Adaptive Finite Element Discretisation of the Spindle Grinding Wheel System

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Summary. In the simulation of the NC-shape grinding process, a finite element model of the grinding machine is included. To enhance the accuracy and efficiency of the finite element computation, a posteriori error estimation and resulting adaptive mesh refinement techniques are used. In this note, a dual weighted a posteriori error estimate for a linear second order hyperbolic model problem is derived. Numerical results illustrate the performance of the presented approach.

1 Introduction

To model the interaction between the grinding process and the machine structure is indispensable in the simulation of the NC-shape grinding process. The coupling of separate machine and process simulations is a common simulation approach. We use an empiric force model in conjuction with a geometric-kinematical simulation to model the process [7]. The machine model is described in [15]. It is based on a finite element simulation, in which the spindle and the grinding wheel are explicitly considered. The remaining parts of the grinding machine are modelled by elastic bearings. The simulations are coupled by the exchange of the predicted grinding force, which is used as Neumann type boundary condition in the finite element simulation, and of the displacement of the grinding wheel, which changes the contact conditions in the geometric-kinematical simulation. Because of the varying length scales, the diameter of the grinding wheel is about 100 mm and the depth of cut is less than 1 mm, adaptive finite element algorithms are an appropriate tool to obtain an efficient simulation.

In general, a posteriori error estimates for second order hyperbolic problems are possible for two different discretisation approaches. One of them uses space time Galerkin methods for discretisation and applies similar techniques for error control as in the static case ([2, 3, 10, 12]). The other one is based on finite differences in time and finite elements in space. Here, separate error estimators are used for the space and time direction ([9, 11, 16]) or error estimates

for the whole problem ([1, 6]) are derived.
Starting from the weak formulation of the wave equation, a space time Galerkin discretisation is introduced in Section 2. In the next Section, the goal-oriented a posteriori error estimate is derived and an adaptive refinement algorithm based on it is discussed. Numerical results, which illustrate the performance of the developed approach, are presented in Section 4. The article concludes with a discussion of the results and an outlook.

2 Continuous Formulation and Space Time Galerkin Discretisation

Based on the weak formulation of the linear wave equation, a space time nonconforming Petrov Galerkin scheme with continuous basis functions in time is introduced. We consider the linear wave equation

$$\rho\ddot{u} - \operatorname{div}(\kappa\nabla u) = f \tag{1}$$

on the domain $\Omega \subset \mathbb{R}^2$ and the time interval $I = [0, T]$ with the initial conditions $u(0) = u_s$ and $\dot{u}(0) = v_s$ as well as homogeneous Dirichlet boundary conditions. For notational simplicity, the density ρ is set equal to 1. The parameter κ describes the elasticity coefficient.
Rewriting equation (1) as a first order system, multiplying by suitable test functions, and spatial integration by parts lead to the weak formulation:

$$\forall\varphi = (\psi, \chi) \in U \times V: \quad A(w, \varphi) = 0 \tag{2}$$

Here, $w = (u, v) \in U \times V$ is the weak solution and

$$\begin{aligned} U &:= \left\{u | u \in L^2(I; H_0^1(\Omega)),\, \dot{u} \in L^2(I; L^2(\Omega))\right\}, \\ V &:= \left\{v | v \in L^2(I; L^2(\Omega)),\, \dot{v} \in L^2(I; H^{-1}(\Omega))\right\} \end{aligned}$$

are the appropriate trial spaces, which are continuously embedded into $C\left(I; L^2(\Omega)\right)$. The bilinear form A is given by

$$\begin{aligned} A(w, \varphi) := & ((\dot{u} - v, \psi)) + ((\dot{v}, \chi)) + (a(u)(\chi)) - ((f, \chi)) \\ & + (u(0) - u_s, \psi(0)) + (v(0) - v_s, \chi(0)) \end{aligned}$$

with $((\psi, \chi)) := \int_0^T \int_\Omega (\psi\chi)\, dx\, dt$ and $a(u, \chi)) := \int_0^T (\kappa\nabla u, \nabla\chi)\, dt$.
The time interval I is decomposed into M subintervals $I_m := (t_{m-1}, t_m]$ with $0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = T$ and $k_m := t_m - t_{m-1}$. The finite element trial space in time step m, V_h^m, is based on the spatial mesh $\mathbb{T}_h^m$ and on bilinear basis functions. In time, piecewise linear continuous basis functions are used for the trial space and piecewise constant functions for the test space:

$$\begin{aligned} V_{kh} &:= \left\{ v_{kh} \in C(I; H_0^1(\Omega)) \left| v_{kh|_{I_m}} \in \tilde{\mathcal{P}}_1(I_m, V_h^m),\, v_{kh}(0) \in V_h^0 \right. \right\} \\ W_{kh} &:= \left\{ v_{kh} \in L^2(I; H_0^1(\Omega)) \left| v_{kh|_{I_m}} \in \mathcal{P}_0(I_m, V_h^m),\, v_{kh}(0) \in V_h^0 \right. \right\} \end{aligned}$$

The space $\tilde{\mathcal{P}}_1(I_m, V_h^m)$ is a slight modification of the space of linear polynomials (see [14]). Eventually, the discrete problem is to find $w_{kh} = (u_{kh}, v_{kh}) \in V_{kh} \times V_{kh}$ with

$$\forall \varphi_{kh} = (\psi_{kh}, \chi_{kh}) \in W_{kh} \times W_{kh} : \quad A(w_{kh}, \varphi_{kh}) = 0. \tag{3}$$

3 A Posteriori Error Estimation

In this section, the a posteriori error estimate is derived. In the first step an abstract result from [5] is applied on the present situation. Then, the error estimate is transformed into a computable estimate by well-tested approximations. Afterwards, it is used as basis for an adaptive refinement process.

Functionals of interest of the form $J(w) := \int_0^T J_1(w(t))\,dt$ are considered, where J_1 is an arbitrary three times continuously differentiable functional. The Lagrangian is defined by $\mathcal{L}(w,z) := J(w) - A(w)(z)$. We say $(w,z) \in (U \times V) \times (V \times U)$ is a stationary point of $\mathcal{L}$, if

$$\forall (\delta w, \delta z) \in (U \times V) \times (V \times U) : \quad \mathcal{L}'(w,z)(\delta w, \delta z) = 0.$$

The discrete stationary point $(w_{kh}, z_{kh}) \in (V_{kh} \times V_{kh}) \times (W_{kh} \times W_{kh})$ is given analogously. Following the results in [5, 13], we obtain the abstract error representation

$$\begin{aligned} J(w) - J(w_{kh}) &= \frac{1}{2}\mathcal{L}'(w_{kh}, z_{kh})(w - \tilde{w}_{kh}, z - \tilde{z}_{kh}) + \mathcal{R}_{kh} \\ &= \frac{1}{2}\rho(w_{kh})(z - \tilde{z}_{kh}) + \frac{1}{2}\rho^\star(w_{kh}, z_{kh})(w - \tilde{w}_{kh}) + \mathcal{R}_{kh}, \end{aligned}$$

with arbitrary $\tilde{w}_{kh} \in V_{kh} \times V_{kh}$ and $\tilde{z}_{kh} \in W_{kh} \times W_{kh}$. In the proof, we have to pay attention to the fact that a nonconforming Petrov Galerkin discretisation scheme is used. Here, the primal and the dual residual are given by

$$\begin{aligned} \rho(w)(\varphi) &:= \mathcal{L}'_z = -A(w, \varphi) \\ \rho^\star(w,z)(\varphi) &:= \mathcal{L}'_w = J'(\varphi) - A(\varphi, z), \end{aligned}$$

respectively. The remainder term $\mathcal{R}_{kh}$ is bounded above by the third power of the error.

The weights, which represent the interpolation error, are approximated by $w - \tilde{w}_{kh} \approx \Pi_{kh} w_{kh}$ and $z - \tilde{z}_{kh} \approx \Pi_{kh} z_{kh}$. Here, the operator Π_{kh} is given by $\Pi_{kh} := i_{kh} - \mathrm{id}$, where i_{kh} is a patchwise interpolation of higher order [4]. The operator Π_{kh} approximates the interpolation error in space and time. The spatial counterpart is defined by $\Pi_h := i_h - \mathrm{id}$ and the temporal one by $\Pi_k := i_k - \mathrm{id}$. Eventually, we obtain the computable error representation

$$J(w) - J(w_{kh}) \approx \frac{1}{2}\left[\rho(w_{kh})(\Pi_{kh} z_{kh}) + \rho^\star(w_{kh}, z_{kh})(\Pi_{kh} w_{kh})\right].$$

Using the identity

$$\Pi_{kh}\varphi_{kh} = i_k\Pi_h\varphi_{kh} + \Pi_k\varphi_{kh},$$

which holds true for tensor product trial functions, the error estimate is split into a temporal part η_k and a spatial part η_h:

$$\begin{aligned} J(w) - J(w_{kh}) &= \frac{1}{2}\left[\rho(w_{kh})(\Pi_k z_{kh}) + \rho^\star(w_{kh}, z_{kh})(\Pi_k w_{kh})\right] \\ &\quad + \frac{1}{2}\left[\rho(w_{kh})(i_k\Pi_h z_{kh}) + \rho^\star(w_{kh}, z_{kh})(i_k\Pi_h w_{kh})\right] \\ &=: \eta_k + \eta_h. \end{aligned}$$

The spatial residual terms are integrated by parts to localise the error estimate as basis for an adaptive refinement process. This process consists of several steps. In the first step, a space time refinement strategy decides, whether a refinement in spatial or temporal direction or in both directions is performed. We use an equlibration strategy, which was developed in [14]. The temporal refinement strategy is a simple fixed fraction strategy [4]. In space, a more complex global fixed fraction strategy [14] is used. There, all refinement indicators of all mesh cells are compared. After the adaptive refinement, the meshes are regularised to ensure a suitable structure, which includes only single hanging nodes in space and time and a patch structure property [8].

4 Numerical Results

The domain of the spindle grinding wheel system contains several re-entrant corners. Furthermore, the material is varying throughout the domain. A model example for this difficulties is an L-shape domain with varying material, which is considered here. The data of the example is choosen as:

$$\begin{aligned} \Omega \times I &:= ([-0.5, 0] \times [-0.5, 0.5]) \cup ([0, 0.5] \times [-0.5, 0]) \times [0, 1] \\ \kappa &:= 1 + \min\{1, 10(x_1 - 0.05)_+\} \\ f &:= 100\mathbb{I}_{x_1 \geq \frac{1}{4} \wedge t \in ([0, \frac{1}{4}] \cup [\frac{1}{2}, \frac{3}{4}])} \\ J(w) &:= \frac{1}{|I||B|}\int_I\int_B u(x,t)\, dx\, dt, \quad B := B^\infty_{\frac{1}{8}}\left(-\frac{1}{4}, \frac{1}{4}\right)^T \end{aligned}$$

In Fig. 1, the spatial meshes of different time steps are depicted. In the beginning, the cells in the area of the acting force are refined. Along the outgoing wave, the mesh is refined. The inner edge of the L-shape domain is especially well resolved. At the end, the domain of interest B gets more and more refined. The second impulse is not considered, since the arising wave does not reach B. In Fig. 2, the development of the error in the functional of interest is shown over the complete number of mesh cells. The calculation with dynamic meshes is most efficient, followed by the calculation with an adaptively

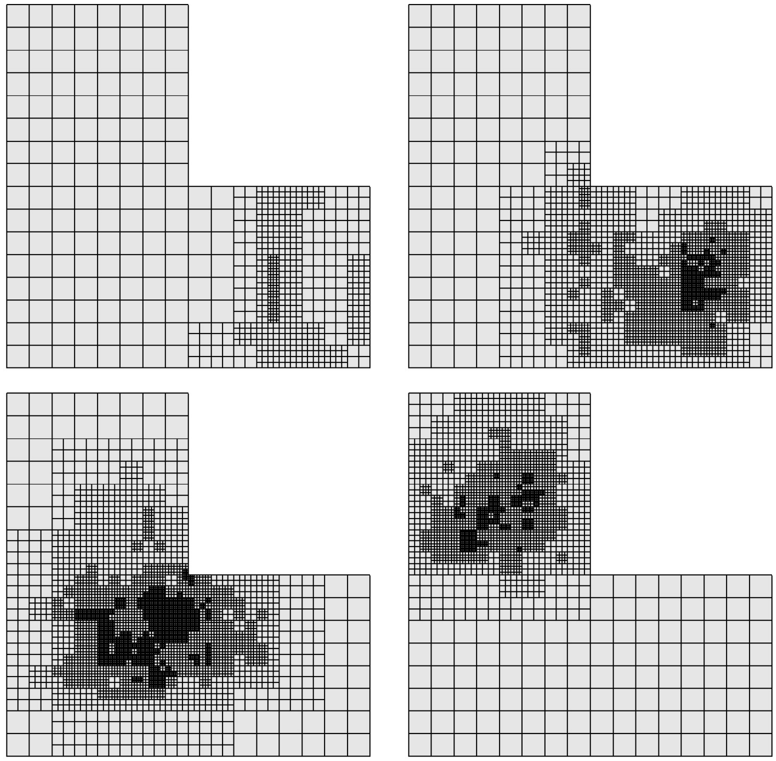

Fig. 1. Meshes for different time steps ($n = 1$, $n = 50$, $n = 100$, $n = 150$)

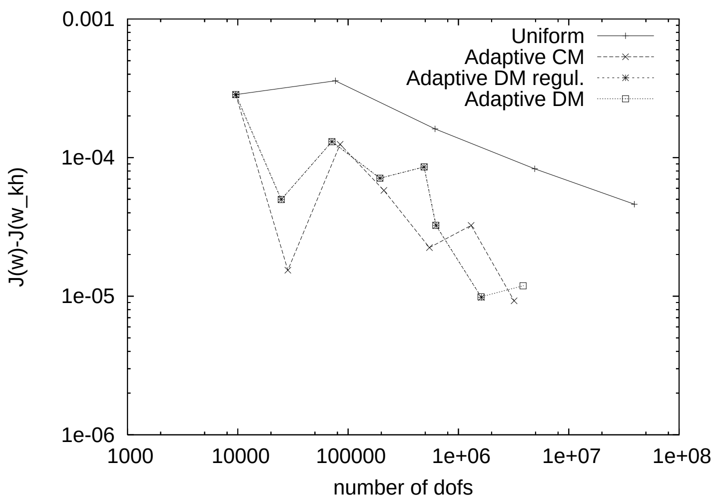

Fig. 2. Convergence of the adaptive method

refined mesh, which is kept constant during one refinement cycle. The graph of the calculation without temporal mesh regularisation shows the need of the algorithm to ensure the proper convergence of the adaptive method. The effectivity indices are in the range of 1.

5 Conclusions and Further Work

In this article, we have presented a new approach to goal-oriented error estimation for the linear wave equation. It leads to well adaptively refined meshes and enhances the efficiency of the finite element discretisation.
The extension of the presented approach to nonlinear second order hyperbolic problems will be considered in a separate article. The mesh refinement and regularisation algorithms will be enhanced further and elaborately analysed.

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