

ON SOLVING FRICTIONAL CONTACT PROBLEMS PART III: UNILATERAL CONTACT

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ABSTRACT. In this paper, we study mixed variational finite element methods for the unilateral problems arising in contact mechanics. The presented discrete spaces can handle the case of non matching meshes in the contact area. We first state the static problem and then extend it to the frictional contact case, and end with the dynamic frictional problem.

1. INTRODUCTION

The third part of the article focuses on the unilateral contact between two elastic bodies. The problem is of interest in engineering applications if, for example, the interaction of a machine with a workpiece is regarded. The problems arising from applications often require to work with non matching meshes on the possible contact zone. There are several technics to cope with the arising problems like for example the mortar method [1],[8] or stabilization technics [7]. The main idea is to formulate the problem as a saddlepoint problem, which allows to handle the contact restraint in an integral form. The discretisation technics differ in the definition of the used lagrange multiplier space. For the contact with a rigid obstacle in [6] piecewise constant functions on a coarser grid are chosen. This approach has been extended for the unilateral contact problem by [11] for the frictionless two dimensional case. For the mortar method there are various publications for the unilateral contact and it's dynamical extension with friction, see [9]. We focus on the approach with piecewise constant functions for the lagrange multiplier, that is used for example in [11],[3] and show it's extension to frictional contact and dynamic contact. The approach has the advantage, that there are no restrictions to the possible contact zone as in the mortar method. We will fit the unilateral contact problem into the abstract framework given in part one in order to apply the solution technics explained there and extend the dynamic methods of part two to the unilateral problem.

The paper is organised as follows. We will first state the static problem and the discretisation technic and then show the extensions to frictional contact and dynamic contact. In contrast to the contact problem with a rigid obstacle examined in the first parts the constraints differ because now two domains have to be taken into account. However, we will see that the problem is of the same kind as the problems in part one and two so that the same solving technics can be applied.

2. NOTATION

Let the two bodies be described by the domains $\Omega^l \in \mathbb{R}^d$ with $l = 1, 2$ and $d = 2, 3$. The boundary $\partial\Omega^l$ of the domains is divided in three parts denoted by $\Gamma_D^l, \Gamma_N^l, \Gamma_C^l$ which are assumed to be disjunct and to have positive measures. We assume dirichlet boundary data on Γ_D^l and neumann data on Γ_N^l . The boundary Γ_C^l stands for the possible contact surface.

We use a linear elastic material model with small deformations. The linearized strain operator is given by $\varepsilon(u^l) := \frac{1}{2}(\nabla u^l + \nabla(u^l)^T)$, where ∇u^l is the gradient of the displacement u^l . Let $n^l(x)$ be the outer normal vector of $x \in \partial\Omega^l$. The stress operator of linear elasticity, which is defined by the modulus of elasticity E^l and by Poisson's number ν^l , is $\sigma(u^l)$. The displacement on the boundary in normal direction is given by $\delta_n(u^l)$ and $\sigma_n(u^l) = \sigma(u^l) \cdot n^l \cdot n^l$ is the stress in normal direction on the boundary. We define the tangential contact stress by $\sigma_t(u^l) = \sigma(u^l) \cdot n^l - \sigma_n(u^l) \cdot n^l$, see [10] for details.

The displacements are assumed to possess weak derivatives in $(L^2(\Omega^l))^d$ and thus are in the Sobolev spaces $\mathcal{H}^1(\Omega^l) := (H^1(\Omega^l))^d$.

The subspaces

$$\mathcal{H}_D^1(\Omega^l) = \left\{ v \mid v \in \mathcal{H}^1(\Omega^l), v|_{\Gamma_D^l} = 0 \right\}$$

are used to apply homogeneous dirichlet boundaries and we set $\mathcal{H}_D^1 := \mathcal{H}_D^1(\Omega^1) \times \mathcal{H}_D^1(\Omega^2)$.

Scalar products are written in the form $(\cdot, \cdot)_{2, \Omega^l} := (\cdot, \cdot)_{(L^2(\Omega^l))^d}$ and $(\cdot, \cdot)_{1, \Omega^l} := (\cdot, \cdot)_{\mathcal{H}^1(\Omega^l)}$ with the induced norms

$$\|v\|_{2, \Omega^l}^2 = (v, v)_{2, \Omega^l} \text{ and } \|v\|_{1, \Omega^l}^2 = (v, v)_{1, \Omega^l}.$$

The norm for $\mathcal{H}_D^1$ is given through

$$\|u\|^2 := \|u^1\|_{1, \Omega^1}^2 + \|u^2\|_{1, \Omega^2}^2,$$

where $u := (u^1, u^2)$.

Following the common notation we denote the trace space on $\Gamma_C^l \subset \partial\Omega^l$ of $H^1(\Omega^l)$ as $H^{\frac{1}{2}}(\Gamma_C^l)$ and dual by $H^{-\frac{1}{2}}(\Gamma_C^l)$, see [10].

We define the norm for functions $\lambda \in H^{1/2}(\Gamma_C)$ by:

$$\|\lambda\|_{1/2} = \inf \{ \|u\|_{1, \Omega^l} \mid u \in \mathcal{H}_D^1(\Omega^1) \text{ and } \gamma_n(u) = \lambda \}$$

and the norm of its dual space is given by

$$\|\mu\|_{-1/2} = \sup_{v \in \mathcal{H}_D^1(\Omega^l)} \frac{\langle \mu, \gamma_n(v) \rangle_{-\frac{1}{2}, \frac{1}{2}}}{\|v\|_{1, \Omega_i}},$$

where $\langle \cdot, \cdot \rangle_{-\frac{1}{2}, \frac{1}{2}}$ is the dual pairing between $(H^{-1/2})(\Gamma_C)$ and $H^{1/2}(\Gamma_C)$.

If the two domains are in contact in the reference configuration then there exists a common contact boundary Γ_C with $\Gamma_C = \Gamma_C^1 \cap \Gamma_C^2$ and the contact conditions can be directly applied on this part of the boundary: At the contact interface the two bodies may come into contact but must not penetrate each other which leads to the non-penetration condition

$$[u \cdot n](x) = u^1(x) \cdot n^1(x) + u^2(x) \cdot n^2(x) \leq 0.$$

Note that for $x \in \Gamma_C$ there holds $n^1(x) = -n^2(x)$.

If the two bodies are not in contact we assume a bijective mapping $\Phi : \Gamma_C^1 \rightarrow \Gamma_C^2$ between the two possible contact surfaces to be given. Further we define

$$n_\Phi = \begin{cases} \frac{\Phi(x)-x}{|\Phi(x)-x|}, & \text{if } x \neq \Phi(x) \\ n^1(x) = -n^2(x), & \text{if } x = \Phi(x) \end{cases}$$

as the normal vector in the contact area. The non-penetration condition for $x \in \Gamma_C^1$ then reads as:

$$[u \cdot n]_\Phi(x) = u^1(x) \cdot n_\Phi(x) - u^2(\Phi(x)) \cdot n_\Phi(x) \leq g.$$

Here g defines the gap function and is given via Φ by:

$$\Gamma_C^1 \ni x \rightarrow g(x) = |x - \Phi(x)| \in \mathbb{R}.$$

Under certain assumptions on Φ and on the geometry of the deformed configuration the above defined non-penetration condition is a close approximation of the geometrical non-penetration condition, see [5]. We will call $[u \cdot n]_\Phi$ the jump of the displacements.

3. STATIC CONTACT

This chapter examines the static frictionless contact problem and its discretisation.

3.1. continuous formulation. With the notation given in chapter one we are able to state the static unilateral contact problem. As we use the problem with an initial gap all contact conditions are expressed on the boundary Γ_C^1 .

The strong formulation is given by

Problem 3.1. The strong formulation of the problem is given by: Find u with

$$\begin{aligned} -\operatorname{div} \sigma(u^l) &= f^l && \text{in } \Omega^l \\ u^l &= 0 && \text{on } \Gamma_D^l, \\ \sigma(u^l) n^l &= p^l && \text{on } \Gamma_N^l, \\ \sigma_t(u^l) &= 0 \\ \sigma_{n_\Phi}(u^1) = -\Phi^* \sigma_{n_\Phi}(u^2) &\leq 0 \\ [u \cdot n]_\Phi &\leq g && \text{on } \Gamma_C^l \\ \sigma_{n_\Phi}(u) \cdot ([u \cdot n]_\Phi - g) &= 0 \end{aligned}$$

In this paper we define the multiplier on Γ_C^1 , the choice is of course arbitrary. We will skip the domain index and simply write Γ_C if we refer to the contact conditions.

The strong problem can be formulated as a mixed variational problem where the contact condition is satisfied in a weak sense, see e.g. [6]. In the mixed formulation the Lagrange multiplier can be interpreted as the normal force in the contact area. Therefore we define the bilinear form $a(\cdot, \cdot)$ as

$$(3.1) \quad a(v, w) = \sum_{k=1,2} \int_{\Omega^l} \sigma(v^l) : \varepsilon(w^l) \, dx, \quad v, w \in \mathcal{H}_D^1$$

The weak contact condition is now defined by:

$$(3.2) \quad b(\lambda, u) = \langle \lambda, [u \cdot n]_{\Phi} \rangle_{-\frac{1}{2}, \frac{1}{2}}.$$

With this notation the mixed formulation of problem 3.1 is given by

Problem 3.2. Find $(u, \lambda) \in \mathcal{H}_D^1 \times H_+^{-\frac{1}{2}}(\Gamma_C)$ with

$$\begin{aligned} a(u, v) + b(\lambda, v) &= (f, v) \quad \forall v \in \mathcal{H}_D^1 \\ b(\mu - \lambda, u) &\leq \langle \mu - \lambda, g \rangle_{-\frac{1}{2}, \frac{1}{2}} \quad \forall \mu \in H_+^{-\frac{1}{2}}(\Gamma_C) \end{aligned}$$

The existence and uniqueness of this problem is given, see e.g. [6]. The problem has the form that is required for the framework of part one.

3.2. discrete spaces. The choice of the discrete spaces is important for the stability of the discrete problem; the stability argument can be found in [2]. We recall the discrete spaces defined in part 1:

We assume a finite element mesh $\mathcal{T}_h^l$ consisting of rectangles in 2d and hexadral in 3d with a meshwidth h_l on each domain Ω^l . On the contact boundary a finite element mesh $\mathcal{T}_{C,H}$ consisting of lines or rectangles with a meshwidth H is given.

Let $\Psi_T^l : [-1, 1]^d \rightarrow T \in \mathcal{T}_h^l, \Psi_{C,T_C} : [-1, 1]^{d-1} \rightarrow T_C \in \mathcal{T}_{C,H}$ be bijective and sufficiently smooth transformations and let $p_T^l, p_{C,T_C} \in \mathbb{N}$ be degree distributions on $\mathcal{T}_h^l$ and $\mathcal{T}_{C,H}$, respectively. Using the polynomial tensor product space S_k^q of order q on the reference element $[-1, 1]^d$, we define

$$S_l^p(\mathcal{T}_h^l) := \{v \in \mathcal{H}_D^1(\Omega^l) \mid \forall T \in \mathcal{T}_h^l : v|_T \circ \Psi_T^l \in S_k^{p_T^l}\}.$$

With this definition the space for the displacements in Ω^l is given by:

$$(3.3) \quad V_{h_l}^{p_l} := \{u \in \mathcal{H}^1(\Omega^l) \mid u|_{T_h} \in S_l^p(\mathcal{T}_h^l)\}.$$

For ease of notation we set $V_h^p := V_{h_1}^{p_1} \times V_{h_2}^{p_2}$. For the Lagrange multiplier we have :

$$(3.4) \quad M_H^{p_C} := \{\nu \in L^2(\Gamma_C) \mid \forall T_C \in \mathcal{T}_{C,H} : \nu|_{T_C} \circ \Psi_{C,T_C} \in S_{k-1}^{p_{C,T_C}}\}$$

In this paper we restrict ourself to $V_{h_l} := V_{h_l}^1$ and $M_H := M_H^0$, that means piecewise (bi-)linear for the displacements and piecewise constant for the Lagrange multiplier. For higher order discrete spaces the technic stays the same, some results are shown in [2].

For the given set of discrete spaces the problem reads:

Problem 3.3. Find $(u_h, \lambda_H) \in V_h \times M_H$

$$\begin{aligned} a(u_h, v_h) + b(\lambda_H, v_h) &= f_{ext}(v_h) & v_h \in V_h \\ b(\nu_H - \lambda_H, u_h) &\leq g & \nu_H \in M_H^+, \end{aligned}$$

where M_H^+ is given by $M_H^+ = \{\nu \in M_H \mid \nu \geq 0\}$.

We have in Matrix notation:

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \cdot \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \lambda = \begin{pmatrix} f^1 \\ f_2 \end{pmatrix}$$

$$\begin{pmatrix} B_1^T & B_2^T \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \leq g,$$

where A_l is the stiffness matrix of Ω^l and the constraint matrix is defined via $B_{ij} := \int_{\Gamma_C} \psi_i(\varphi_j \cdot n_\Phi) d\Gamma$. The basis functions of the multiplier space are given by $\psi_i \in M_H$. The basis functions of V_h^l are given by φ_k^l , $1 \leq k \leq N^l$, for the constraint matrix we set:

$$\varphi_j(x) \cdot n_\Phi(x) := \begin{cases} \varphi_n^1(x) \cdot n_\Phi(x) & \text{for } j \leq N^1 \\ \varphi_m^2(\Phi(x)) \cdot n_\Phi(x) & \text{for } j > N^1 \end{cases}.$$

Finally the entries of the gap vector are given by $g_i := \int g \cdot \psi_i d\Gamma$

remark 1:

The problem can be solved by applying the Schur Complement technic explained in part 1. The blockstructure can be used for solving the inner problem parallel.

remark 2:

If we only regard the upper part of the block structure we have the contact problem with a rigid obstacle.

3.3. Numerical examples. example 1: We consider a hertz contact problem: A circular disc is being pressed against a plane by a point force on it's top pole. For this example the contact forces p can be computed analytically by:

$$p(x) = \frac{2 \cdot f}{\pi \cdot b^2} \sqrt{(b^2 - x^2)}, \quad b \leq x, \quad b = \sqrt{\frac{f \cdot r \cdot (1 - \nu^2)}{E \cdot \pi}}$$

Here b is the half width of the contact surface and x is the distance from the center of the contact surface which is at the lower pole of the disc. As material constants for the circle we set $E = 7000$ and $\nu = 0.3$ and replace the plane by a rectangular linear elastic body with zero Dirichlet boundary conditions on the bottom. To contain the properties of the original problem we use larger Young's modulus on the rectangle than on the disc. We set $E = 10^{10}$ and $\nu = 0.45$. In order to avoid a strong irregularity on the upper pole we replace the point force by a surface force. In picture 3.1 (right) the deformed bodies with the von Mises stress are shown and on the left the analytical contact forces against the lagrange multiplier of our solution is plotted, thus showing a good approximation.

example 2: As a second example we show an application from mechanical engineering simulating a grinding process. Here the machine work piece interaction is simulated by the contact problem. The grinding tool has homogeneous Dirichlet boundary conditions at the end of it's shank. The workpiece is clamped at it's bottom, thus introducing Dirichlet boundary conditions. The depth of cut is given by $40\mu\text{m}$ which results in a local overlapping of the domains. The grinding tool has a diameter of 12 mm.

We see the initial situation in 3.2 on the left and the deformation in x-

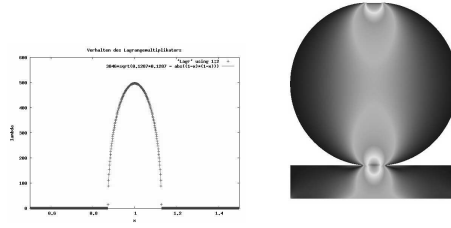


FIGURE 3.1. problem setting / von Mises stress

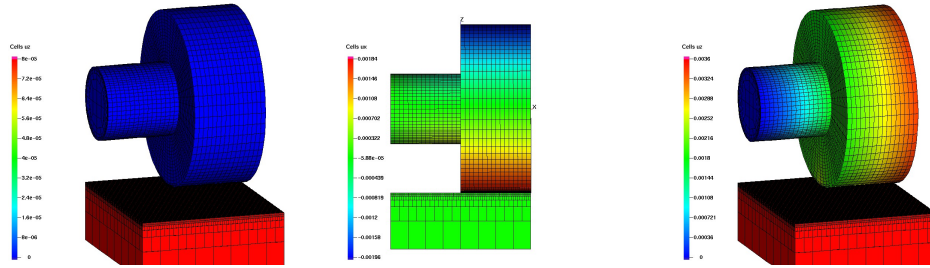


FIGURE 3.2. problem setting at start / displacement in x-direction / displacement in z-direction

direction and z-direction in the middle and on the left. As the displacement of the shank which is mainly in z-direction induces a displacement in x-direction the resulting contact areas between work piece and grinding tool differ from the results of a rigid obstacle simulation. Thus the simulation becomes more accurate.

The resulting von mises stresses of the contact situation and the are shown in figure 3.3(left and middle). The contact stress which is given via the Lagrange multiplier is plotted on the right.

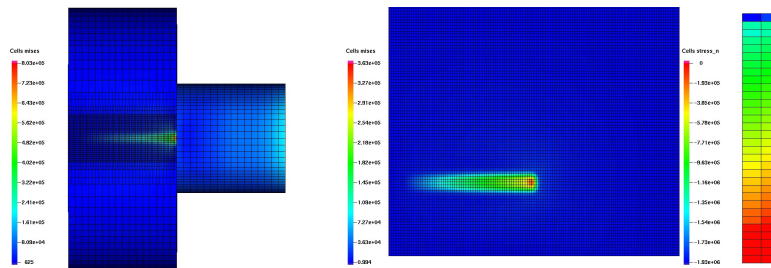


FIGURE 3.3. von Mises stress on grinding tool (left), on work piece (middle) and contact forces

4. STATIC FRICTIONAL CONTACT

The extension to frictional contact can be done by the abstract framework described in part one. There is a difference though: As now there are two bodies to be considered, the frictional condition now depends on the contact situation of the two domains. In this chapter we will first discuss the

frictional condition, then state the problem formulation. After that we will show an example.

4.1. Continuous formulation and discretisation. For frictional contact we will use Tresca's law which states:

Frictional conditions are modelled by assuming that there is no sliding when the magnitude of the tangential forces are below a critical value given by a function s . If the tangential forces reach this critical value, sliding is obtained in a direction of the tangential forces.

In Coulomb friction the function s is set to the magnitude of the normal forces times a friction coefficient. To solve this, the problem is put into a fixed point scheme. For solving technics we refer to part one.

To state the problem we need to focus on the tangential forces and formulate the interacting of the two bodies. That means we need to find the effective tangential force by summing up the forces of the two domains. Therefore we make again use of the mapping ϕ and gain a condition which is similar to the contact condition (number).

With $\gamma_t : H^1(\Omega^l)^d \rightarrow H^1(\Gamma_C^l)^{d-1}$ we denote $\gamma_t(u^l)$ the displacement of u^l in tangential direction for each domain, where $\sigma_{nt}(u^l)$ are the tangential stresses of u^l . To state the problem we need an extension to the definition of the normal vector defined in section 1. We will split every $x \in \Gamma_C^1$ into it's normal and tangential part according to $\{n_\phi(x), t_\Phi(x)\}$, see also part one (?!). The resulting tangential stress for the frictional contact is given by $\sigma_{nt_\Phi}(u^1) = -\sigma_{nt_\Phi}(u^2)$. The effective tangential displacements are given by $\gamma_{t_{eff}}(u) = \gamma_{t_\Phi}(u^1) - \gamma_{t_\Phi}(u^2)$, where γ_{t_Φ} is the mapping on the tangential trace space via Φ , as described for the normal mapping.

The strong formulation of the frictional contact problem reads as:

Problem 4.1. Find $u = (u^1, u^2)$ with

$$\begin{aligned}
 -\operatorname{div} \sigma(u^l) &= f^l && \text{in } \Omega^l \\
 u^l &= 0 && \text{on } \Gamma_D^l, \\
 \sigma(u^l)n^l &= p^l && \text{on } \Gamma_N^l, \\
 &\text{and} \\
 \sigma_{n_\Phi} &\leq 0 && \text{on } \Gamma_C^1 \\
 [un]_\Phi &\leq g && \text{on } \Gamma_C^1 \\
 \sigma_{nt_\Phi}(u) \cdot ([un]_\Phi - g) &= 0 && \text{on } \Gamma_C^1 \\
 |\sigma_{nt_\Phi}(u)| &\leq s && \text{on } \Gamma_C^1 \\
 |\sigma_{nt_\Phi}(u)| < s &\Rightarrow \gamma_{t_{eff}}(u) = 0 && \text{on } \Gamma_C^1 \\
 |\sigma_{nt_\Phi}(u)| = s &\Rightarrow \exists \zeta \in \mathbb{R}_{\geq 0} : \gamma_{t_{eff}}(u) = -\zeta \sigma_{t_{eff}}(u) && \text{on } \Gamma_C^1.
 \end{aligned}$$

Now, with this definition of $\gamma_{t_{eff}}$ we can directly apply the methods of part one for frictional problems and we gain the variational inequality

$$\begin{aligned}
 a(u, v - u) + \int_{\Gamma_C^1} s (|\gamma_{t_{eff}}(v)| - |\gamma_{t_{eff}}(u)|) d\Gamma &\geq \\
 (f, v - u) + (p, v - u)_{\Gamma_N}. &
 \end{aligned}$$

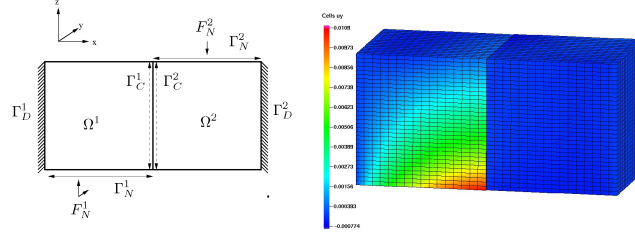


FIGURE 4.1. problem setting / von Mises stress

for all $v \in \mathcal{K}$.

We define the functional $j : V \rightarrow \mathbb{R}$ by $j(v) = (s, |\gamma_{t_{eff}}(v)|)_{0, \Gamma_C^1}$, and set

$$\Lambda_t := \left\{ \mu_t \in (L^2(\Gamma_C^1))^{d-1} \mid |\mu_t| \leq s \right\}.$$

Following part one we define the saddle point problem for frictional contact as:

Problem 4.2. Find $(u, \lambda_n, \lambda_t)$ with

$$\forall v \in \mathcal{H}_D^1 :$$

$$a(u, v) + b_n(\lambda_n, v) + b_t(\lambda_t, v) = f_{ext}(v)$$

$$\forall (\mu_n, \mu_t) \in \Lambda_n \times \Lambda_t :$$

$$b_n(\mu_n - \lambda_n, u) + b_t(\mu_t - \lambda_t, u) - g(\mu_n) \leq 0 \quad .$$

where the bilinear form b_n is given by $b_n := b$ as defined in section 2 and the bilinear form b_t is defined via

$$b_t(\mu, v) := (\mu, \gamma_{t_{eff}}(v))_{0, \Gamma_C^1}.$$

4.2. Numerical example: As a numerical example we take two three dimensional blocks that are in contact in reference configuration and are deformed by opposing surface forces in z-direction. On domain one an additional force in y-direction is set, as shown in figure 4.1 (left). On the right side of figure 4.1 the deformed blocks in z-direction are shown.

Of more interest are the contact forces represented by λ_n and the tangential forces λ_t that show the stick and slip part of the contact zone. illustrated in figure 4.2

5. DYNAMIC CONTACT PROBLEM

In section two and three we demonstrated how the bilateral contact problem can be solved via the ideas of part one. For the dynamical contact problem we use the ideas of part two. The bilateral dynamic contact problem differs from the dynamic rigid contact problem in the definition of the contact constraint. However the structure of the problem remains the same, so we can apply the time discretisation technics of part two. We will start with the problem definition by the strong and weak formulation. For details of the time discretisation technics we refer to part two.

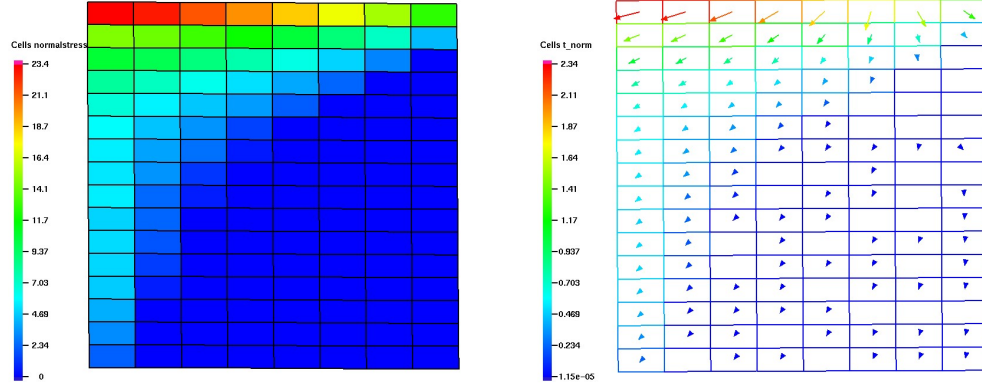


FIGURE 4.2. normal (left) and tangential (right) stress

5.1. problem formulation. For the dynamic frictional problem we define a time interval $I := \{0, T\}$ and search for a solution $u \in \mathcal{H}_D^1 \times I$ for which holds the strong formulation:

Problem 5.1. Find $u \in \mathcal{H}_D^1 \times I$ with

$$\begin{aligned}
 \rho^l \ddot{u}^l - \operatorname{div}(\sigma(u^l)) &= f^l && \text{in } \Omega^l \times I \\
 u^l &= 0 && \text{on } \Gamma_D^l, \\
 \sigma(u^l) n^l &= p^l && \text{on } \Gamma_N^l, \\
 &\text{and} \\
 \sigma_{n_\Phi} &\leq 0 && \text{on } \Gamma_C^1 \\
 [un]_\Phi &\leq g && \text{on } \Gamma_C^1 \\
 \sigma_{nt_\Phi}(u) \cdot ([un]_\Phi - g) &= 0 && \text{on } \Gamma_C^1 \\
 |\sigma_{nt_\Phi}(u)| &\leq s && \text{on } \Gamma_C^1 \\
 |\sigma_{nt_\Phi}(u)| < s &\Rightarrow \gamma_{t_{eff}}(\dot{u}) = 0 && \text{on } \Gamma_C^1 \\
 |\sigma_{nt_\Phi}(u)| = s &\Rightarrow \exists \zeta \in \mathbb{R}_{\geq 0} : \gamma_{t_{eff}}(\dot{u}) = -\zeta \sigma_{nt}(u) && \text{on } \Gamma_C^1.
 \end{aligned}$$

Here the first time derivative is denoted with a dot and the second time derivative with two dots and ρ^l is the density of the domain Ω^l . Like described in part two we use a Rothe's method by first discretising in time via a Newmark scheme and solving the spatial problems by low order galerkin methods. This leads to the following quasi static problem which has to be solved for every time step.

Problem 5.2. Find $(u, \lambda_n, \lambda_t)$ with $u^0 = u_s$, $v^0 = v_s$ and $a^0 = a_s$, such that $(u^n, \lambda_n^n, \lambda_t^n) \in V^n \times \Lambda_n^n \times \Lambda_t^n$ is the solution of the system

$$\begin{aligned}
 c(u^n, \varphi) + b_n(\lambda_n^n, \varphi) + b_t(\lambda_t^n, \varphi) &= (F^n, \varphi) \\
 \langle \mu_n - \lambda_n^n, [u \cdot n] - g^n \rangle \\
 + \left\langle \mu_t - \lambda_t^n, \frac{1}{k} (\gamma_{t_{eff}}(u^n) - r^n) \right\rangle &\leq 0.
 \end{aligned}$$

The bilinear form $c(\cdot, \cdot)$, the exterior forces F^n and the function r^n depend on the used time stepping scheme. For the newmark scheme which is used

in the following example we have according to part 2:

$$\begin{aligned} r^n &:= \gamma_{t_{eff}}(u^{n-1} - \frac{1}{2}k^2 a^{n-1}) \\ &= \gamma_{t_\Phi}(u^{1,n-1} - \frac{1}{2}k^2 a^{1,n-1}) - \gamma_{t_\Phi}(u^{2,n-1} - \frac{1}{2}k^2 a^{2,n-1}), \end{aligned}$$

with the definition of $\gamma_{t_{eff}}$ from section 3. For the discrete quasi static problems we use the same spaces as used in the static case. Following Part 2 we get the saddlepoint problem in $\mathbb{R}^m$:

Problem 5.3. Find $(\bar{u}, \bar{\lambda}) \in \mathbb{R}^m \times \mathbb{R}^{\tilde{m}}$ with:

$$\begin{aligned} A\bar{u}^n + B_n\bar{\lambda}_n^n + B_t\bar{\lambda}_t^n &= \bar{F}^n \\ (\bar{\mu}_n - \bar{\lambda}_n^n)^T (B_n^T \bar{u}^n - \bar{g}^n) \\ + \frac{1}{k} (\bar{\mu}_t - \bar{\lambda}_t^n)^T (B_t^T \bar{u}^n - \bar{r}^n) &\leq 0, \end{aligned}$$

which must hold for all $\bar{\mu}_n \in \mathbb{R}_{\leq 0}^{\tilde{m}}$ and all $\bar{\mu}_t \in \mathbb{R}_{\leq 0}^{\tilde{m}}$. Here, $A := M + \frac{1}{2}k^2 K$ is the generalised stiffness matrix, $M \in \mathbb{R}^{m \times m}$ is the mass matrix and $K^n \in \mathbb{R}^{m \times m}$ is the stiffness matrix. The matrix $B_n \in \mathbb{R}^{m \times \tilde{m}}$ represents the the contact conditions and the matrix $B_t \in \mathbb{R}^{m \times \tilde{m}}$ the friction conditions, respectively.. Of course the matrices M, K, B_n, B_t have the same block structure as explained in section 2.

Though the quasistatic problem depends on the used time stepping method the structure of problem remains the same. We see that problem 5.2 fits into the abstract framework, so the solution technics can be applied.

remark1:For the twobody contact problem there is often need for stabilization because the nonpenetration condition may lead to artificial oscillations, see [4], [8]. These stabilization technics do not change the quasistatic problem so they can be incorporated into the presented schemes.

remark 2:For the frictionless dynamic contact all parts in the above sceme that include the tangential multiplier λ_t can be dropped.

5.2. numerical example. As a numerical example we use a contact between 2 blocks in 2d where sliding occurs. First we show the frictionless contact then then the contact under coulomb's law of friction.

For an analysis The initial situation is shown in picture 5.1. The upper block is given by $\Omega^1 = [4.1, 12.1] \times [0, 8]$, with a circular bottom which is defined by $r = 80, m = (8.1, 80)$, the second block is defined as $\Omega^2 = [2, 12] \times [-5, -0.02]$. The bottom part of the second block has restrictions in y -direction, thus allowing only movement in x -direction.

Both blocks are given an initial velocity $v_1 = (0.2, -0.1)$ for the first and $v_2 = (-0.1, 0)$ for the second. The vertical part of v_1 leads to a contact.situation which is illustrated in 5.1(right). The contact situation shown in 5.1(middle) results from frictionless contact.

The time intervall I is given by $[0, 2]$, the material parametrs are $E = 500, \nu = 0.3$ and the densities on domain 1 and 2§ are $\rho_1 = 1, \rho_2$. In figure 5.2 we illustrate the energy of the frictionless case. As expected the overall energy is conserved. The contact stressover time, which is represented by the lagrangemultiplier is shown on the right. Here we use the the total stress given

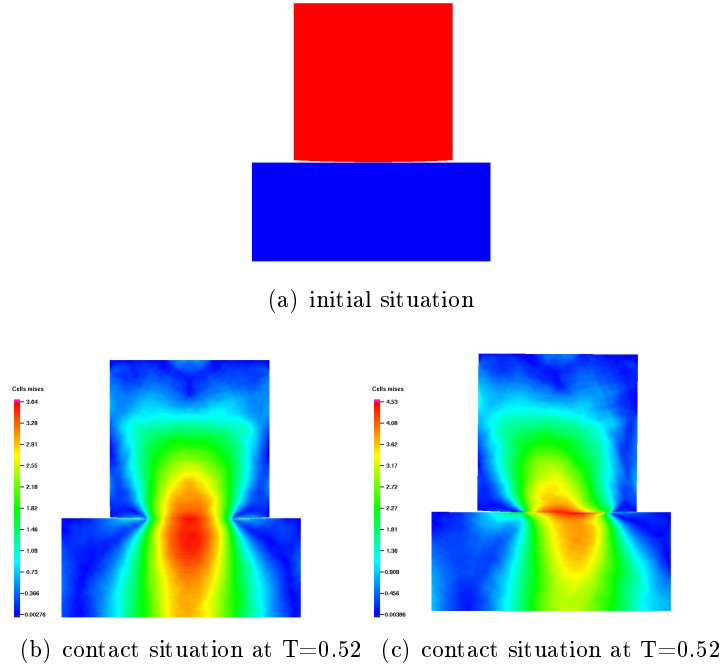


FIGURE 5.1. Dynamic Contact, initial condition(left),frictionless contact situation(middle),frictional contact situation(right)

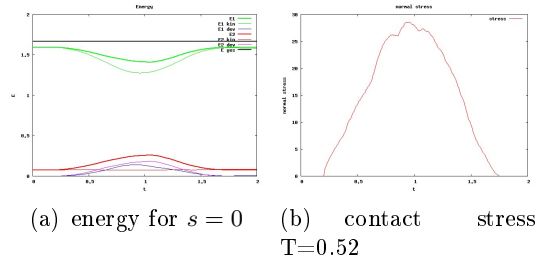


FIGURE 5.2. energy diagram (left) and contact stress (right) for $s = 0$

by $\lambda_{tot} = \int_{\Gamma_C} \lambda d\Gamma_C$.

In figure 5.3 we illustrate the energy of the frictional case (with $s = 0.5$). We see the decline of the total energy due to the frictional contact . The contact stress over time, which is represented by the lagrangemultiplier is shown on the right. The total stress over time for $s = 0.5$ seems to correspond to the frictionless case $s = 0$. However, the contact forces differ which is illustrated in 5.4. Here we see the Lagrange multiplier at $T = 0.52$ for the frictionless and the frictional contact. On the right the tangential stress is illustrated.

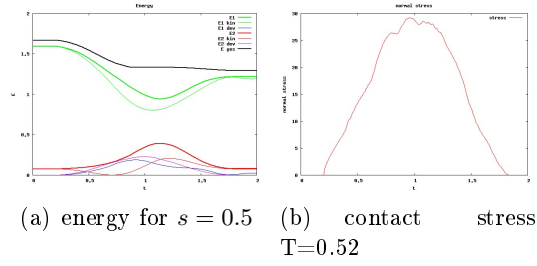


FIGURE 5.3. energy diagram (left) and contact stress (right) for $s = 0.5$

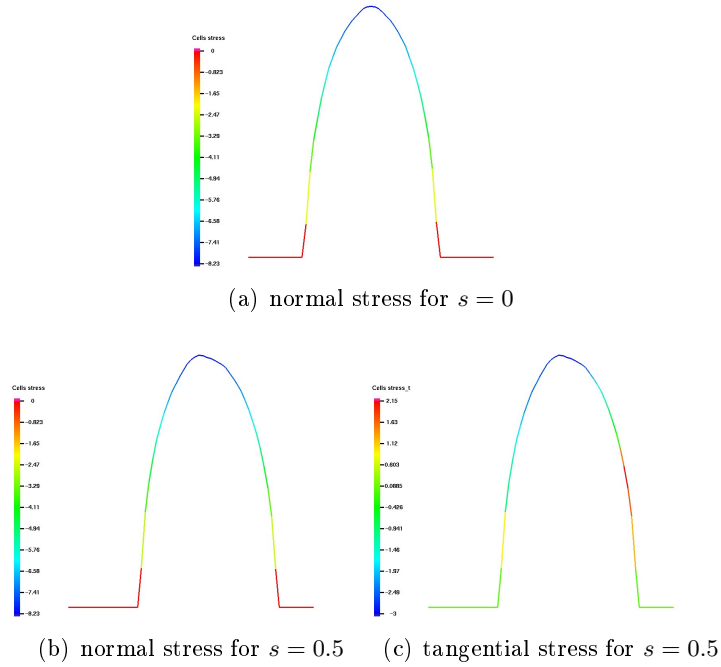


FIGURE 5.4. normal and tangential stress at $T = 0.52$

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