0 *be a cone with vertex at the origin and let* $\Phi_0: V\times\Lambda_0\to\mathbb{R}$ *fulfill*

$$\forall\alpha\geq 0,\forall(v,\mu_0)\in V\times\Lambda_0:\ \Phi_0(v,\alpha\mu_0)=\alpha\Phi_0(v,\mu_0), \tag{7}$$

$$v\in K\Leftrightarrow\forall\mu_0\in\Lambda_0:\ \Phi_0(v,\mu_0)\leq 0. \tag{8}$$

Then, Φ_0 *also fulfill* (3).

In the following, let a be a symmetric, continuous and V-elliptic bilinear form and $\ell\in V'$. Furthermore, let U_i be reflexive Banach spaces, $\Lambda_1\subset U_1'$ be closed, convex and bounded, $\beta_i\in L(V,U_i)$, $G\subset U_0$ be a closed and convex cone with vertex at the origin and $g\in U_0$. We consider the class of minimimization problems which is defined by

$$H(v):=\frac{1}{2}a(v,v)-\langle\ell,v\rangle,\quad j(v):=\sup_{\mu_1\in\Lambda_1}\langle\mu_1,\beta_1(v)\rangle, \tag{9}$$
$$K:=\{v\in V\mid g-\beta_0(v)\in G\}.$$

Note that j is well-defined due to Theorem 1. Moreover, H is convex, continuous and, therefore, weakly semicontinuous. Due to its ellipticity, H is strictly convex. The set K is closed and convex, and the functional j is convex and lower semicontinuous. As a consequence of the closedness and convexity of the epigraph $\operatorname{epi}(j)$ (Prop I.2,3 in [10]) and the separation theorem of Hahn-Banach, there exist a $\phi\in V'$ and a $c\in\mathbb{R}$ such that $j(v)\geq\langle\phi,v\rangle+c$. Therefore, $(H+j)(v)\geq\gamma\|v\|^2-(\|\ell\|+\|\phi\|)\|v\|+c$ which implies that $H+j$ is coercive. Due to its convexity and lower semicontinuity, $H+j$ is weakly lower semicontinuous. Applying Theorem 1 yields

Theorem 4 *There exists a unique minimizer.*

Let G' denote the dual cone of G which is defined by $G':=\{\mu_0\in U_0'\mid\forall v\in G:\langle\mu_0,v\rangle\geq 0\}$. Moreover, let $\Lambda_0:=G'$.

Theorem 5 *The triple* $(u,\lambda_0,\lambda_1)\in V\times U_0\times U_1$ *is a saddle point if and only if,*

$$\begin{aligned}&a(u,v)=\langle\ell,v\rangle-\langle\lambda_0,\beta_0(v)\rangle-\langle\lambda_1,\beta_1(v)\rangle,\\&\langle\mu_0-\lambda_0,\beta_0(u)-g\rangle+\langle\mu_1-\lambda_1,\beta_1(u)\rangle\leq 0\end{aligned} \tag{10}$$

for all $v\in V$ *and* $(\mu_0,\mu_1)\in\Lambda_0\times\Lambda_1$.

Theorem 6 *There exists a saddle point* $(u,\lambda_0,\lambda_1)\in V\times\Lambda_0\times\Lambda_1$*, if there exists an* $\alpha\in\mathbb{R}_{>0}$ *such that*

$$\alpha\|\mu_0\|_{U_0'}\leq\sup_{v\in V,\,\|v\|=1}\langle\mu_0,\beta_0(v)\rangle. \tag{11}$$

for all $\mu_0\in U_0'$.

Remark 1 It is easy to see, that the Lagrange multipliers λ_0 and λ_1 are unique if $\beta_1(\ker\beta_0)$ is dense in U_1.

Remark 2 Condition (11) is fulfilled, if the mapping β_0 is surjective. This is a direkt consequence of the closed range theorem, cf. [34].

Remark 3 If $G = U_0$, then $G' = \{0\}$, and we can omit all terms in (10) concerning λ_0. If $\Lambda_1 = \{0\}$ or $\beta_1 := 0$, all terms in (10) concerning λ_1 can be omitted.

Let $V_h \subset V$, $U'_{0,H} \subset U'_0$ and $U'_{H,1} \subset U'_1$ be finite dimensional subspaces and $\Lambda_{i,H} \subset U'_{i,H}$, $i = 0,1$, where $\Lambda_{0,H}$ is a closed and convex cone with vertex at the origin and $\Lambda_{1,H}$ is closed, convex and bounded. The discrete saddle problem consists in finding a triple $(u_h, \lambda_{0,H}, \lambda_{1,H}) \in V_h \times \Lambda_{0,H} \times \Lambda_{1,H}$ such that

$$\mathscr{L}(u_h, \lambda_{0,H}, \lambda_{1,H}) = \inf_{v_h \in V_h} \sup_{\mu_{0,H} \in \Lambda_{0,H}, \mu_{1,H} \in \Lambda_{1,H}} \mathscr{L}(v_h, \mu_{0,H}, \mu_{1,H}). \tag{12}$$

It is easy to see that the first component is the unique minimizer of the minimization problem $(H + j_{hH})(u_h) = \min_{v_h \in K_{hH}} (H + j_{hH})(v_h)$ with $K_{hH} := \{v_h \in V_h \mid \forall \mu_{0,H} \in \Lambda_{0,H} : \langle \mu_{0,H}, \beta_0(v_h) - g\rangle \leq 0\}$ and $j_{hH} := \sup_{\mu_{1,H} \in \Lambda_{1,H}} \langle \mu_{1,H}, \beta_1(v_h)\rangle$. Furthermore, $(u_h, \lambda_{0,H}, \lambda_{1,H}) \in V_h \times \Lambda_{0,H} \times \Lambda_{1,H}$ is a discrete saddle point if and only if

$$\begin{aligned} &a(u_h, v_h) = \langle \ell, v_h\rangle - \langle \lambda_{0,H}, \beta_0(v_h)\rangle - \langle \lambda_{1,H}, \beta_1(v_h)\rangle, \\ &\langle \mu_{0,H} - \lambda_{0,H}, \beta_0(u_h) - g\rangle + \langle \mu_{1,H} - \lambda_{1,H}, \beta_1(u_h)\rangle \leq 0 \end{aligned} \tag{13}$$

for all $v_h \in V_h$ and $(\mu_{0,H}, \mu_{1,H}) \in \Lambda_{0,H} \times \Lambda_{1,H}$. The first component u_h is uniquely determined.

Theorem 7 *There exists a discrete saddle point, if $g \in \beta_0(V_h)$.*

Since uniqueness of the Lagrange multipliers is not guaranteed, Theorem 7 is somewhat unsatisfactory. Furthermore, the existence depends on the assumption $g \in \beta_0(V_h)$ which is not fulfilled in general. The proof of the theorem is based on the closedness of $\beta_0(V_h)$ as a finite dimensional subspace of U_1 which enforces us to consider a saddle point problem in quotient spaces (see the proof in the appendix). Of course, it is more natural to consider a saddle point problem in the discretization space directly.

Theorem 8 *Let $\tilde{U}'_1$ be a Banach space and U'_1 be a dense subspace of $\tilde{U}'_1$. Assume that there exists an $\alpha \in \mathbb{R}_{>0}$ such that*

$$\alpha \|(\mu_{0,H}, \mu_{1,H})\|_{U'_0 \times \tilde{U}'_1} \leq \sup_{v_h \in V_h, \|v_h\|=1} \left(\langle \mu_{0,H}, \beta_0(v_h)\rangle + \langle \mu_{1,H}, \beta_1(v_h)\rangle\right) \tag{14}$$

for all $(\mu_{0,H}, \mu_{1,H}) \in U'_{0,H} \times U'_{1,H}$, then there exists a unique discrete saddle point.

To proof the inf-sup condition (14), we will make use of the following general result:

Lemma 2 *Let $\hat{a}$ be a continuous and V-elliptic bilinear form on $V \times V$ and let $\beta \in L(V,U)$ be a surjective mapping onto the Banach space U. For $\mu \in U'$, there exists a unique $u^\mu \in V$ such that*

$$\hat{a}(u^\mu, v) = \langle \mu, \beta(v)\rangle \tag{15}$$

for all $v \in V$. Additionally, there holds $C_1\|\mu\|_U \leq \|u^\mu\|_V$ for some constant $C_1 > 0$.

3 Signorini's problem with Tresca's friction and its higher-order finite element discretizations

Let $\Omega \subset \mathbb{R}^k$, $k \in \mathbb{N}$, be a domain with sufficiently smooth boundary $\Gamma := \partial\Omega$. Moreover, let $\Gamma_D \subset \Gamma$ be closed with positive measure and let $\Gamma_C \subset \Gamma\backslash\Gamma_D$ with $\overline{\Gamma}_C \subsetneq \Gamma\backslash\Gamma_D$. $L^2(\Omega)$, $H^k(\Omega)$ with $k \geq 1$, and $H^{1/2}(\Gamma_C)$ denote the usual Sobolev spaces and $H^1_D(\Omega) := \{v \in H^1(\Omega) \mid \gamma(v) = 0 \text{ on } \Gamma_D\}$ with the trace operator γ. The space $H^{-1/2}(\Gamma_C)$ denotes the topological dual space of $H^{1/2}(\Gamma_C)$ with the norms $\|\cdot\|_{-1/2,\Gamma_C}$ and $\|\cdot\|_{1/2,\Gamma_C}$, respectively. Let $(\cdot,\cdot)_{0,\omega}$, $(\cdot,\cdot)_{0,\Gamma'}$ be the usual L^2-scalar products on $\omega \subset \Omega$ and $\Gamma' \subset \Gamma$, respectively. We define the gradient operator ∇ in the weak sense. Note that the linear and bounded mapping $\gamma_C := \gamma_{|\Gamma_C} : H^1_D(\Omega) \to H^{1/2}(\Gamma_C)$ is surjective due to the assumptions on Γ_C, cf. [24]. For functions in $L^2(\Omega)$ or $L^2(\Gamma_C)$, the inequality symbols $\geq$ and $\leq$ are defined as "almost everywhere". We set $H^{1/2}_{\pm}(\Gamma_C) := \{v \in H^{1/2}(\Gamma_C) \mid \pm v \geq 0\}$ and $L^2_l(\Gamma_C) := \{\mu \in (L^2(\Gamma_C))^l \mid |\mu| \leq 1 \text{ on } \operatorname{supp} s,\ v = 0 \text{ on } \Gamma_C \backslash \operatorname{supp} s\}$ with the euclidian norm $|\cdot|$ and $s \in L^2(\Gamma_C)$, $s \geq 0$. Furthermore, we define the dual cones $H^{-1/2}_{\pm}(\Gamma_C) := (H^{1/2}_{\pm}(\Gamma_C))'$.

We propose a higher-order finite element discretization based on quadrangles or hexahedrons as follows: Let $\mathscr{T}$ be a finite element mesh of Ω with mesh size h and let $\mathscr{T}_C$ be a finite element mesh of Γ_C with mesh size H. We assume that a submesh of $\mathscr{T}_C$ is a mesh of $\operatorname{supp} s$. Furthermore, let $\Psi_T : [-1,1]^k \to T \in \mathscr{T}$ and $\Psi_{C,T} : [-1,1]^{k-1} \to T \in \mathscr{T}_C$ be bijective and sufficiently smooth transformations and let $p_T \in \mathbb{N}$ be a degree distribution on $\mathscr{T}$ and $q_T \in \mathbb{N}$ be ones on $\mathscr{T}_C$. Using the polynomial tensor product space S^r_k of order r on the reference element $[-1,1]^k$, we define

$$\mathscr{S}^p_h := \left\{v_h \in H^1_D(\Omega) \mid \forall T \in \mathscr{T} :\ v_{h|T} \circ \Psi_T \in S^{p_T}_k\right\},$$
$$\mathscr{M}^q_H := \left\{\mu \in L^2(\Gamma_C) \mid \forall T \in \mathscr{T}_C :\ \mu_{|T} \circ \Psi_{C,T} \in S^{q_T}_{k-1}\right\}.$$

For a finite subset $M \subset [-1,1]^k$, we define

$$\mathscr{M}^q_{H,\pm} := \{\mu \in \mathscr{M}^q_H \mid \forall T \in \mathscr{T}_C : \forall x \in M :\ \pm\mu(\Psi_{C,T}(x)) \geq 0\},$$
$$\mathscr{M}^q_{H,l} := \{\mu \in (\mathscr{M}^q_H)^l \mid \forall T \in \mathscr{T}_C, T \subset \operatorname{supp} s : \forall x \in M :\ |\mu(\Psi_{C,T}(x))| \leq 1,$$
$$\mu = 0 \text{ on } \Gamma_C \backslash \operatorname{supp} s\}.$$

Contact problems in mechanical engineering with small deformations are often modelled by *Signorini's problem with Tresca's friction* where a linear elastic material law is used to describe the deformation of elastic bodies through linearized stress and strain tensors. We consider a body which is described by $\overline{\Omega} \subset \mathbb{R}^k$, $k \in \{2,3\}$. The body is clamped at the boundary part Γ_D, volume and surface forces given by functions $f \in (L^2(\Omega))^k$ and $b \in (L^2(\Gamma_N))^{k-1}$ with $\Gamma_N \subset \Gamma\backslash(\Gamma_D \cup \overline{\Gamma}_C)$ act on the body leading to a deformation. For the displacement field v we define the strain tensor $\varepsilon(v) := \frac{1}{2}(\nabla v + (\nabla v)^\top)$ and the stress tensor $\sigma(v)_{ij} := \mathscr{C}_{ijkl}\varepsilon(v)_{kl}$ where $\mathscr{C}_{ijkl} \in L^\infty(\Omega)$ with $\mathscr{C}_{ijkl} = \mathscr{C}_{jilk} = \mathscr{C}_{klij}$ and $\mathscr{C}_{ijkl}\tau_{ij}\tau_{kl} \geq \kappa\tau^2_{ij}$ for $\tau \in L^2(\Omega)^{k\times k}_{\text{sym}}$ and a $\kappa > 0$. We assume that Γ_C and the section of the obstacle's surface which possibly gets in contact are parameterized by sufficiently smooth functions $\psi, \varphi : \mathbb{R}^{k-1} \to \mathbb{R}$. Provided that the body is located under the obstacle, we obtain

$$\varphi(x) + v_3(x, \varphi(x)) \le \psi(x_1 + v_1(x, \varphi_1(x)), \ldots, x_{k-1} + v_2(x, \varphi(x))) \qquad (16)$$

with $x := (x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1}$. In general, the geometrical contact condition (16) is non-linear. An appropriate linearization is introduced in [24] by $g - v_n \ge 0$ with $g(x, \psi(x)) := (\psi(x) - \varphi(x))(1 + (\nabla\phi(x))^\top \nabla\phi(x))^{-1/2}$ with the outer normal n.
Frictional contact conditions can be introduced assuming that sliding does not occur if the magnitude of the tangential forces is below a critical value described by a frictional function $s \in L^2(\Gamma_C)$ with $s \ge 0$. If the tangential forces reach this critical value, sliding is obtained in the direction of the tangential forces. Such Tresca friction can be extended to Coulomb's friction setting s to the magnitude of the normal forces times a friction coefficient and integrating the problem into a fixed point scheme, see Section 6. Taking the linearized geometrical as well as frictional contact conditions into account, Signorini's problem with Tresca's friction is to find a displacement field $u \in W := \{v \in (H^1(\Omega))^k \mid \sigma(v) \in H(\mathrm{div}, \Omega),\ u = 0 \text{ on } \Gamma_D\}$ such that

$$\begin{aligned}
-\operatorname{div}(\sigma(u)) = f \text{ in } \Omega, \quad \sigma_n(u) = b \text{ on } \Gamma_N, \\
u_n - g \le 0,\ \sigma_{nn}(u) \le 0,\ \sigma_{nn}(u)(u_n - g) = 0 \text{ on } \Gamma_C, \\
|\sigma_{nt}(u)| \le s \text{ with } \left\{ \begin{array}{l} |\sigma_{nt}(u)| < s \Rightarrow u_t = 0, \\ |\sigma_{nt}(u)| = s \Rightarrow \exists \zeta \in \mathbb{R}_{\ge 0} :\ u_t = -\zeta \sigma_{nt}(u) \end{array} \right\} \text{ on } \Gamma_C.
\end{aligned}$$

Here, t denotes the matrix containing the tangential vectors and $\sigma_{n,j} := \sigma_{ij} n_i$, $\sigma_{nn} := \sigma_{ij} n_i n_j$, $\sigma_{nt,k} := \sigma_{ij} n_i t_{jk}$, $u_n := u_i n_i$ and $u_{t,j} := u_i t_{ij}$.
The function $u \in W$ is a solution if and only if the variational inequality

$$(\sigma(u), \varepsilon(v - u))_0 + (s, |\gamma_t(v)| - |\gamma_t(u)|)_{0,\Gamma_C} \ge (f, v - u)_0 + (b, \gamma_N(v - u))_{0,\Gamma_N} \qquad (17)$$

is fulfilled for all $v \in K := \{v \in H^1(\Omega, \Gamma_D)^k \mid g - \gamma_n(v) \ge 0\}$, cf. [8]. Here, we define $\gamma_n(v) := \gamma_C(v_i) n_i$, $\gamma_t(v)_j := \gamma_C(v_i) t_{ij}$ and $\gamma_N := \gamma_{|\Gamma_N}$. Using the notation of Section 2, we set $V := (H^1_D(\Omega))^k$, $\beta_0 := \gamma_n$, $U_0 := H^{1/2}(\Gamma_C)$, $G := H^{1/2}_+(\Gamma_C)$ and $\langle \ell, v \rangle := (f, v)_0 + (b, \gamma_N(v))_{0,\Gamma_N}$. Furthermore, we define the bilinear form a as $a(v, w) := (\sigma(v), \varepsilon(w))_0$ which is symmetric, continuous, and, due to Korn's inequality, elliptic. It is easy to see, that $j(v) := (s, |\gamma_t(v)|)_{0,\Gamma_C}$ is continuous, convex and can be expressed through $j(v) = \sup_{\mu \in \Lambda_1} (\mu_1, s\gamma_t(v))_{0,\Gamma_C}$ with $\Lambda_1 := L^2_{k-1}(\Gamma_C)$ (see Section 4). Setting $\beta_1 := s\gamma_t$, $U_1 := (L^2(\Gamma_C))^{k-1}$ and applying the results of Section 2, we obtain u as the unique minimizer of (1). Again, from Lemma 6 and Remark 2, we obtain a unique saddle point $(u, \lambda_0, \lambda_1) \in (H^1_D(\Omega))^k \times H^{1/2}_+(\Gamma_C) \times L^2_{k-1}(\Gamma_C)$ which is equivalently characterized by the mixed variational formulation

$$\begin{aligned}
&(\sigma(u), \varepsilon(v))_0 = (f, v)_0 + (b, \gamma_N(v))_{0,\Gamma_N} - \langle \lambda_0, \gamma_n(v) \rangle - (\lambda_1, \gamma_t(v))_{0,\Gamma_C}, \\
&\langle \mu_0 - \lambda_0, \gamma_n(v) - g \rangle + (\mu_1 - \lambda_1, s\gamma_t(u))_{0,\Gamma_C} \le 0
\end{aligned}$$

for all $v \in (H^1_D(\Omega))^k$ and $(\mu_0, \mu_1) \in H^{1/2}_+(\Gamma_C) \times L^2_{k-1}(\Gamma_C)$. The discretization is to find $(u_h, \lambda_{0,H}, \lambda_{1,H}) \in S^p_h \times \mathscr{M}^q_{H,+} \times \mathscr{M}^q_{H,k-1}$ such that

$$\begin{aligned}
&(\sigma(u_h), \varepsilon(v_h))_0 = (f, v_h)_0 + (b, \gamma_N(v_h))_{0,\Gamma_N} - (\lambda_{0,H}, \gamma_n(v_h))_{0,\Gamma_C} - (\lambda_{1,H}, \gamma_t(v_h))_{0,\Gamma_C}, \\
&(\mu_{0,H} - \lambda_{0,H}, \gamma_n(u_h) - g)_{0,\Gamma_C} + (\mu_{1,H} - \lambda_{1,H}, s\gamma_t(u_h))_{0,\Gamma_C} \le 0
\end{aligned}$$

for all $v \in (\mathscr{S}_h^p)^k$ and $(\mu_{0,H}, \mu_{1,H}) \in \mathscr{M}_{H,+}^q \times \mathscr{M}_{H,k-1}^q$. Note that both the geometrical obstacle function g and frictional function s are included in this formulation in a weak sense.

If the contact area and normal force known a priori, Signorini's problem can be simplified to *Signorini's problem with prescribed normal force* which is to find a displacement field $u \in W$ such that

$$-\operatorname{div}(\sigma(u)) = f \text{ in } \Omega, \quad \sigma_n(u) = q \text{ on } \Gamma_N, \quad \sigma_{nn}(u) = s \text{ on } \Gamma_C,$$
$$|\sigma_{nt}(u)| \leq s \text{ with } \left\{ \begin{array}{l} |\sigma_{nt}(u)| < s \Rightarrow u_t = 0, \\ |\sigma_{nt}(u)| = s \Rightarrow \exists \zeta \in \mathbb{R}_{\geq 0} : \; u_t = -\zeta \sigma_{nt}(u) \end{array} \right\} \text{ on } \Gamma_C.$$

The function $u \in W$ is a solution, if and only if the variational inequality (17) is fulfilled with $K := (H_D^1(\Omega))^k$. We use the same notation as for Signorini's problem with Tresca's friction, but here, we set $G := H^{1/2}(\Gamma_C)$. Due to the results of Section 2, we obtain u as the unique minimizer of (1). A unique saddle point $(u, \lambda_1) \in (H_D^1(\Omega))^k \times L_{k-1}^2(\Gamma_C)$ is equivalently characterized by the mixed variational formulation

$$\begin{aligned} &(\sigma(u), \varepsilon(v))_0 = (f, v)_0 + (b, \gamma_N(v))_{0,\Gamma_N} + (s, \gamma_n(v))_{0,\Gamma_C} - (\lambda_1, \gamma_t(v))_{0,\Gamma_C}, \\ &(\mu_1 - \lambda_1, s\gamma_t(u))_{0,\Gamma_C} \leq 0 \end{aligned}$$

for all $v \in (H_D^1(\Omega))^k$ and $\mu_1 \in L_{k-1}^2(\Gamma_C)$. The discretization is to find $(u_h, \lambda_{1,H}) \in (\mathscr{S}_h^p)^k \times \mathscr{M}_{H,k-1}^q$ such that

$$\begin{aligned} &(\sigma(u_h), \varepsilon(v_h))_0 = (f, v_h)_0 + (q, \gamma_N(v_h))_{0,\Gamma_N} + (s, \gamma_n(v_h))_{0,\Gamma_C} - (\lambda_{1,H}, \gamma_t(v_h))_{0,\Gamma_C}, \\ &(\mu_{1,H} - \lambda_{1,H}, s\gamma_t(u_h))_{0,\Gamma_C} \leq 0 \end{aligned}$$

for all $v \in (\mathscr{S}_h^p)^k$ and $\mu_{1,H} \in \mathscr{M}_{H,k-1}^q$.

4 Simplications of Signorini's problem

Both the geometrical part and the frictional part of Signorini's problem with Tresca's friction can be studied separately considering model problems. A *simplified version of Signorini's problem*, which only captures the geometrical condition, is to find a function $u \in H_D^1(\Omega) \cap H^2(\Omega)$ such that

$$\begin{aligned} -\Delta u = f \text{ in } \Omega, \quad \partial_n u = 0 \text{ on } \Gamma_N, \\ u \geq g, \; \partial_n u \geq 0, \; \partial_n u\,(u - g) = 0 \text{ on } \Gamma_C, \end{aligned} \tag{18}$$

where $f \in L^2(\Omega)$. The function $g \in H^{1/2}(\Gamma_C)$ represents an obstacle on the boundary Γ_C. Multiplying with a test function and integrating by parts yield that $u \in H_D^1(\Omega) \cap H^2(\Omega)$ is a solution if and only if $u \in K := \{v \in H_D^1(\Omega) \mid \gamma_C(v) \geq g\}$ and

$$(\nabla u, \nabla(v - u))_0 \geq (f, (v - u))_0 \tag{19}$$

for all $v \in K$. Using the notation of Section 2, we set $V := H^1_D(\Omega)$, $U_0 := H^{1/2}(\Gamma_C)$, $\beta_0 := \gamma_C$, $G := H^{1/2}_-(\Gamma_C)$, $j := 0$, $a(v,w) := (\nabla v, \nabla w)_0$ and $\langle \ell, v\rangle := (f,v)_0$. The bilinear form a is symmetric, continuous, and V-elliptic, due to Poincare's inequality. Therefore, u is the unique minimizer of (1). Due to Theorem 6 and Remark 2, we obtain a unique saddle point $(u,\lambda_0) \in H^1_D(\Omega) \times H^{-1/2}_-(\Gamma_C)$ which is equivalently characterized by the mixed variational formulation

$$\begin{aligned}&(\nabla u, \nabla v)_0 = (f,v)_0 - \langle \lambda_0, \gamma_C(v)\rangle,\\ &\langle \mu_0 - \lambda_0, \gamma_C(u) - g\rangle \le 0\end{aligned}$$

for all $v \in H^1_D(\Omega)$ and $\mu_0 \in H^{-1/2}_-(\Gamma_C)$. A discretization is given by setting $V_h := \mathscr{S}^p_h$ and $U'_{0,H} := \mathscr{M}^q_{H,-}$. Due to Theorem 7, we obtain a saddle point $(u_h, \lambda_{0,H}) \in \mathscr{S}^p \times \mathscr{M}^q_{H,-}$ which is equivalently characterized by

$$\begin{aligned}&(\nabla u_h, \nabla v_h)_0 = (f,v_h)_0 - (\lambda_{0,H}, \gamma_C(v_h))_{0,\Gamma_C},\\ &(\mu_{0,H} - \lambda_{0,H}, \gamma_C(u_h) - g)_{0,\Gamma_C} \le 0\end{aligned}$$

for all $v_h \in \mathscr{S}^p_h$ and $\mu_{0,H} \in \mathscr{M}^q_{H,-}$.

An idealized frictional problem is to find a function $u \in H^1_D(\Omega) \cap H^2(\Omega)$ such that

$$\begin{aligned}&-\Delta u = f \text{ in } \Omega, \quad \partial_n u = 0 \text{ on } \Gamma_N,\\ &|\partial_n u| \le s \text{ with } \left\{\begin{array}{ll} |\partial_n u| & < s \Rightarrow u = 0,\\ \partial_n u & = s \Rightarrow u \ge 0,\\ -\partial_n u & = s \Rightarrow u \le 0 \end{array}\right\} \text{ on } \Gamma_C\end{aligned}$$

with $f \in L^2(\Omega)$ and $s \in L^2(\Gamma_C)$, $s \ge 0$. Again, multiplying by a test function and integrating by parts, we obtain that $u \in H^1(\Omega,\Gamma_D) \cap H^2(\Omega)$ is a solution if and only if

$$(\nabla u, \nabla(v-u))_0 + (s, |\gamma(v)| - |\gamma(u)|)_{0,\Gamma_C} \ge (f, v-u)_0 \tag{20}$$

for all $v \in H^1(\Omega,\Gamma_D)$. Here, we set $V := H^1_D(\Omega)$, $U_0 := H^{1/2}(\Gamma_C)$, $\beta_0 := \gamma_C$, $G := H^{1/2}(\Gamma_C)$, and $j(v) := (s, |\gamma(v)|)_{0,\Gamma_C}$. Futhermore, we define a and ℓ as above and conclude that u is the unique minimizer of (1). For a mixed variational formulation, we have to ensure that j can be expressed as in (9). To this end, we define $\beta_1 := s\gamma_C$, $U_1 := L^2(\Gamma_C)$, $\Lambda_1 := L^2_1(\Gamma_C)$. For $\mu_1 \in L^2_1(\Gamma_C)$ and $v \in H^1(\Omega,\Gamma_D)$, there holds $(\mu_1, s\gamma_C(v))_{0,\Gamma_C} \le (|\mu_1|, s|\gamma_C(v)|)_{0,\Gamma_C} \le j(v)$. Furthermore, we have

$$j(v) = \int_{\tilde{\Gamma_C}} s|\gamma_C(v)|^{-1}\gamma_C(v)^2\, d\Gamma \le \sup_{\mu\in\Lambda_1} (\mu_1, s\gamma_C(v))_{0,\Gamma_C}$$

with $\tilde{\Gamma_C} := \Gamma_C \backslash \{x \in \Gamma_C \mid \gamma(v(x)) = 0\}$. Altogether, we obtain (4). Due to Lemma 6 and Remark 2, we obtain a unique saddle point $(u,\lambda_1) \in H^1_D(\Omega) \times L^2_1(\Gamma_C)$ which is equivalently characterized by the mixed variational formulation

$$\begin{aligned}&(\nabla u, \nabla v)_0 = (f,v)_0 - (\lambda_1, s\gamma_C(v))_{0,\Gamma_C},\\ &(\mu_1 - \lambda_1, s\gamma_C(u))_{0,\Gamma_C} \le 0\end{aligned}$$

for all $v \in H^1_D(\Omega)$ and $\mu_1 \in L^2_1(\Gamma_C)$. The discrete mixed variational formulation is to find $(u_h, \lambda_{1,H}) \in \mathscr{S}^p_h \times \mathscr{M}^q_{H,1}$ such that

$$(\nabla u_h, \nabla v_h)_0 = (f, v_h)_0 - (\lambda_{1,H}, s\gamma_C(v_h))_{0,\Gamma_C},$$
$$(\mu_{1,H} - \lambda_{1,H}, s\gamma_C(u_h))_{0,\Gamma_C} \leq 0$$

for all $v_h \in \mathscr{S}^p_h$ and $\mu_{1,H} \in \mathscr{M}^q_{H,1}$.

5 The inf-sup condition for Signorini's problem with Tresca's friction

In this section, we prove the unique existence of a discrete saddle point for Signorini's problem with Tresca's friction. According to Theorem 8, we have to show the discrete inf-sup condition (14). Signorini's problem with prescribed normal force is likewise included. Similar results for the simplified version of Signorini's problem and the idealized frictional problem can be found in [31].
In particular, we show that the constant α in (14) can be chosen independently from h, H, p and q. For the proof, we make use of an higher order approximation result (Lemma 3) and of an inverse inequality for negative norms (Lemma 4). We follow the proof of Lemma 3.1 in [18] where this condition is derived for discretization schemes of lower-order and combine it with the proof given for the idealized frictional problem as shown in [31].

The interpolation spaces $H^{1+\theta}(\Omega)$ and $H^{-1/2+\theta}(\Gamma_C)$ are defined as $H^{1+\theta}(\Omega) := [H^1(\Omega), H^2(\Omega)]_{\theta,2}$ and $H^{-1/2+\theta}(\Gamma_C) := [H^{-1/2}(\Gamma_C), H^{1/2}(\Gamma_C))]_{\theta,2}$, $0 < \theta \leq 1$, with norms $\|\cdot\|_{1+\theta}$ and $\|\cdot\|_{-1/2+\theta,\Gamma_C}$, respectively, see [28,32]. We assume that $\mathscr{T}$ and $\mathscr{T}_C$ are quasi-uniform and p and q are constant degree distributions. With $\hat{a}(v,w) := (\varepsilon(v), \varepsilon(w))_0 + (v,w)_0$, $v,w \in (H^1(\Omega))^k$, and $\beta := (\gamma_n, \gamma_t)$ with $V := H^1_D(\Omega)$ and $U := H^{1/2}(\Gamma_C) \times (H^{-1/2}(\Gamma_C))^2$, we call the variational problem (15) regular, if $u^\mu_i \in H^1_D(\Omega) \cap H^{1+\theta}(\Omega)$, $i = 1, \ldots, k$, and

$$\|u^\mu_i\|_{1+\theta} \leq C_4 \sum_{i=1}^{k} \|\mu_i\|_{-1/2+\theta,\Gamma_C} \tag{21}$$

for all $\mu \in (H^{-1/2+\theta}(\Gamma_C))^k$ and a constant $C_4 > 0$. For $k = 2$ and parallelogram meshes, there holds

Lemma 3 *Let $\mu \in (L^2(\Gamma_C))^k$ and $u^\mu_i \in H^1_D(\Omega) \cap H^{1+\theta}(\Omega)$, $i = 1, \ldots, k$, be the solution of* (15)*, then there exists a function $u^\mu_I \in (\mathscr{S}^p_h)^k$ and a constant $C_2 > 0$, independent of u^μ, h and p, such that*

$$\|u^\mu - u^\mu_I\|_1 \leq C_2 \frac{h^\theta}{p^\theta} \sum_{i=1}^{k} \|u^\mu_i\|_{1+\theta}.$$

Proof See [1, Thm. 4.6]. □

For $k > 2$, we refer to [2].

Lemma 4 *There exists a constant $C_3 > 0$ which is independent of H and q, such that*

$$\|\mu_H\|_{-1/2+\theta,\Gamma_C} \leq C_3 \frac{\max\{1,q\}^{2\theta}}{H^\theta} \|\mu_H\|_{-1/2,\Gamma_C}$$

for all $\mu_H \in \mathscr{M}_H^q$.

Proof See [11, Thm. 3.5., Thm. 3.9]. □

Lemma 5 *Let $\tilde{L}^2(\Gamma_C) := \{\mu \in (L^2(\Gamma_C))^{k-1} \mid \mu = 0$ on $\Gamma_C \setminus \operatorname{supp} s\}$ and $C, C' > 0$. There exists a $\kappa > 0$, such that for h, H, p and q satisfying*

$$\Pi(h,H,p,q) := \left(hH^{-1}\max\{1,q\}^2 p^{-1}\right)^\theta < \kappa$$

there holds

$$\sum_{i=1}^{k-1} \left(C\|s\mu_{1,H,i}\|_{-1/2,\Gamma_C} - C'\Pi(h,H,p,q)\|\mu_{1,H,i}\|_{-1/2,\Gamma_C}\right) \geq \kappa \sum_{i=1}^{k-1} \|\mu_{1,H,i}\|_{-1/2,\Gamma_C}$$

for all $\mu_{1,H} \in (\mathscr{M}_H^q)^{k-1} \cap \tilde{L}^2(\Gamma_C)$.

Proof Assume that for all $\kappa > 0$ there exist h_κ, H_κ, p_κ and q_κ such that

$$\Pi_\kappa := \Pi(h_\kappa, H_\kappa, p_\kappa, q_\kappa) < \kappa$$

and there exists a function $\mu_\kappa \in (\mathscr{M}_{H_\kappa}^{q_\kappa})^{k-1} \cap \tilde{L}^2(\Gamma_C)$, such that

$$\sum_{i=1}^{k-1} \left(C\|s\mu_{\kappa,i}\|_{-1/2,\Gamma_C} - C'\Pi_\kappa\|\mu_{\kappa,i}\|_{-1/2,\Gamma_C}\right) < \kappa \sum_{i=1}^{k-1} \|\mu_{\kappa,i}\|_{-1/2,\Gamma_C}. \tag{22}$$

Obviously, $\mu_\kappa \neq 0$. Defining $\tilde{\mu}_\kappa := \|\mu_\kappa\|_{-1/2,\Gamma_C}^{-1}\mu_\kappa \in \tilde{L}^2(\Gamma_C)$, we obtain $\|\tilde{\mu}_\kappa\|_{-1/2,\Gamma_C} = 1$. Due to the reflexivity of $L^2(\Gamma_C)$ and the convexity as well as the closedness of $\tilde{L}^2(\Gamma_C)$, there exists some $\tilde{\mu} \in \tilde{L}^2(\Gamma_C)$ such that $\tilde{\mu}_{\kappa_n} \rightharpoonup \tilde{\mu}$ for a sequence $\kappa_n \to 0$. This also implies $\tilde{\mu}_{\kappa_n} \to \tilde{\mu}$ in the norm $\|\cdot\|_{-1/2,\Gamma_C}$ using a well known compactnes result. Therefore, $\|\tilde{\mu}\|_{-1/2,\Gamma_C} = 1$ and $\tilde{\mu} \neq 0$ on $\operatorname{supp} s$. From (22), we have

$$C\sum_{i=1}^{k-1} \|s\tilde{\mu}_{\kappa_n,i}\|_{-1/2,\Gamma_C} < (k-1)(1+C')\kappa_n$$

which implies $\sum_{i=1}^{k-1} \|s\tilde{\mu}_i\|_{-1/2,\Gamma_C} = 0$ and therefore, $s\tilde{\mu} = 0$, which is a contradiction to $\tilde{\mu} \neq 0$ on $\operatorname{supp} s$. □

Using Lemma 2, Lemma 3 and Lemma 4 as well as the regularity assumption (21) on u^μ and Lemma 5, we are able to prove the main theorem.

Theorem 9 *Let the variational problem* (15) *be regular for $\theta \leq 1/2$ and $s \in L^\infty(\Gamma_C)$. Furthermore, let $\Pi(h,H,p,q)$ be sufficiently small. Then, the inf-sup condition* (14) *with $\tilde{U}_1' := (H^{-1/2}(\Gamma_C))^{k-1}$ is fulfilled with α independent from h, H, p und q.*

Proof Let $\mu_H := (\mu_{0,H}, \mu_{1,H}) \in \mathscr{M}_H^q \times (\mathscr{M}_H^q)^{k-1}$ and $u_h^\mu \in (\mathscr{S}_h^p)^k$ be the solution of (15) with $V := (\mathscr{S}_h^p)^k$ and $\mu := \mu_{s,H} := (\mu_{0,H}, s\mu_{1,H}^1, \dots, s\mu_{1,H}^{k-1})$. Using the Galerkin orthogonality, Lemma 3, the regularity assumption and Lemma 4, we obtain

$$\begin{aligned}
\|u^{\mu_{s,H}} - u_h^{\mu_{s,H}}\|_1 &\leq \|u^{\mu_{s,H}} - u_I^{\mu_{s,H}}\|_1 \leq C_2 \frac{h^\theta}{p^\theta} \sum_{i=1}^{k} \|u^{\mu_{s,H,i}}\|_{1+\theta} \\
&\leq C_2 C_4 \frac{h^\theta}{p^\theta} \sum_{i=1}^{k} \|\mu_{s,H,i}\|_{-1/2+\theta,\Gamma_C} \\
&\leq C_2 C_4 \frac{h^\theta}{p^\theta} \max\{1, \|s\|_{\infty,\Gamma_C}\} \sum_{i=1}^{k} \|\mu_{H,i}\|_{-1/2+\theta,\Gamma_C} \\
&\leq C_2 C_3 C_4 \left(\frac{h^\theta}{p^\theta} \frac{\max\{1,q\}^{2\theta}}{H^\theta} \right) \max\{1, \|s\|_{\infty,\Gamma_C}\} \sum_{i=1}^{k} \|\mu_{H,i}\|_{-1/2,\Gamma_C} \\
&= C_2 C_3 C_4 \Pi(h,H,p,q) \max\{1, \|s\|_{\infty,\Gamma_C}\} \sum_{i=1}^{k} \|\mu_{H,i}\|_{-1/2,\Gamma_C}.
\end{aligned}$$

From Lemma 2 and the norm equivalence

$$C_0 \|\mu\|_{H^{-1/2}(\Gamma_C)\times(H^{-1/2}(\Gamma_C))^{k-1}} \leq \sum_{i=0}^{k} \|\mu_i\|_{-1/2,\Gamma_C} \leq C_0^{-1} \|\mu\|_{H^{-1/2}(\Gamma_C)\times(H^{-1/2}(\Gamma_C))^{k-1}}$$

with a constant $C_0 > 0$, we obtain

$$\begin{aligned}
&\sup_{v_h \in S^p(\mathscr{T}_h)\setminus\{0\}} \frac{(\mu_{0,H}, \gamma_n(v_h))_{0,\Gamma_C} + (\mu_{1,H}, s\gamma_t(v_h))_{0,\Gamma_C}}{\|v_h\|_1} \\
&\geq \frac{(\mu_{0,H}, \gamma_n(u_h^{\mu_{s,H}}))_{0,\Gamma_C} + (s\mu_{1,H}, \gamma_t(u_h^{\mu_{s,H}}))_{0,\Gamma_C}}{\|u_h^{\mu_{s,H}}\|_1} = \|u_h^{\mu_{s,H}}\|_1 \\
&\geq \|u^{\mu_{s,H}}\|_1 - \|u^{\mu_{s,H}} - u_h^{\mu_{s,H}}\|_1 \\
&\geq C_0 C_1 \sum_{i=1}^{k} \|\mu_{s,H,i}\|_{-1/2,\Gamma_C} - C_2 C_3 C_4 \Pi(h,H,p,q) \max\{1, \|s\|_{\infty,\Gamma_C}\} \sum_{i=1}^{k} \|\mu_{H,i}\|_{-1/2,\Gamma_C} \\
&\geq (C_0 C_1 - C_2 C_3 C_4 \Pi(h,H,p,q) \max\{1, \|s\|_{\infty,\Gamma_C}\}) \|\mu_{0,H}\|_{-1/2,\Gamma_C} \\
&\quad + \sum_{i=1}^{k-1} \big(C_1 \|\mu_{1,H,i}\|_{-1/2,\Gamma_C} - C_2 C_3 C_4 \Pi(h,H,p,q) \max\{1, \|s\|_{\infty,\Gamma_C}\} \|\mu_{1,H,i}\|_{-1/2,\Gamma_C} \big) \\
&\geq (C_0 C_1 - C_2 C_3 C_4 \Pi(h,H,p,q) \max\{1, \|s\|_{\infty,\Gamma_C}\}) \|\mu_{0,H}\|_{-1/2,\Gamma_C} + \kappa \sum_{i=1}^{k-1} \|\mu_{1,H,i}\|_{-1/2,\Gamma_C} \\
&\geq (C_0 C_1 - C_2 C_3 C_4 \varepsilon \max\{1, \|s\|_{\infty,\Gamma_C}\}) \|\mu_{0,H}\|_{-1/2,\Gamma_C} + \varepsilon \sum_{i=1}^{k-1} \|\mu_{1,H,i}\|_{-1/2,\Gamma_C} \\
&\geq \min\{C_0 C_1 - C_2 C_3 C_4 \varepsilon \max\{1, \|s\|_{\infty,\Gamma_C}\}, \varepsilon\} \sum_{i=0}^{k} \|\mu_{H,i}\|_{-1/2,\Gamma_C} \\
&\geq C_0 \min\{C_0 C_1 - C_2 C_3 C_4 \varepsilon \max\{1, \|s\|_{\infty,\Gamma_C}\}, \varepsilon\} \|\mu_H\|_{H^{-1/2}(\Gamma_C)\times(H^{-1/2}(\Gamma_C))^{k-1}}
\end{aligned}$$

with $\Pi(h,H,p,q) \leq \varepsilon < \min\{C_0C_1(C_2C_3C_4\max\{1,\|s\|_\infty\})^{-1},\kappa\}$. □

Hence, from Theorem 8 and Theorem 9 we obtain

Corollary 1 *Under the assumptions of Theorem 9, there exists a unique discrete saddle point of Signorini's problem with Tresca's friction.*

Remark 4 The assumptions of Theorem 9 seem hard to be verified in practice as it is not clear when $\Pi(h,H,p,q)$ is sufficiently small. Furthermore, it is often unclear whether the regularity assumption (21) holds. For convex domains, this assumption is fulfilled. Nevertheless, Theorem 9 justifies the modification of the discretization scheme by coarsening the mesh $\mathscr{T}_C$ or by decreasing the polynomial degree q to obtain a stable scheme. In Section 8, numerical results confirm this theoretical observation.

Remark 5 The choice $\tilde{U}_1' = ((H^{1/2}(\Gamma_C))^{k-1})'$ is important. To use Theorem 8 we might choose $\tilde{U}_1' = (L^2(\Gamma_C))^{k-1}$. However, in this case, the mapping β would not be surjective and Lemma 2 could not be applied in the proof of Theorem 9.

6 Solution scheme based on the dual formulation

In this section, we propose a solution scheme which is based on the dual formulation of the discrete mixed variational formulation and is, in particular, convenient to handle discretizations of higher-order. We first introduce the scheme within the abstract framework of Section 2. Thereafter, we discuss the application of the scheme to the higher-order discretization of Section 3.

Introducing a basis $\{\varphi_j\}_{0\leq j<n}$ of V_h and bases $\{\psi_{ij}\}_{0\leq j<m_i}$ of $U_{i,H}'$ with $n := \dim V_i$ and $m_i := \dim U_{i,H}'$ and setting $\bar{\Lambda}_i := \{z \in \mathbb{R}^{m_i} \mid z_j\psi_{ij} \in \Lambda_{i,H}\}$, the discretization (13) is to find $(x,y_0,y_1) \in \mathbb{R}^n \times \bar{\Lambda}_0 \times \bar{\Lambda}_1$ such that

$$\begin{aligned} &\mathscr{A}x = \mathscr{L} - \mathscr{B}_0^\top y_0 - \mathscr{B}_1^\top y_1, \\ &(y_0 - z_0)^\top(\mathscr{B}_0 x - \mathscr{G}) + (y_1 - z_1)^\top \mathscr{B}_1 x \leq 0 \end{aligned} \tag{23}$$

for all $(z_0,z_1) \in \bar{\Lambda}_0 \times \bar{\Lambda}_1$. Here, $\mathscr{A} \in \mathbb{R}^{n\times n}$, $\mathscr{L} \in \mathbb{R}^n$, $\mathscr{B}_i \in \mathbb{R}^{m_i\times n}$ and $\mathscr{G} \in \mathbb{R}^{m_0}$ are defined as $\mathscr{A}_{jk} := a(\varphi_k,\varphi_j)$, $\mathscr{L}_j := \langle \ell,\varphi_j\rangle$, $\mathscr{B}_{i,jk} := \langle \psi_{i,j}, \beta_i(\varphi_k)\rangle$ and $\mathscr{G}_j := \langle \psi_{0,j}, g\rangle$. The solution is given by $(u_h,\lambda_{0,H},\lambda_{1,H}) = (x_i\varphi_i, y_{0,j}\psi_{0,j}, y_{1,j}\psi_{1,j})$. With

$$\mathscr{B} := \begin{pmatrix} \mathscr{B}_0 \\ \mathscr{B}_1 \end{pmatrix}, \quad \tilde{\mathscr{G}} := \begin{pmatrix} \mathscr{G} \\ 0 \end{pmatrix},$$

and $\bar{\Lambda} := \bar{\Lambda}_0 \times \bar{\Lambda}_1$, the system (23) is equivalent to find $(x,y) \in \mathbb{R}^n \times \bar{\Lambda}$ such that

$$\begin{aligned} &\mathscr{A}x = \mathscr{L} - \mathscr{B}^\top y, \\ &(y - z)^\top(\mathscr{B}x - \tilde{\mathscr{G}}) \leq 0 \end{aligned} \tag{24}$$

for all $z \in \bar{\Lambda}$. A simple iterative scheme with projection is often refered to solve the system (24), cf. [15]. With a suitable projection $P : \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \to \bar{\Lambda}$ and $\mathscr{S}^{-1} \in \mathbb{R}^{n\times n}$, this scheme reads

$$x^{n+1} = x^n - \rho_1 \mathscr{S}^{-1}(Ax^n + \mathscr{B}^\top y^n - \mathscr{L}),$$
$$y^{n+1} = P(y^n + \rho_2(\mathscr{B}x^{n+1} - \tilde{\mathscr{G}})).$$

Usually, $\mathscr{S}^{-1}$ is chosen as $\mathscr{A}^{-1}$ or as an appropriate approximation of $\mathscr{A}^{-1}$. Since it is not obvious to define the projection P for higher-order discretizations with possibly non-nodal basis functions, we consider an alternative scheme based on the dual formulation of (24). The basic idea is to reformulate (24) into a minimzation problem in terms of the Lagrange multipliers using a Schur complement ansatz.

Theorem 10 *The pair (x,y) fulfills* (24) *if and only if*

$$F(y) = \min_{z\in\bar{\Lambda}} F(z), \quad F(z) := \frac{1}{2} z^\top \mathscr{B}\mathscr{A}^{-1}\mathscr{B}^\top z - z^\top \left(\mathscr{B}\mathscr{A}^{-1}\mathscr{L} - \tilde{\mathscr{G}}\right) \tag{25}$$

and $x = \mathscr{A}^{-1}(\mathscr{L} - \mathscr{B}^\top y)$.

Proof Resolving the equation in (24) leads to $x = \mathscr{A}^{-1}(\mathscr{L} - \mathscr{B}^\top y)$. Replacing x in the inequality, we obtain

$$\left(\mathscr{B}\mathscr{A}^{-1}\mathscr{B}^\top y - (\mathscr{B}\mathscr{A}^{-1}\mathscr{L} - \tilde{\mathscr{G}})\right)^\top (z-y) \geq 0$$

for all $z \in \bar{\Lambda}$. Applying the general Theorem 1 completes the proof. □

To solve Problem (25), within an optimization scheme of quadratic programming, we usually have to specify an evaluation routine for the objective function F which is given as follows

(i) $b = \mathscr{B}^\top z$
(ii) Solve $A\tilde{x} = b$
(iii) $\tilde{z} = \mathscr{B}\tilde{x}$
(iv) $F = 0.5 z^\top \tilde{z} - z^\top w$

with some auxiliary vectors $b, \tilde{x} \in \mathbb{R}^n$ and $z, w \in \mathbb{R}^m$, $m := m_0 + m_1$. The vector w can be evaluated in a preprocessing step by

(i) Solve $A\tilde{x} = \mathscr{L}$
(ii) $\tilde{z} = \mathscr{B}\tilde{x}$
(iii) $w = \tilde{z} - \tilde{\mathscr{G}}$

Using a direct solver, only a single factorization of the matrix $\mathscr{A}$ is necessary. Instead of a direct solver, which may be more suited to higher-order discretizations, iterative or multigrid schemes can be used, too.
Note that the dimension m of the opimization variable given by the Lagrange multipliers is, in general, much smaller than the dimension of the discrete displacement variable n. Therefore, the total amount to solve the system mainly depends on m and

on an efficient matrix-vector computation to evaluate the objective function F. In the end, this fact makes this approach applicable. It may be, therefore, also an alternative to other very efficient approaches for solving contact problems. We refer to some recent works [6,22,23,25,33].
For lower-order finite elements, the introduced approach is widely studied and enhanced for many applications in frictional contact problems. We refer to [7,17,19] for more details. In particular, the block structure of the matrix $\mathscr{B}^\top \mathscr{A}^{-1}\mathscr{B}$ can be further exploited using splitting type algorithms [20] as well as domain decomposition techniques can be applied [7]. Also the application of this approach to multibody contact problems is possible. Especially, the discrete mixed variational formulation allows for the use of non-matching grids which can be directly included in the solution scheme. We refer to [4] for more details.
In our case, we prefer this general approach since it seems to be very convenient for higher-order finite element discretizations and, in particular, for the discrete mixed variational formulation proposed in this work. An advantage of the approach is that the additional implementational effort is small, if one uses a standard optimization tool based on QP- or SQP-techniques. In particular, for varying polynomial degrees, for instance in hp-adaptive schemes, cf. [30], the constraints can be profoundly complicated so that the derivation of more sophisticated algorithms which capture the specific properties of the higher-order discretization is not obvious.

The application of the solution scheme to higher-order discretizations is given as follows. Using the discretization as introduced in Section 3, we have $\Lambda_{0,H} = \mathscr{M}^q_{H,\pm}$ and $\Lambda_{1,H} = \mathscr{M}^q_{H,l}$. To determine $\bar{\Lambda}_i$, $i = 0,1$, suppose that $\{\kappa_j\}_{0\le j<m^r}$ is a basis of S^r_{k-1} with $m^r := \dim S^r_{k-1}$. With $\zeta(T_l) := \sum_{i=0}^{l-1} m^{qT_i}$, a basis of $\mathscr{M}^q_H$ is simply given by

$$\tilde{\psi}_{\zeta(T_l)+j} := \kappa_j \circ \Psi^{-1}_{T_l}$$

on $T_l \in \mathscr{T}_C = \{T_0, T_1, \ldots, T_{\tilde{m}-1}\}$ and 0 on $\Gamma_C \backslash T_l$. Assuming $M = \{x_0, \ldots, x_{d-1}\}$, we define a matrix $C \in \mathbb{R}^{d\tilde{m}\times m_0}$, $m_0 := \dim \mathscr{M}^q_H$, by

$$C_{ld+\nu,\zeta(T_l)+j} := \kappa_j(x_\nu),$$

$j = 0, \ldots, m^{qT_l}$, $\nu = 0, \ldots, d$, and 0 otherwise. Thus, we have

$$\bar{\Lambda}_0 = \{z \in \mathbb{R}^{m_0} \mid \pm Cz \le 0\}, \quad \bar{\Lambda}_1 = \{z \in \mathbb{R}^{m_1} \mid f(z) \le 1\}$$

with $m_1 := (m_0)^{k-1}$ and

$$f(z_{1,0}, \ldots, z_{1,k-2})_j := \sum_{i=0}^{k-2} ((Cz_{1,i})_j)^2,$$

$j = 0, \ldots, d\tilde{m} - 1$. Hence, (25) reads

$$F(y) = \min_{\substack{z=(z_0,z_1)\in\mathbb{R}^{m_0}\times\mathbb{R}^{m_1}, \\ \pm Cz_0 \le 0,\ f(z_1)\le 1}} F(z). \tag{26}$$

Note that the set M should be chosen so that the additional numerical error is minimized. We use Chebycheff points to ensure the additional error to be small. We refer

to [9] for a further justification of this choice.
In view of (26), we have linear constraints for the variable z_0. For the variable z_1, we also have linear constraints in the case $k = 2$ and non-linear constraints in the case $k = 3$. In our implementation, we use the `sqopt`-method of the SQP-package `Snopt` by Gill et. al [12, 13] to include the linear constraints and the `snopt`-method of this package for the general non-linear constraints.

It should be mentioned that the solution scheme is also convenient to implement Coulomb friction law, where the frictional function s is defined as $s := \mathscr{F}|\sigma_{nn}(u)|$ with some frictional coefficient $\mathscr{F} > 0$. Under certain regularity assumptions, the Lagrange multiplier λ_0 coincides with the normal contact stress $-\sigma_{nn}(u)$. However, setting $s := \mathscr{F}|\lambda_0|$ would lead to a formulation which is not captured by the introduced framework of Section 2. Instead, we can embed Coulomb's friction into our framework using a simple fix point scheme: For an arbitrary frictional function $s \in L^2(\Gamma_C)$ with $s \geq 0$, we define $(u(s), \lambda_0(s), \lambda_1(s))$ as the unique saddle point of Signorini's problem with Tresca's friction, and furthermore, the operator $\mathscr{H}(s) := \mathscr{F}|\lambda_0(s)|$. Assuming that $\mathscr{H}$ has a fix point, i.e., $\mathscr{H}(\bar{s}) = \bar{s}$, the saddlepoint $(u(\bar{s}), \lambda_0(\bar{s}), \lambda_1(\bar{s}))$ fulfills Coulomb friction law. Transfering this concept to the discrete mixed variational formulation, we obtain $(x(s), y_0(s), y_1(s))$ as the solution of (23) and define $\tilde{\mathscr{H}}(s) := \mathscr{F}|y_{0,j}(s)\psi_{0,j}|$. Again, a fix point $\tilde{s}$ of $\tilde{\mathscr{H}}$ (or a suitable approximation) leads to solution vectors $(x(\tilde{s}), y_0(\tilde{s}), y_1(\tilde{s}))$ yielding a discrete saddlepoint $(u_h(\tilde{s}), \lambda_{0,H}(\tilde{s}), \lambda_{1,H}(\tilde{s}))$ which approximatively fulfills Coulomb friction law. We refer to [20, 17] and reference therein for more details on this well-known proceeding.

7 An extension to time-dependent problems

The use of the solution scheme as proposed in Section 6 is not restricted to the static case, which is, in a sense, uninteresting in many applications of engineering. It is also applicable to dynamic contact problems. To demonstrate this, we extend the simplified version of Signorini's problem of Section 4 to a time-dependent model problem which is to find a time-dependent function $u \in H^2(I; H^1_D(\Omega) \cap H^2(\Omega))$ on $\Omega \times I$, $I := [0, T]$, with $u(0) = u_0 \in H^1_D(\Omega)$, $\dot{u}(0) = v_0 \in H^1_D(\Omega)$ such that

$$\ddot{u} - \Delta u = f \text{ in } I \times \Omega, \quad \partial_n u = 0 \text{ on } \Gamma_N \times I,$$
$$u \geq g,\ \partial_n u \geq 0,\ \partial_n u\,(u - g) = 0 \text{ on } \Gamma_C \times I$$

with a time-dependent load function f and a time-dependent obstacle function g on $\Omega \times I$ and $\Gamma_C \times I$, respectively. Again, multiplying by a test function and integrating by parts yield, that u is a solution if and only if $u \in \tilde{K} := \{v \in V \mid \gamma_C(v(t)) \geq g,\ t \in [0, T]\}$ and

$$(\ddot{u}(t), v(t) - u(t))_0 + (\nabla u(t), \nabla(v(t) - u(t)))_0 \geq (f(t), v(t) - u(t))_0$$

for almost all $t \in [0, T]$ and all $v \in \tilde{K}$, cf. [29]. Here, we set $V := W^{2,\infty}([0, T]; L^2(\Omega)) \cap L^\infty([0, T]; H^1_D(\Omega))$. To discretize this variational problem in time, we may use Rothe's method on the basis of a Newmark scheme. Setting $u^0 := u_0$, $\dot{u}^0 := v_0$, we successively seek a function $u^n := u(t_n) \in K^n := \{v \in H^1_D(\Omega) \mid \gamma_C(v) \geq g^n := g(t_n)\}$ such that

$$a(u^n, v - u^n) \geq (F^n, v - u^n)_0 \tag{27}$$

for all $v \in K^n$ in each time step $t_n := nk$, $k := T/N$, $N \in \mathbb{N}$. Here, the bilinear form a is defined as $a(u,v) := 2k^{-2}(u,v)_0 + (\nabla u, \nabla v)_0$. Furthermore, we set

$$\ddot{u}^n := 2k^{-2}(u^n - u^{n-1}) - 2k^{-1}\dot{u}^{n-1}, \quad \dot{u}^n := \dot{u}^{n-1} + 2^{-1}k(\ddot{u}^{n-1} + \ddot{u}^n),$$
$$F^n := f(t_n) + 2k^{-2}u^{n-1} + 2k^{-1}\dot{u}^{n-1}.$$

Note that the bilinear form a is symmetric, continuous and V-elliptic. Therefore, using the same notations as introduced in Section 4 for the static problem and the general results of Section 2, we obtain existence and uniqueness of the solution u^n of (27). In particular, we obtain an appropriate mixed variational formulation with a unique saddle point (u^n, λ_0^n)

$$a(u^n, v)_0 = (F^n, v)_0 - \langle \lambda_0^n, \gamma_C(v) \rangle,$$
$$\langle \mu_0 - \lambda_0^n, \gamma_C(u) - g^n \rangle \leq 0$$

for all $v \in H^1_D(\Omega)$ and $\mu_0 \in H^{-1/2}_-(\Gamma_C)$.
To discretize in space, we set $u_h^0 := i_h u_0$, $\dot{u}_h^0 := i_h v_0$ with some interpolation operator i_h and successively determine the discrete saddle point $(u_h^n, \lambda_{0,H}^n) \in \mathscr{S}_h^p \times \mathscr{M}_{H,-}^q$ of the discrete mixed variational formulation

$$a(u_h^n, v_h)_0 = (F_h^n, v_h)_0 - (\lambda_{0,H}^n, \gamma_C(v_h))_{0,\Gamma_C},$$
$$(\mu_{0,H} - \lambda_{0,H}^n, \gamma_C(u_h^n) - g)_{0,\Gamma_C} \leq 0$$

for all $v_h \in \mathscr{S}_h^p$ and $\mu_{0,H} \in \mathscr{M}_{H,-}^q$. Here, we set $F_h^n := f(t_n) + 2k^{-2}u_h^{n-1} + 2k^{-1}\dot{u}_h^{n-1}$.
Again, sufficiently small quotients h/H and $\max\{1,q\}^2 p^{-1}$ guarantees the discrete inf-sup condition (14) to be valid and therewith the unique existence of the discrete saddle point, cf. Section 5 and [31].
In the end, having the discrete mixed variational formulation at hand, we are able to use the solution scheme based on the dual formulation of Section 6. In [3], we apply the general solution scheme to time-dependent problems, which we briefly outline in this Section, on a broad range where we study dynamic problems including frictional, thermo-mechanical and rolling contact problems. Similar to the dynamic model problem as discussed in this section, the key to derive a solution scheme for more complex dynamic contact problems is to discretize in time and, then, to use a discretization in space based on a discrete mixed variational formulation.

8 Numerical Results

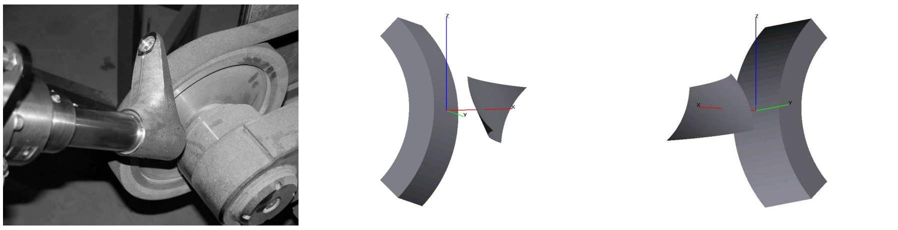
In our numerical experiments, we study Signorini's problem with Tresca's friction by means of an example in production engineering which is given by a robot-based belt grinding process, see Figure 1(a). The domain, which corresponds to a quarter of the contact wheel of the belt grinding machine, is given by

$$\Omega := \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{c} r(x,z) \in (1.295, 1.625), \\ \varphi(x,z) \in [0,\pi/4) \cup (7\pi/4, 0], \\ y \in (-0.575, 0.575) \end{array} \right\}$$

where (r,φ) are the polar coordinates with the origin in $(-1.625, 0)$. We set $\Gamma_D := \{(x,y,z) \in \overline{\Omega} \mid r(x,z) = 1.295\}$ and $\Gamma_C := \{(x,y,z) \in \overline{\Omega} \mid r(x,z) = 1.625\}$. Furthermore, we set $f := 0$ and $b := 0$. The obstacle function describing the surface of a workpiece (here a water tap) is defined as

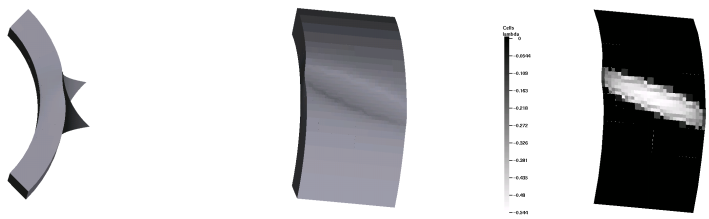
$$\psi(y,z) := \begin{cases} d + 1 - \sqrt{1 - (z + 0.5y)^2}, & |z + 0.5y| \leq r \\ d + 1, & |z + 0.5y| > r \end{cases}$$

where the parameter $d \in \mathbb{R}$ denotes the infeed of the obstacle along x-axis, cf. Figure 1(b,c). We use Hooke's law with Young's modulus $E := 2mN/dm^2$ and Poisson's number $\nu := 0.42$.



(a) (b) (c)

Fig. 1 **(a)** A robot is pressing a workpiece (water tap) against the contact wheel of the belt grinding machine, **(b)**,**(c)** quarter of the contact wheel, surface of the workpiece.



(a) (b) (c)

Fig. 2 **(a)**,**(b)** Deformable body and obstacle's surface in contact with infeed $d := -0.05\,\mathrm{dm}$, **(b)** normal contact force $\sigma_{nn}(u)$ on Γ_C.

In Figure 2(a,b) the deformation of the body is depicted for frictionless contact where $s := 0$. The deformation reflects the geometrical contact condition $u_n - g \leq 0$ on Γ_C. The complementary condition $\sigma_{nn}(u)(u_n - g) = 0$ and the condition $\sigma_{nn}(u) \leq 0$ are shown in Figure 2(c). The normal contact forces $\sigma_{nn}(u)$ describes pressure in the contact zone and is zero outside.

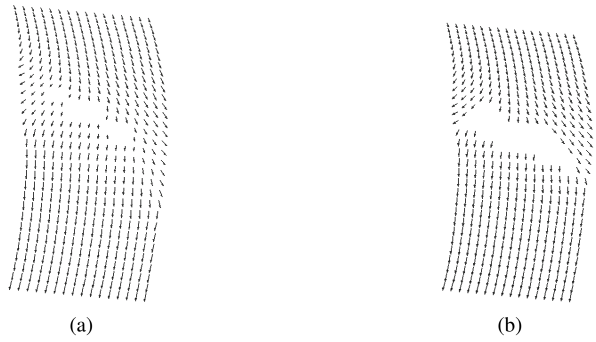


Fig. 3 Tangential displacements on Γ_C for **(a)** Signorini's problem with prescribed normal force and **(b)** Signorini's problem with Coulomb friction law.

In Figure 3(a) the tangential displacements on Γ_C for Signorini's problem with prescribed normal forces are depicted. Here, the prescribed normal force

$$q_n := \begin{cases} -0.2, & |z + 0.5y| \leq r \\ 0, & |z + 0.5y| > r \end{cases}$$

are applied. Hence, the contact zone is given by $|z + 0.5y| \leq r$. We define $s := \mathscr{F}|q_n|$ with the coefficient of friction $\mathscr{F} := 0.5$. To obtain considerable tangential forces and displacements on Γ_C, we insert additional tangential forces b_t by exchanging $\sigma_{nt}(u)$ with $b_t - \sigma_{nt}(u)$. This leads to the additional integral $(b_t, \gamma_t(v))_{0,\Gamma_C}$ within the mixed variational formulations. In our numerical experiments, we set $b_t := (0, -0.05)^\top$. The numerical results are based on descritizations with uniform h, H, p and q for which the validation of the discrete inf-sup condition is numerically verified, see below. Furthermore, the solution scheme using the dual formulation as described in Section 6 is applied.

Figure 3(a) shows that outside of the contact zone the tangential displacements correspond to the tangential forces b_t. In the contact zone, we observe areas with gliding indicated by the logarithmically scaled displacement vectors. The displacements are zero in areas with sticking which are located in the center of the contact zone. Displacement vectors are not depicted there. In Figure 3(b) the tangential displacements on Γ_C for Signorini's problem with Coulomb friction law are shown, where areas with gliding and sticking are depicted.

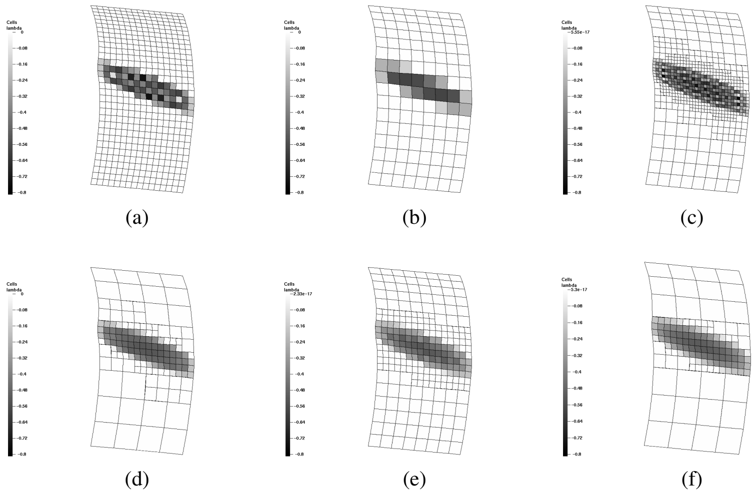


Fig. 4 $-\lambda_{0,H}$ with $p \equiv 1$, $q \equiv 0$ and **(a,c)** $h/H = 1$ and **(b,d)** $h/H = 0.5$, furthermore $-\lambda_{n,H}$ with $p \equiv 2$, $q \equiv 1$ and **(e)** $h/H = 1$ and **(f)** $h/H = 0.5$ on an adaptive mesh.

As stated in Section 2, the discretization with mixed finite elements admits a unique solution if the discrete inf-sup condition (14) is fulfilled. In Theorem 9 it is proven, that (14) holds if $\Pi(h,H,p,q)$ is sufficiently small. This theoretical statement can also be observed in numerical experiments. Figures 4(a,c) show $-\lambda_{0,H}$ with $p \equiv 1$ and $q \equiv 0$. In the case $h/H = 1$, we observe checkerboard patterns which typically indicate that the discrete inf-sup condition is not fulfilled. In the case $h/H = 0.5$, these patterns do not occur which shows that $\Pi(h,H,p,q)$ is small enough so that the discrete inf-sup condition holds, see Figure 4(b,d). In Figure 4 (c,d), an adaptive mesh is applied in order to resolve the contact zone more accurately. Also in the case $p \equiv 2$ und $q \equiv 1$ and $h/H = 1$, the checkerboard patterns occur. Again, using $h/H = 0.5$, these patterns vanish, see Figure 4 (e,f). For $p > 2$, we observe similar results. Consequently, the combination $q = p-1$ and $H = 2h$ seems to be convenient to obtain a stable scheme.
However, the use of different mesh sizes h and H leads to a certain implementational effort. Obviously, it is much simpler to use the mesh $\mathscr{T}_C := \{F \mid F \in \mathscr{E},\ F \subset \Gamma_C\}$ where $\mathscr{E}$ is the set of all faces (or edges) of $\mathscr{T}$. In this case, we have $h = H$. Thus, we can only vary the polynomial degree p and q to ensure that $\Pi(h,H,p,q)$ is sufficiently small. In Figure 5, we choose $p = 2$ and $q = 0$ and obtain stable numerical results for the discrete Lagrange multipliers $\lambda_{0,H}$ and $\lambda_{1,H}$. Here, Coulomb friction law is used which is incorparated via the fix point method as described in Section 6. In this numerical experiment, the infeed d is set to -0.25 which results in a slightly larger contact zone.

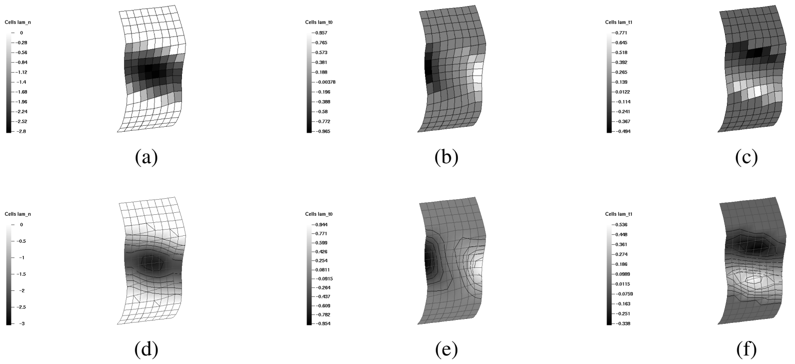


Fig. 5 **(a)** $\lambda_{0,H}$, **(b)** $\lambda^{1}_{1,H}$ and **(c)** $\lambda^{2}_{1,H}$ with $p \equiv 2$, $q \equiv 0$, **(d,e,f)** smoothed values of the discrete Lagrange multipliers.

9 Conclusions

In this work, we study contact problems based on Signorini's problem with Tresca's friction and introduce a mixed variational formulation and its discretization with higher-order finite elements. In particular, the frictional function of Tresca's friction is included in a weak variational sense. For the existence and uniqueness of the discrete saddle point, a discrete inf-sup condition is considered which is motivated within an abstract framework of convex analysis. To prove the discrete inf-sup condition we use an higher-order approximation result and an inverse inequality for negative norms. The main result is that stability can be ensured if one reduces the quotient of the mesh sizes for the displacement variable and the Lagrange multipliers or the quotient of their polynomial degrees.
The discrete mixed variational formulation can be solved using its dual formulation which is given by a reformulation as a minimization problem. The approach is justified by the small number of variables capturing the Lagrange multipliers. Our main interest is to extend this approach to discretizations of higher-order where we use a standard tool of quadratic programming to capture the complicated higher-order constraints. We point out that the proposed solution scheme can also be used for time-dependent problems. Finally, numerical results are presented which show the applicability of the discrete mixed discretization by means of an example in production engineering. In particular, we demonstrate the influence of varying the quotient of mesh sizes and, therewith, the prediction of the theoretical findings.

Appendix

Proof (Lemma 1) Since 0 is contained in Λ_0, there holds $\Phi_0(v,0) = 0$ for all $v \in V$ due to (7). The condition (8) yields $\sup_{\mu_0 \in \Lambda_0} \Phi_0(v,\mu_0) = 0$ for $v \in K$. If $v \notin K$, then there exist a $\tilde{\mu}_0 \in \Lambda_0$, so that $\Phi(v,\tilde{\mu}_0) > 0$. Therefore, $\sup_{\mu_0 \in \Lambda_0} \Phi_0(v,\mu_0) \geq \sup_{\alpha>0} \Phi_0(v,\alpha\tilde{\mu}_0) = \sup_{\alpha>0} \alpha\Phi_0(v,\tilde{\mu}_0) = \infty$. $\square$

Proof (Theorem 5) H is Fréchet-differentiable in V with the Fréchet-derivative $H'(v) = A(v) - \ell$ where the functional $A \in L(V,V')$ is defined as $\langle A(v), w\rangle := a(v,w)$ for $v,w \in V$. For $\Phi_0(v,\mu_0) := \langle \mu_0, \beta_0(v) - g\rangle$, the condition (7) obviously holds. Let $v \in V$ with $\Phi_0(v,\mu_0) \leq 0$ for all $\mu_0 \in \Lambda_0$. Assuming, that $g - \beta_0(v) \notin G$. Due to the closedness and convexity of G and the separation theorem of Hahn-Banach there exists a $\tilde{\mu}_0 \in U_0'$ with

$$\langle \tilde{\mu}_0, g - \beta_0(v)\rangle < \inf_{w\in G} \langle \tilde{\mu}_0, w\rangle. \tag{28}$$

Since $0 \in G$, there holds

$$\langle \tilde{\mu}_0, g - \beta_0(v)\rangle < 0. \tag{29}$$

For $t \geq 0$ and $w \in G$, we obtain $tw \in G$. Assuming, that $\inf_{w\in G}\langle \tilde{\mu}_0, w\rangle < 0$, then we have $\inf_{w\in G}\langle \tilde{\mu}_0, tw\rangle = t \inf_{w\in G}\langle \tilde{\mu}_0, w\rangle \to -\infty$ for $t \to \infty$ in contradiction to (28). Therefore, there holds $\tilde{\mu}_0 \in G'$ which is a contradiction to (29). Thus, condition (8) is also fulfilled. From Lemma 1, we obtain that Φ_0 fulfills (3). By defining $\Phi_1(v,\mu_1) := \langle \mu_1, \beta_1(v)\rangle$, we finally obtain the assertion from Theorem 2. □

Proof (Theorem 6) The proof is standard, e.g., [24, Lem. 3.2]. For completeness, we present a proof including the boundedness of Λ_1. Evidently, the conditions (i)-(iii) of Theorem 3 hold with Φ_0 and Φ_1 as defined in the proof of Theorem 5. If $G' = \{0\}$, then $\Lambda_0 \times \Lambda_1$ is bounded and we immediately obtain the assertion from Theorem 3. If $G' \neq \{0\}$, then G' is unbounded and we need to varify that the mapping

$$(\mu_0,\mu_1) \mapsto \sup_{v\in V} -(\frac{1}{2}a(v,v) - \langle \ell, v\rangle + \langle \mu_0, \beta_0(v) - g\rangle + \langle \mu_1, \beta_1(v)\rangle) \tag{30}$$

is coercive. For this purpose, let ν_0, ν_1 be the constants of continuity and ellipticity, and $\mu := (\mu_0,\mu_1) \in \Lambda_0 \times \Lambda_1$. From Theorem 1, we obtain a v_μ with $\mathscr{L}(v_\mu,\mu_0,\mu_1) = \inf_{v\in V} \mathscr{L}(v,\mu_0,\mu_1)$ and $a(v_\mu, v) = \langle \ell, v\rangle - \langle \mu_0, \beta_0(v)\rangle - \langle \mu_1, \beta_1(v)\rangle$ for all $v \in V$. Due to the boundedness of Λ_1, there exists a $c \in \mathbb{R}_+$ with $\|\mu_1\|_{U_1'} \leq c$. Therefore, we obtain $\alpha\|\mu_0\|_{U_0'} \leq \sup_{v\in V, \|v\|_V = 1} \langle \ell, v\rangle - a(v_\mu, v) - \langle \mu_1, \beta_1(v)\rangle \leq \|\ell\|_{V'} + \nu_0\|v_\mu\|_V + c\|\beta_1\|_{L(V,U_1)}$. Thus, we have $\|v_\mu\|_V \to \infty$ for $\|\mu\|_{U_0'\times U_1'} \to \infty$. The assertion follows from $-\mathscr{L}(v_\mu,\mu) = \frac{1}{2}a(v_\mu,v_\mu) - \langle \mu_0, g\rangle \geq \frac{1}{2}\nu_1\|v_\mu\|_V^2 - \alpha^{-1}\|g\|_{U_0}(\|\ell\|_{V'} + \nu_0\|v_\mu\|_V + c\|\beta_1\|_{L(V,U_1)})$. □

Proof (Theorem 7) Since $\beta_0(V_h)$ is closed in U_0, we obtain from the closed range theorem, [34], that there exists an $\alpha \in \mathbb{R}_{>0}$

$$\alpha\|[\mu_0]\|_{U_0'/\ker\beta_{0|V_h}'} \leq \sup_{v\in V_h, \|v_h\|=1} \langle \mu_0, \beta_0(v_h)\rangle \tag{31}$$

for all $[\mu_0] \in U_0'/\ker\beta_{0|V_h}'$ where $[\mu_0] := \mu_0 + \ker\beta_{0|V_h}'$ and $\beta_{0|V_h}' : U_0' \to V_h'$ denotes the transpose of $\beta_{0|V_h}$. With $\tilde{\Lambda}_{0,H} := \{[\mu_{0,H}] \in U_0'/\ker\beta_{0|V_h}' \mid \mu_{0,H} \in \Lambda_{0,H}\}$ we define

$$\tilde{\mathscr{L}}(v_h,[\mu_{0,H}],\mu_{1,H}) := \mathscr{L}(v_h,\mu_{0,H},\mu_{1,H})$$

which is well-defined in $V_h \times \tilde{\Lambda}_{0,H} \times \Lambda_{1,H}$ due to the assumption $g \in \beta_0(V_h)$. By the same arguments as in the proof of Lemma 6, we obtain that

$$\tilde{\Lambda}_{0,H} \times \Lambda_{1,H} \ni ([\mu_{0,H}],\mu_{1,H}) \mapsto \sup_{v_h\in V_h} -\tilde{\mathscr{L}}(v_h,[\mu_{0,H}],\mu_{1,H})$$

is coercive. By Theorem 3 there exists $(u_h, [\lambda_{0,H}], \lambda_{1,H}) \in V_h \times \tilde{\Lambda}_{0,H} \times \Lambda_{1,H}$ with

$$\tilde{\mathscr{L}}(u_h, [\lambda_{0,H}], \lambda_{1,H}) = \inf_{v_h \in V_h} \sup_{[\mu_{0,H}] \in \tilde{\Lambda}_{0,H}, \lambda_{1,H} \in \Lambda_{1,H}} \tilde{\mathscr{L}}(v_h, [\mu_{0,H}], \mu_{1,H}).$$

Thus, $(u_h, \lambda_{0,H}, \lambda_{1,H})$ fulfills (12). □

Proof (Theorem 8) In the same way as in the proof of Theorem 6, we conclude that

$$\Lambda_{0,H} \times \Lambda_{1,H} \ni (\mu_{0,H}, \mu_{1,H}) \mapsto \sup_{v_h \in V_h} -\mathscr{L}(v_h, \mu_{0,H}, \mu_{1,H})$$

is coercive and, thus, a saddle point exists. The uniqueness is a direct consequence of (14) and the density of U_1' in $\tilde{U}_1'$. □

Proof (Lemma 2) The unique existence of $u^\mu \in V$ is guaranteed by the Lax-Milgram Lemma. The mapping $\hat{\beta} : V/\ker\beta \to U$ with $\hat{\beta}([v]) := \beta(v)$ and $[v] := v + \ker\beta \in V/\ker\beta$ is bijective and continuous. Since V and U are Banach spaces, the inverse $\hat{\beta}^{-1}$ is continuous, too. Let $\tilde{V} := \{v \in V \mid \ \|v\|_V \le \|\hat{\beta}^{-1}\|_{L(U,V/\ker\beta)} \|\beta(v)\|_U\}$. In order to show that $\tilde{V}$ is a non-empty set, let $w \in U$ and $v \in V$ with $\hat{\beta}^{-1}(w) = [v]$. If $\bar{z} \in \ker\beta$ such that $\|v - \bar{z}\|_V = \inf_{z \in \ker\beta} \|v - z\|_V$ and $v^* := v - \bar{z}$, we obtain

$$\beta(v^*) = \beta(v - \bar{z}) = \beta(v) = \hat{\beta}([v]) = w. \tag{32}$$

Therefore, we have

$$\begin{aligned}\|v^*\|_V = \inf_{z \in \ker\beta} \|v - z\|_V = \|\hat{\beta}^{-1}(w)\|_{V/\ker\beta} &\le \|\hat{\beta}^{-1}\|_{L(U,V/\ker\beta)} \|w\|_U \\ &= \|\hat{\beta}^{-1}\|_{L(U,V/\ker\beta)} \|\beta(v^*)\|_U,\end{aligned}$$

which implies that $v^* \in \tilde{V}$. Moreover, there is a $v^* \in \tilde{V}$ for each $w \in U$ such that (32) is valid, i.e., $\beta(\tilde{V}) = U$. Using these preparations, we conclude from the definition of the dual norm and continuity of $\hat{a}$ with constant C, that

$$\begin{aligned}\|\mu\|_{U'} = \sup_{w \in U \setminus \{0\}} \frac{\langle \mu, w \rangle}{\|w\|_U} = \sup_{v \in V \setminus \{0\}} \frac{\langle \mu, \beta(v) \rangle}{\|\beta(v)\|_U} &= \sup_{v \in \tilde{V} \setminus \{0\}} \frac{\hat{a}(u^\mu, v)}{\|\beta(v)\|_U} \\ \le \sup_{v \in \tilde{V} \setminus \{0\}} \frac{C\|u^\mu\|_V \|v\|_V}{\|\beta(v)\|_U} &\le C\|\hat{\beta}^{-1}\|^{-1}_{L(U,V/\ker\beta)} \|u^\mu\|_V.\end{aligned}$$

Setting $C_1 := C\|\hat{\beta}^{-1}\|_{L(U,V/\ker\beta)}$, we obtain the assertion. □

References

1. Babuška, I., Suri, M.: The h-p version of the finite element method with quasiuniform meshes. M^2AN **21**, 199–238 (1987)
2. Bernardi, C., Maday, Y.: Spectral methods.. (1997). In: Handbook of Numerical Analysis, vol. V. Handb. Numer. Anal., V, North-Holland, Amsterdam pp. 209–485
3. Blum, H., Kleemann, H., Rademacher, S., Schröder, A.: On solving frictional contact problems part ii: Dynamic case. Tech. rep., Fakultät für Mathematik, TU Dortmund (2008). Ergebnisberichte des Instituts für Angewandte Mathematik, Nummer 378

4. Blum, H., Kleemann, H., Rademacher, S., Schröder, A.: On solving frictional contact problems part iii: Unilateral contact. Tech. rep., Fakultät für Mathematik, TU Dortmund (2008). Ergebnisberichte des Instituts für Angewandte Mathematik, Nummer 379
5. Cea, J.: Lectures on optimization - theory and algorithms. Tata Institute of Fundamental Research Lectures on Mathematics and Physics. Mathematics. Springer-Verlag, Berlin (1978)
6. Dickopf, T., Krause, R.: Efficient simulation of multi-body contact problems on complex geometries: A flexible decomposition approach using constrained minimization. Int. J. Numer. Methods Eng. **77**(13), 1834–1862 (2009)
7. Dostál, Z., Horák, D., Kučera, R., Vondrák, V., Haslinger, J., Dobiáš, J.r., Pták, S.: FETI based algorithms for contact problems: Scalability, large displacements and 3D Coulomb friction. Comput. Methods Appl. Mech. Eng. **194**(2-5), 395–409 (2005)
8. Duvaut, G., Lions, J.L.: Inequalities in mechanics and physics. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin (1976)
9. Ehlich, H., Zeller, K.: Schwankungen von polynomen zwischen gitterpunkten. Math. Z. **86**, 41–44 (1964)
10. Ekeland, I., Temam, R.: Convex analysis and variational problems. Studies in Mathematics and its Applications. North-Holland Publishing Company, Amsterdam (1976)
11. Georgoulis, E.H.: Inverse-type estimates on hp-finite element spaces and applications. Math. Comput. **77**(261), 201–219 (2008)
12. Gill, P.E., Murray, W., Saunders, M.A.: Snopt: An sqp algorithm for large-scale constrained optimization. SIAM J. Optim. **12**(4), 979–1006 (2002)
13. Gill, P.E., Murray, W., Saunders, M.A.: SNOPT: An SQP algorithm for large-scale constrained optimization. SIAM Rev. **47**(1), 99–131 (2005)
14. Glowinski, R.: Numerical methods for nonlinear variational problems. Springer Series in Computational Physics. Springer-Verlag, New York (1984)
15. Glowinski, R., Lions, J.L., Trémolierès, R.: Numerical analysis of variational inequalities. Studies in Mathematics and its Applications. North-Holland Publishing Company, Amsterdam (1981)
16. Haslinger, J.: Mixed formulation of elliptic variational inequalities and its approximation. Apl. Mat. **26**, 462–475 (1981)
17. Haslinger, J., Kučera, R., Dostál, Z.: An algorithm for the numerical realization of 3D contact problems with Coulomb friction. J. Comput. Appl. Math. **164-165**, 387–408 (2004)
18. Haslinger, J., Lovíšek, J.: Mixed variational formulation of unilateral problems. Commentat. Math. Univ. Carol. **21**, 231–246 (1980)
19. Haslinger, J., Sassi, T.: Mixed finite element approximation of 3d contact problems with given friction: error analysis and numerical realization. Math. Mod. Numer. Anal. **38**, 563–578 (2004)
20. Haslinger, J., Z., D., Kučera, R.: On a splitting type algorithm for the numerical realization of contact problems with coulomb friction. Comput. Methods Appl. Mech. Eng. **191**(21-22), 2261–2281 (2002)
21. Hlávaček, I., Haslinger, J., Nečas, J., Lovíšek, J.: Solution of variational inequalities in mechanics. Applied Mathematical Sciences. Springer-Verlag, New York (1988)
22. Hüeber, S. and Matei, A. and Wohlmuth, B.I.: Efficient algorithms for problems with friction. SIAM J. Sci. Comput. **29**(1), 70–92 (2007)
23. Hüeber, S. and Stadler, G. and Wohlmuth, B.I.: A primal-dual active set algorithm for three-dimensional contact problems with Coulomb friction. SIAM J. Sci. Comput. **30**(2), 572–596 (2008)
24. Kikuchi, N., Oden, J.: Contact problems in elasticity: A study of variational inequalities and finite element methods. SIAM Studies in Applied Mathematics. SIAM, Society for Industrial and Applied Mathematics, Philadelphia (1988)
25. Kornhuber, R., Krause, R.: Adaptive multigrid methods for Signorini's problem in linear elasticity. Comput. Vis. Sci. **4**(1), 9–20 (2001)
26. Krebs, A.: On solving nonlinear variational inequalities by p-version finite elements. Ph.D. thesis, Fachbereich Mathematik der Universität Hannover (2004)
27. Krebs, A., Stephan, E.P.: A p-version finite element method for nonlinear elliptic variational inequalities in 2D. Numer. Math. **105**(3), 457–480 (2007)
28. Lions, J.L., Magenes, E.: Non-homogeneous boundary value problems and applications. Vol. I. Translated from the French by P. Kenneth. Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag (1972)
29. Panagiotopoulos, P.D.: Inequality Problems in Mechanics and Applications, Convex and Nonconvex Energy Functions. Birkhuser, Basel (1985)
30. Schröder, A.: Error control in h- and hp-adaptive FEM for Signorini's problem. J. Numer. Math. **17**(4), 299–318 (2009)

31. Schröder, A.: Mixed finite element methods of higher-order for model contact problems. (2009). Humboldt Universität zu Berlin, Institute of Mathematics, Preprint 09-16, submitted to SINUM
32. Triebel, H.: Interpolation theory, function spaces, differential operators. North-Holland Mathematical Library. North-Holland Publishing Company, Amsterdam (1978)
33. Wohlmuth, B.I., Krause, R.H.: Monotone multigrid methods on nonmatching grids for nonlinear multibody contact problems. SIAM J. Sci. Comput. **25**(1), 324–347 (2003)
34. Yosida, K.: Functional analysis. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin (1980)