Monolithic Newton-Multigrid FEM techniques for nonlinear problems with special emphasis on viscoelastic fluids

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Modelling, Optimization and Simulation of Complex Fluid Flow
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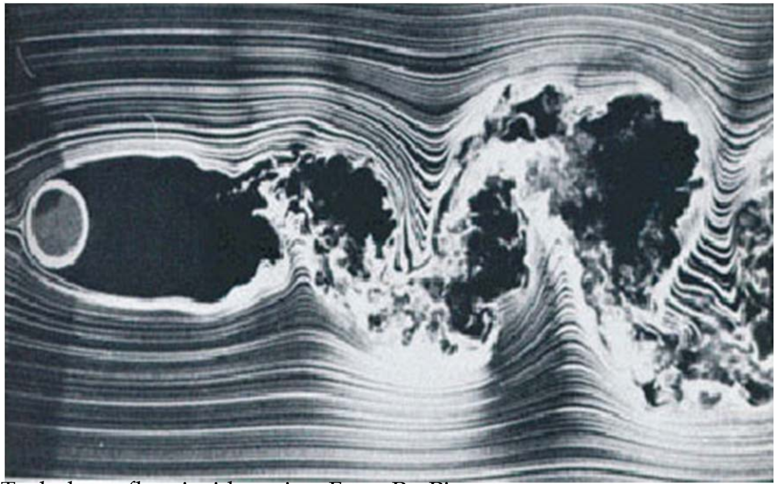
Multiscale CFD Problems

- **Characteristics:**
  - **Complex temporal behaviour and spatially disordered**
  - **Broad range of spatial/temporal scales**

- **Inertia turbulence**
  - **Re>>1**
  - **Numerical instabilities and problems**



Turbulence flow inside a pipe. From ProPipe

  - **Special turbulence models required**
  - **Special stabilization techniques required**

Multiscale CFD Problems

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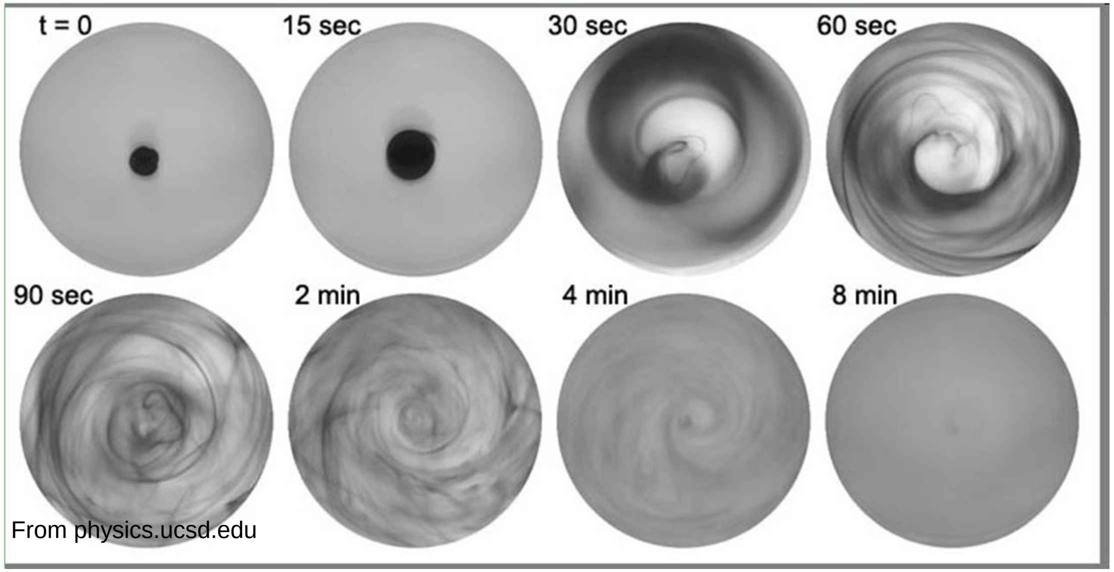
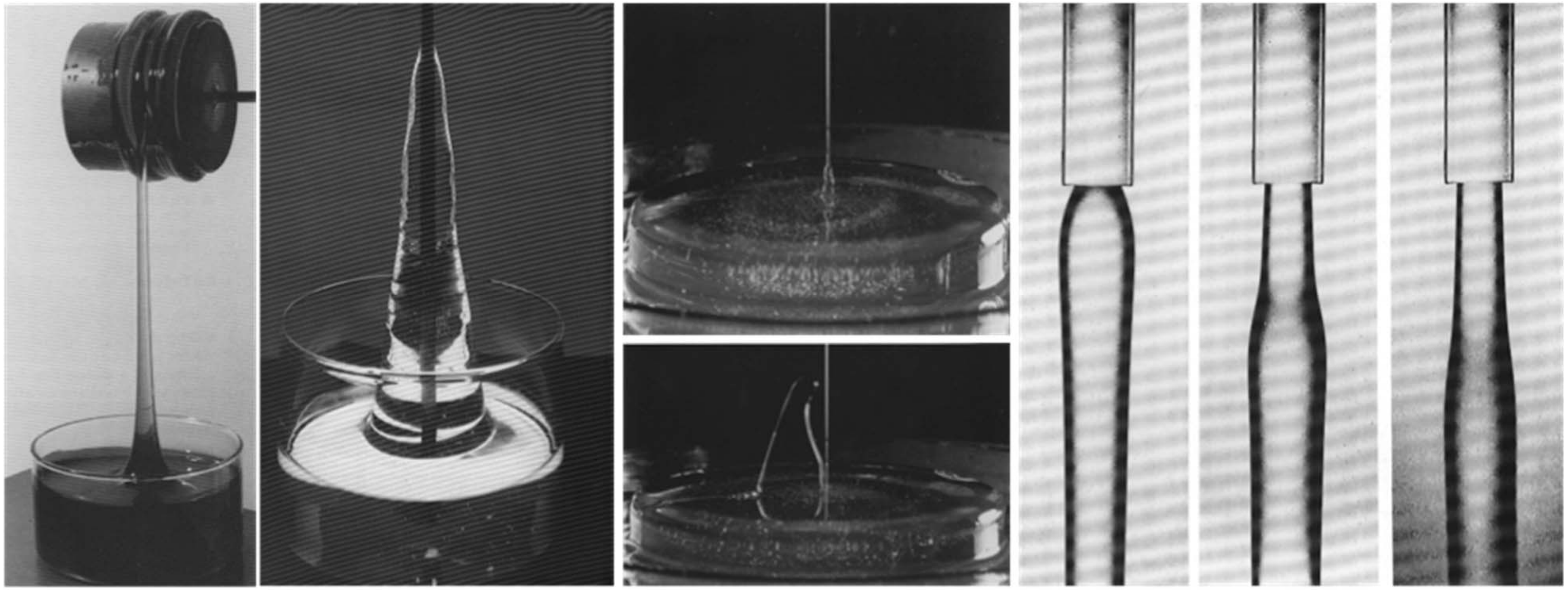
- **Elastic turbulence**
  - **Re<<1, We>>1 (less inertia, more elasticity)**
  - **Numerical instabilities and problems (HWNP)**



  - **Special flow models: Oldroyd-B, Giesekus, Maxwell,…**
  - **Special stabilization: EEME, EEVS, DEVSS/DG, SD, SUPG,…**

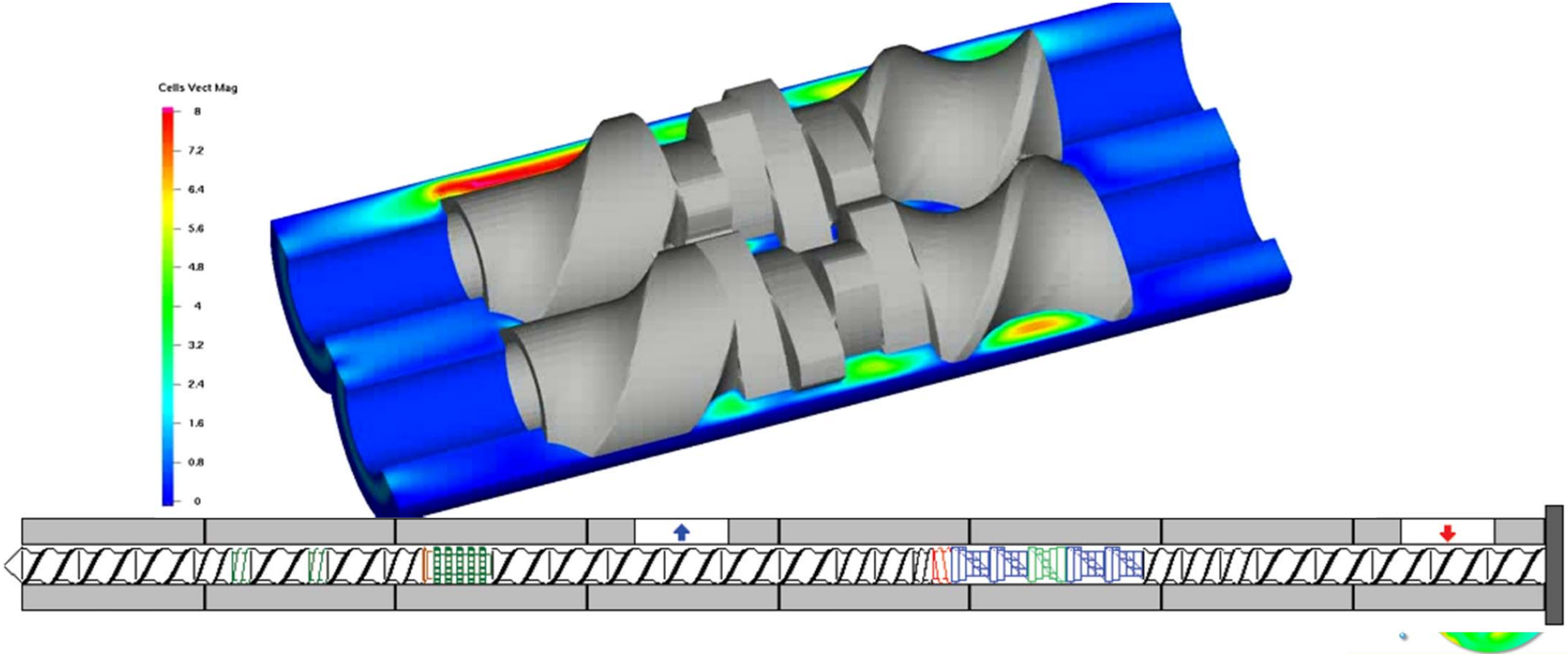
Viscoelastic Fluids

- Special effects due to normal stresses
- Special effects due to elongational viscosity
- The drag reduction phenomenon
- …

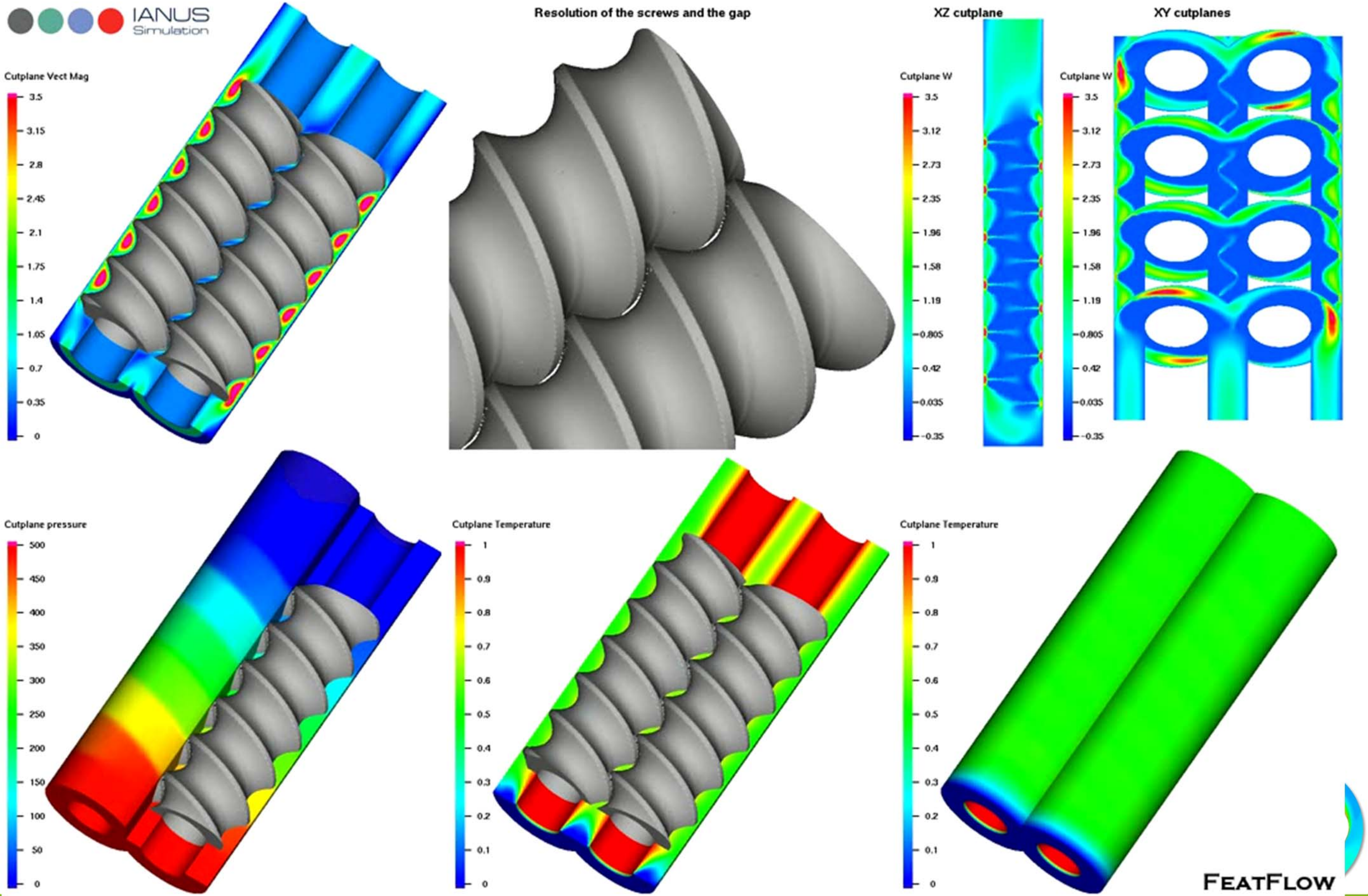


Application: Twinscrew Extruder

- *Viscoelastic rheological* models (additionally shear & temperature dependent) with *non-isothermal* conditions (cooling from outside, heat production, melting, solidification)
- Multiphase flow behaviour due to partially filled and transported granular material
- Complex time dependent geometry and meshes



FeatFlow Simulations with FBM



Governing Equations

- **Generalized Navier-Stokes equations**

$$\rho\left(\frac{\partial}{\partial t}+u\cdot\nabla\right)u = -\nabla p + \nabla\cdot\sigma, \quad \nabla\cdot u = 0,$$

$$\rho c_p\left(\frac{\partial}{\partial t}+u\cdot\nabla\right)\Theta = k_1 \nabla^2\Theta + k_2 \mathrm{D}:\mathrm{D},$$

$$D(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right),$$

$$\sigma = \sigma_s + \sigma_p .$$

- **Viscous stress**

$$\sigma_s = 2\eta_s(\dot\gamma, \Theta, p)D, \dot\gamma = \sqrt{\mathrm{tr}(D(u)^2)}.$$

- **Elastic stress**

$$f_1(L,\sigma_p)\sigma_p + \Lambda\overset{\nabla}{\sigma}_p + F_2(\sigma_p, D) + F_3(\sigma_p) = 2\eta_p D(u).$$

- **Viscous stress**

$$\sigma_s = 2\eta_s(\dot{\gamma},\Theta,p)D,\ \dot{\gamma}=\sqrt{\mathrm{tr}(D(u)^2)}.$$

- **Power Law model**

$$\eta_s(\dot{\gamma},\Theta,p) = \eta_0(\varepsilon+\dot{\gamma}^2)^{(\frac{r}{2}-1)},(\eta_0>0,r>1).$$

- **Generalized Cross model**

$$\eta_s(\dot{\gamma},\Theta,p) = \eta_\infty + \frac{(\eta_0-\eta_\infty)}{(1+(\lambda\dot{\gamma})^{r_1})^r}\exp(\alpha p + (a_1+\frac{a_2}{a_3+\Theta})),$$

$$(\eta_0>\eta_\infty\ge 0, r>1, \lambda>0).$$

- **Generalized upper convective constitutive model**

$$f_1(L_k, tr(\sigma_p), \Lambda, \eta_p)\sigma_p + \Lambda \overset{\nabla}{\sigma}_p + F_2(\sigma_p, D) + F_3(\sigma_p) = 2\eta_p D(u),$$

$$\overset{\nabla}{\sigma}_p := \frac{\partial \sigma_p}{\partial t} + u \cdot \nabla \sigma_p - \nabla u \cdot \sigma_p - \sigma_p \cdot \nabla u^T.$$

	f_1	F_2	F_3
Oldroyd-B/UCM	1	0	0
Giesekus	1	0	$\alpha\sigma_p^2$
FENE-P/-CR	$f_1(L_k, tr(\sigma_p))$	0	0
White & Metzner $\Lambda = \Lambda(\dot{\gamma}), \eta_p = \eta_p(\dot{\gamma})$	1	0	0
PTT	$f_1(\eta_p, tr(\sigma_p), \Lambda)$	$\xi(D\sigma_p + \sigma_p D)$	0
Pom-Pom	$f_1(tr(\sigma_p), \Lambda)$	$F_2(G, \sigma_p, \Lambda)$	$F_3(G, \sigma_p^2, \alpha)$

- **Exemplary model: White-Metzner**

$$\sigma_p + \Lambda(\dot{\gamma})\overset{\nabla}{\sigma}_p = 2\eta_p(\dot{\gamma})D(u), \qquad \dot{\gamma} = \sqrt{2D(u):D(u)}$$

  - **Larson:**

$$\Lambda(\dot{\gamma}) = \frac{\Lambda}{1 + a\Lambda\dot{\gamma}} \qquad \eta_p(\dot{\gamma}) = \frac{\eta_p}{1 + a\Lambda\dot{\gamma}}$$

  - **Cross:**

$$\Lambda(\dot{\gamma}) = \frac{\Lambda}{1 + (L\dot{\gamma})^{1-n}} \qquad \eta_p(\dot{\gamma}) = \frac{\eta_p}{1 + (k\dot{\gamma})^{1-m}}$$

  - **Carreau-Yasuda:**

$$\Lambda(\dot{\gamma}) = \Lambda\left[1 + (L\dot{\gamma})^b\right]^{\frac{n-1}{b}} \qquad \eta_p(\dot{\gamma}) = \eta_p\left[1 + (k\dot{\gamma})^a\right]^{\frac{m-1}{a}}$$

Numerical Challenges

- **Discretizations have to handle the following challenges points**
  - **Stable FEM spaces for velocity/pressure and velocity/stress interpolation** $Q_2/Q_2/P_1^{disc}$ **or** $\tilde{Q}_1/\tilde{Q}_1/P_0$ **or the new** $\tilde{Q}_2/\tilde{Q}_2/P_1^{disc}$
  - **Special treatment of the convective terms: edge-oriented/interior penalty (EO-FEM), TVD/FCT**
  - **High Weissenberg number problem (HWNP): LCR**

- **Solvers have to deal with different sources of nonlinearity**
  - **Nonlinearity: Newton method**
  - **Strong coupling of equations: monolithic multigrid approach**

- **Complex geometries (and meshes)**
  - **FBM + distance based Level Set FEM for free interfaces**

Problem Reformulation (I)

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Elastic stress $\rightarrow (u, p, \sigma_p)$

$$\left.\begin{array}{l}\rho\left(\frac{\partial}{\partial t}+u\cdot\nabla\right)u=-\nabla p+2\nabla\cdot\eta_s D+\nabla\cdot\sigma_p,\ \nabla\cdot u=0\\ f_1(L,\sigma_p)\sigma_p+\Lambda\overset{\nabla}{\sigma}_p+F_2(\sigma_p,D)+F_3(\sigma_p)=2\eta_p D(u)\end{array}\right\}\ (1)$$

Conformation stress $\rightarrow (u, p, \sigma_c)$ is positive definite by design !!

Replace σ_p in (1) with $\boxed{\sigma_p=\frac{\eta_p}{\Lambda}(\sigma_c-I)}$ → special discretization: TVD

$$\left.\begin{array}{l}\rho\left(\frac{\partial}{\partial t}+\mathrm{u}\cdot\nabla\right)u=-\nabla p+2\nabla\cdot\eta_s D+\frac{1}{\Lambda}\nabla\cdot\eta_p\sigma_c,\ \nabla\cdot\mathrm{u}=0,\\ \overset{\nabla}{\sigma}_c+F_4(\sigma_c,u)=0\end{array}\right\}\ (2)$$

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$$\sigma_c(t) = \int_{\infty}^{t} \frac{1}{\mathrm{We}^2} \exp\left(\frac{-(t-s)}{\mathrm{We}}\right) F(s,t)\, F(s,t)^T\, ds$$

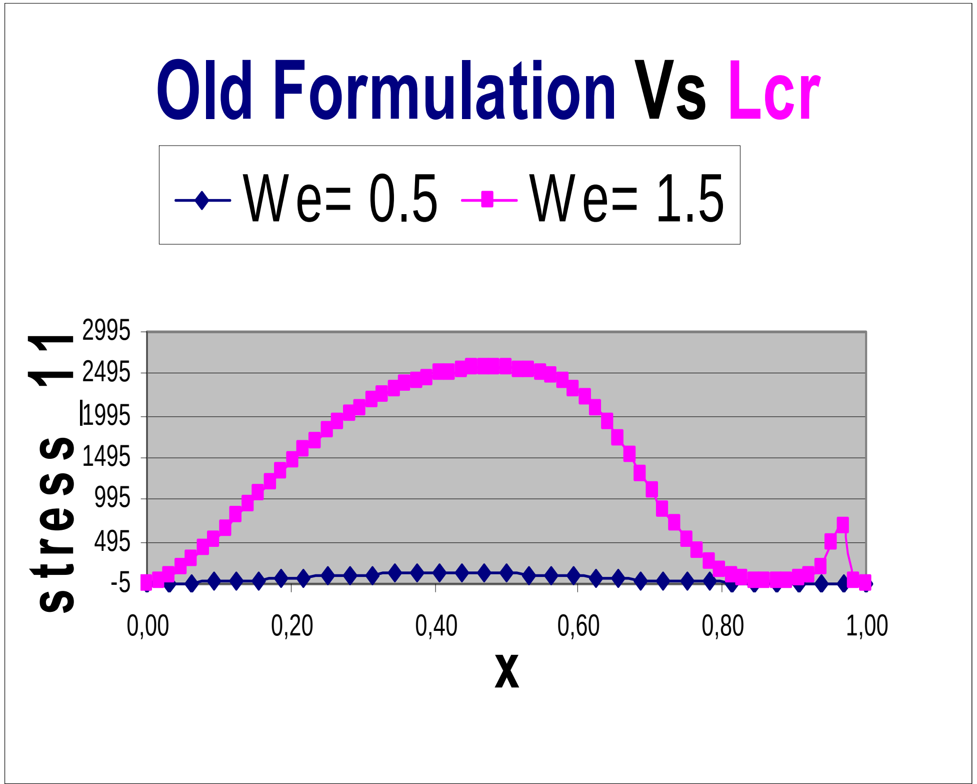
Positive by design,
so we can take its logarithm

<u>2 Observations:</u>

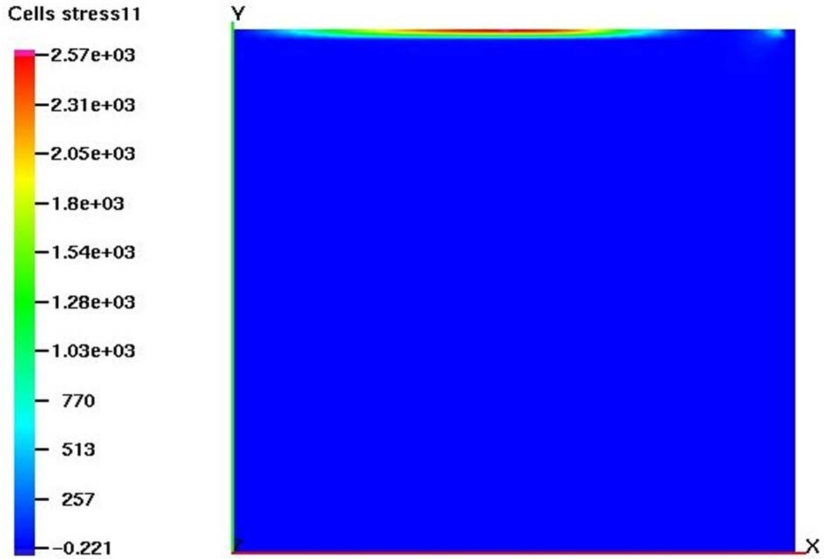
- positive definite → special discretizations like FCT/TVD
- exponential behaviour → approximation by polynomials???

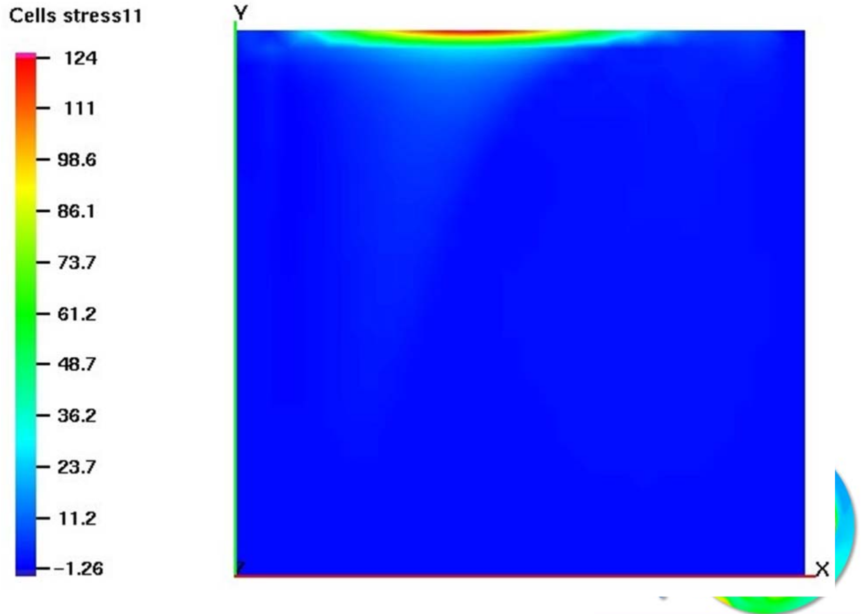
Exponential Behaviour

Driven Cavity:
as We number changes from We=0.5 to We=1.5, the stress value jumps significantly



Cutline of Stress_11 component at y = 1.0





Problem Reformulation (II)

- **Experience:**
  - **Stresses grow exponentially**
  - **Conformation tensor is positive by design**

- **Fattal and Kupferman:**
  - **Take the logarithm as a new variable** $\sigma_{LCR} = \log \sigma_c$ **using the eigenvalue decomposition**

$$\sigma_{LCR} = R \log(\lambda_{\sigma_c}) R^T$$

  - **Decompose the velocity gradient inside the stretching part**

$$\nabla u = B + \Omega + N \sigma_c^{-1}$$

Remark for PTT only $L = B + \Omega + N \sigma_c^{-1}, L = \nabla u - \xi D$

LCR can be applied to all upper convective models !!

LCR Reformulation

$$f_1(L,\sigma_p)\sigma_p + \Lambda \overset{\nabla}{\sigma}_p + F_2(\sigma_p, D) + F_3(\sigma_p) = 2\eta_p D(u)$$

$$\sigma_p = \frac{\eta_p}{\Lambda}(\sigma_c - I)$$

$$\overset{\nabla}{\sigma}_c + F_4(\sigma_c, u) = 0$$

$$\nabla u = \Omega + B + N\sigma_c^{-1}$$

$$\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)\sigma_c - (\Omega\sigma_c - \sigma_c\Omega) + 2B\sigma_c = \frac{1}{\Lambda}(\mathrm{I} - \sigma_c)$$

$$\sigma_c = \exp\ \sigma_{LCR}$$

$$\sigma_{LCR} = R\log(\lambda_{\sigma_c})\,R^T$$

$$\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)\sigma_{LCR} - (\Omega\sigma_{LCR} - \sigma_{LCR}\Omega) - 2B = F_4(\sigma_{LCR}, u).$$

Full Set of Equations (LCR)

- **Generalized Newtonian (VP)**

$$\rho\left(\frac{\partial}{\partial t}+u\cdot\nabla\right)u=-\nabla p+\nabla\cdot\left(2\eta_s(\dot{\gamma},\Theta,p)D(u)\right)+\frac{1}{\Lambda}\nabla\cdot\eta_p e^{\sigma_{LCR}},\quad \nabla\cdot u=0,$$

- **Non-isothermal effect (T)**

$$\rho c_p\left(\frac{\partial}{\partial t}+u\cdot\nabla\right)\Theta=k_1\nabla^2\Theta+k_2\mathrm{D}:\mathrm{D},$$

- **LCR equation (S)**

$$\left(\frac{\partial}{\partial t}+u\cdot\nabla\right)\sigma_{LCR}-\left(\Omega\sigma_{LCR}-\sigma_{LCR}\Omega\right)-2B=F_4\left(\sigma_{LCR},u\right).$$

$$F_4\left(\sigma_{LCR},u\right)$$

Refers to all upper convective constitutive models

Examplary Models (LCR)

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Oldroyd-B/UCM $$F_4 = \frac{1}{\Lambda}(e^{-\sigma_{LCR}} - I)$$

Giesekus $$F_4 = \frac{1}{\Lambda}(e^{-\sigma_{LCR}} - I - \alpha e^{\sigma_{LCR}} - (e^{-\sigma_{LCR}} - I)^2)$$

FENE-P/-CR $$F_4 = \frac{1}{\Lambda} f(R)(e^{-\sigma_{LCR}} - I) \bigg/ F_4 = \frac{1}{\Lambda}(e^{-\sigma_{LCR}} - f(R)I)$$

White-Metzner $$F_4 = \frac{1}{\Lambda(\dot{\gamma})}(e^{-\sigma_{LCR}} - I)$$

Linear PTT $$F_4 = \frac{1}{\Lambda}(1 + \varepsilon(tr(e^{\sigma_{LCR}} - 3)))(e^{-\sigma_{LCR}} - I)$$

Exponential PTT $$F_4 = \frac{1}{\Lambda}\exp(\varepsilon(tr(e^{\sigma_{LCR}} - 3)))(e^{-\sigma_{LCR}} - I)$$

Pom-Pom $$F_4 = -\frac{1}{\Lambda}([f(\sigma_{LCR}) - 2\alpha]e^{\sigma_{LCR}} + \alpha e^{2\sigma_{LCR}} + (\alpha - 1)I)$$

FEM Discretization

- **High order $Q_2 / Q_2 / P_1^{disc}$ for velocity-stress-pressure**

  - **Advantages:**
    - **Inf-sup stable for velocity and pressure**
    - **High order: good for accuracy**
    - **Discontinuous pressure: good for solver & physics**

  - **Disadvantages:**
    - **Stabilization for same spaces for stress-velocity**
    - **a single d.o.f. belongs to four elements (in 2D)**

Compatibility condition between the stress and velocity spaces via EO-FEM !

Variational Formulations

- **Standard Navier-Stokes bilinear forms**

$$a(u,v) = \int_\Omega \frac{1}{\Delta t} u \cdot v \, d\Omega + \int_\Omega 2\eta_s D(u) : D(v) \, d\Omega$$

$$b(p,v) = -\int_\Omega p \, \nabla \cdot v \, d\Omega$$

- **Non-symmetric bilinear forms due to LCR**

$$c(\sigma_{LCR}, v) = \int_\Omega \exp(\sigma_{LCR}) : D(v) \, d\Omega$$

$$\tilde{c}(\tau, u) = -2\int_\Omega B(\nabla u, \sigma_c) : \tau \, d\Omega$$

- **Nonlinear tensor variational form due to LCR**

$$d(\sigma_{LCR},\tau)=\int_\Omega\left(\frac{1}{\Delta t}+(u\cdot\nabla)\right)\sigma_{LCR}:\tau\,dx$$

$$-\int_\Omega(\Omega\sigma_{LCR}-\sigma_{LCR}\Omega):\tau\,dx-\int_\Omega F_4\left(\sigma_{LCR},u\right):\tau\,dx$$

- **Non-symmetric bilinear forms due to LCR**

$$e(\Theta,\Phi)=\int_\Omega\left(\frac{1}{\Delta t}+u\cdot\nabla\right)\Theta\Phi\,dx+\int_\Omega k\nabla\Theta\cdot\nabla\Phi\,dx$$

$$-\int_\Omega 2\eta_s\left[D(u):D(u)\right]\Phi\,dx-\int_\Omega D(u):\exp(\sigma_{LCR})\Phi\,dx$$

- **Source term**

$$l(u,\sigma_{LCR},\Theta,p)$$

- **Set** $X := \left[H_0^1(\Omega)\right]^2 \times \left[L^2(\Omega)\right]^4 \times H^1(\Omega),\ Q := L_0^2(\Omega)$

$$\tilde{u} := (u, \sigma_{LCR}, \Theta) \qquad \tilde{A} := \begin{bmatrix} A & C & 0 \\ \tilde{C}^T & D & 0 \\ E_{fD} & E_{\sigma_{LCR}} & E \end{bmatrix}$$

- **Find** $(\tilde{u}, p) \in X \times Q$ **such that**

$$\left\langle K(\tilde{u}, p), (\tilde{v}, q) \right\rangle = \left\langle l(\tilde{v}, q) \right\rangle \quad \forall (\tilde{v}, q) \in X \times Q$$

$$K = \begin{bmatrix} \tilde{A} & B \\ B^T & 0 \end{bmatrix}$$

Typical saddle point problem !

- **Compatibility condition (for classical approach)**

$$\sup_{u \in [H_0^1(\Omega)]^2} \frac{\int_\Omega \nabla \cdot u \; q \; dx}{\|u\|_{1,\Omega}} \geq \beta_1 \|q\|_{0,\Omega} \qquad \forall q \in L_0^2(\Omega)$$

$$\sup_{\sigma \in [L^2(\Omega)]^4} \frac{\int_\Omega \sigma : \nabla u \; dx}{\|\sigma\|_{0,\Omega}} \geq \beta_2 \|u\|_{1,\Omega} \qquad \forall u \in [H_0^1(\Omega)]^2$$

What about LCR ?

- **The non-symmetric bilinear forms due to LCR**

$$\tau \in T_{PD} \subset \left[L^2(\Omega)\right]^4$$ **such that** τ **is positive definite**

$$c(\tau, v) = \int_\Omega \exp(\tau) : D(v)\, d\Omega$$
$$\geq \beta_2 \|\exp(\tau)\|_{0,\Omega} \|v\|_{1,\Omega}$$
$$\geq \beta_2 \|\tau\|_{0,\Omega} \|v\|_{1,\Omega} \quad \forall \tau \in T_{PD}, \quad \forall v \in \left[H_0^1(\Omega)\right]^2$$

$$\tilde{c}(\tau, u) = -2\int_\Omega B(\nabla u, \sigma_c) : \tau\, d\Omega$$
$$\geq \beta_2 \|\tau\|_{0,\Omega} \|v\|_{1,\Omega} \quad \forall \tau \in \left[L^2(\Omega)\right]^4, \quad \forall v \in \left[H_0^1(\Omega)\right]^2$$

- **Edge-oriented stabilization for**

  - **Equal order finite element interpolation for velocity and stress**

  - **Convective dominated problem**

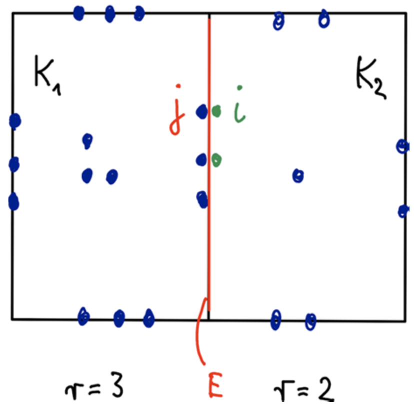
$$\left\langle J_{\tilde{u}} \tilde{u}, \tilde{v} \right\rangle = \sum_{edgeE} \max\left(\gamma_u \eta_p h_E, \gamma_{\tilde{u}} h_E^2\right) \int_E [\nabla \tilde{u}][\nabla \tilde{v}] ds$$

with $\tilde{u} = (u, \sigma, \Theta)$, $\tilde{v} = (v, \tau, \Phi)$ and $[\nabla \tilde{u}][\nabla \tilde{v}] = \sum_i [\nabla \tilde{u}_i][\nabla \tilde{v}_i]$

Efficient Newton-type and multigrid solvers can be easily applied !

Higher Order Nonconforming FEM

- **Larger FE space which allows high order approximation**



- **d.o.f.s belong to at most two elements which is good for parallelisation**

- **Coupling of different polynomial orders**

  - **Mortar condition: test space ≈ order at slave side**

$$|E|^{-1}\int_E v_h|_{K_2} L_{E,k}\,ds = |E|^{-1}\int_E v_h|_{K_1} L_{E,k}\,ds, \quad 0 \le k < 2$$

  - **No hanging nodes**

- **Damped Newton results in the solution of the form**

$$R(x) = 0, \; x = (u, \sigma_{LCR}, \Theta, p)$$

$$x^{l+1} = x^{l} + \omega^{l} \left[\frac{\partial R(x^{l})}{\partial x} \right]^{-1} R(x^{l})$$

- **Inexact Newton:** Jacobian is approximated using finite differences

$$\left[\frac{\partial R(x^{l})}{\partial x} \right]_{ij} \approx \frac{R_j(x + \varepsilon e_i) - R_j(x - \varepsilon e_i)}{2\varepsilon}$$

Jacobian Matrix

- **The Jacobian matrix takes the form**

$$J = \left[\frac{\partial R(x^n)}{\partial x} \right] = \begin{bmatrix} A & \tilde{B}^T \\ B & 0 \end{bmatrix}$$

- **Generalized non-isothermal non-Newtonian problem**

$$A = \begin{bmatrix} A_u & \tilde{C}^T & \tilde{H}^T \\ C & A_\sigma & 0 \\ H & 0 & A_\Theta \end{bmatrix}$$

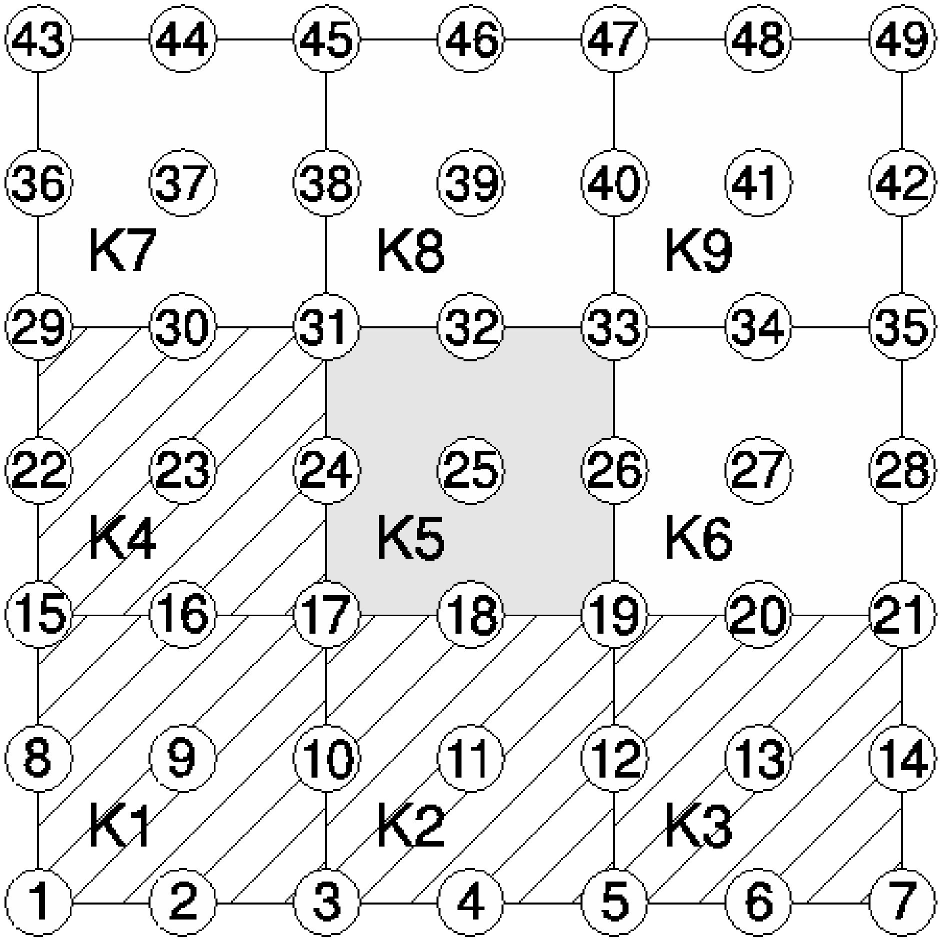
Typical saddle point problem !

Linear Solver

- **Monolithic multigrid solver**
  - **Standard geometric multigrid approach**
  - **Full** Q_2, P_1^{disc} **restrictions and prolongations**
  - **Local MPSC via Vanka-like smoother**

$$\begin{bmatrix} \tilde{u}^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} \tilde{u}^{l} \\ p^{l} \end{bmatrix} + \omega^l \sum_{T \in T_h} \left[J_{|T} \right]^{-1} \begin{bmatrix} R_u(\tilde{u}^l, p^l) \\ R_p(\tilde{u}^l, p^l) \end{bmatrix}_{|T}$$

Fully implicit Monolithic FEM-Multigrid Solver !



Vanka-like Smoother!

Algorithm 1: Smoothing steps

Input: Predefined constant: $\omega > 0$, Number of smoothing steps: NSM

for $j = 1$ **to** NSM **do**
  for $K \in \Omega_h$ **do**
    $x_{I(K)} \leftarrow x_{I(K)} + \omega \mathcal{C}_K^{-1}(\mathbf{b} - A\mathbf{x})_{I(K)}$

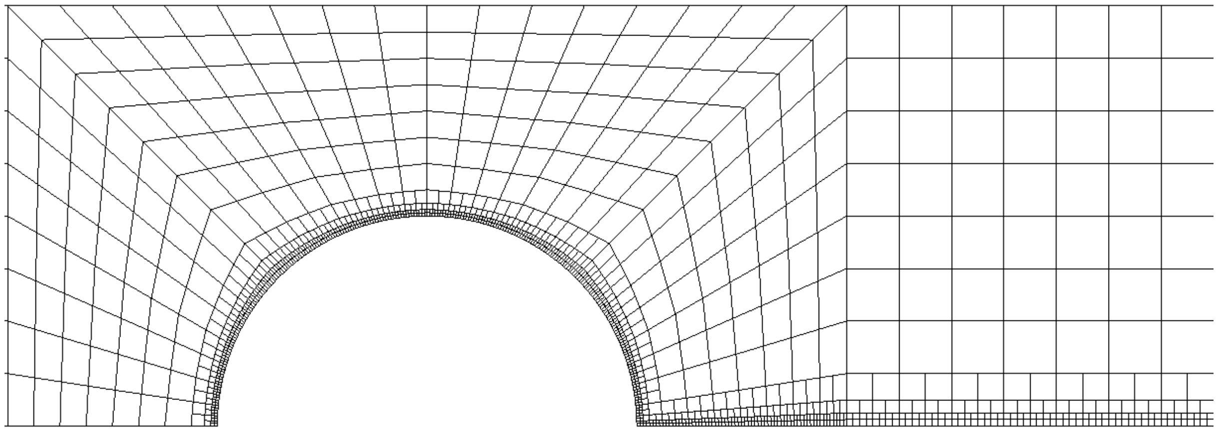
Return x

end

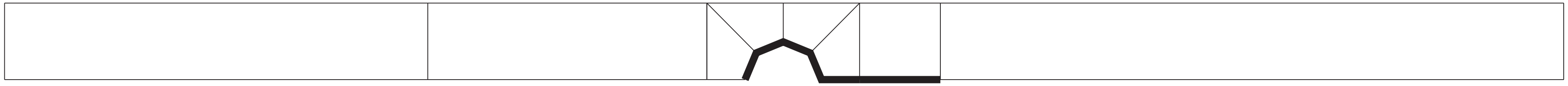
$$\begin{pmatrix} x_{17} \\ x_{18} \\ x_{19} \\ x_{24} \\ x_{25} \\ x_{26} \\ x_{31} \\ x_{32} \\ x_{33} \end{pmatrix} \leftarrow \begin{pmatrix} x_{17} \\ x_{18} \\ x_{19} \\ x_{24} \\ x_{25} \\ x_{26} \\ x_{31} \\ x_{32} \\ x_{33} \end{pmatrix} + \omega \mathcal{C}_5^{-1} \left(\begin{pmatrix} b_1 \\ . \\ . \\ . \\ b_{49} \end{pmatrix} - \begin{pmatrix} a_{1,1} & \dots & a_{1,49} \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ a_{49,1} & \dots & a_{49,49} \end{pmatrix} \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_{49} \end{pmatrix} \right)_{I(K5)}$$

Vanka-like Smoother!

Coarse grid and mesh information

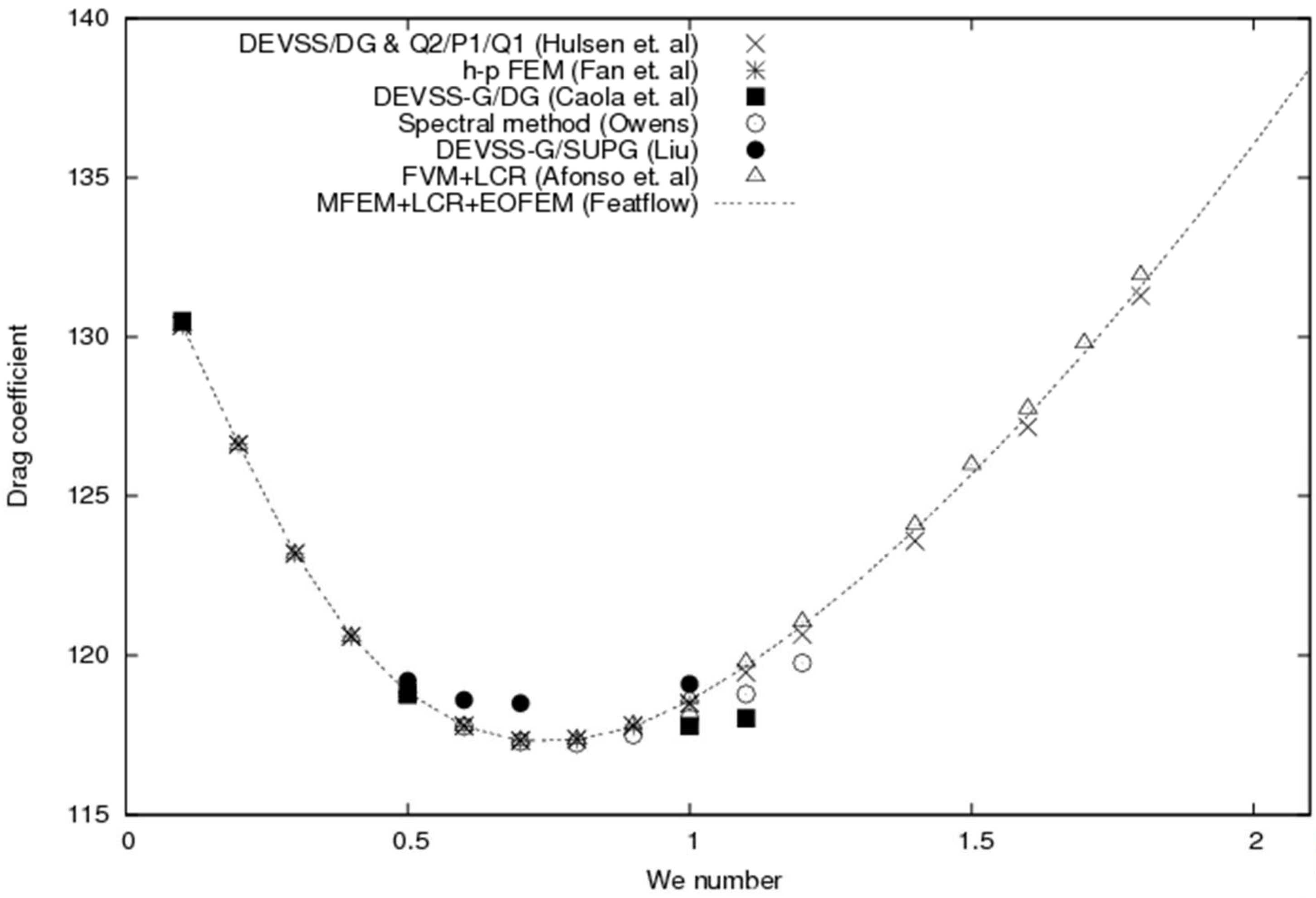


Level	NEL	DOF
R3a1	656	15823
R3a2	944	22457
R3a3	1520	35715
R3a4	2672	62221
R3a5	4976	115223



Local refinement via hanging nodes

- **Planar flow around cylinder (Oldroyd-B)**



Flow around Cylinder Benchmark

- **Oldroyd-B**

We	Drag	NL	We	Drag	NL	We	Drag	NL
0.1	130.366	8	0.8	117.347	4	1.5	125.665	4
0.3	123.194	4	1.0	118.574	6	1.7	129.494	4
0.5	118.828	4	1.2	120.919	5	1.9	133.754	4
0.6	117.779	4	1.3	122.350	4	2.0	136.039	5
0.7	117.321	4	1.4	123.936	4	2.1	138.438	5

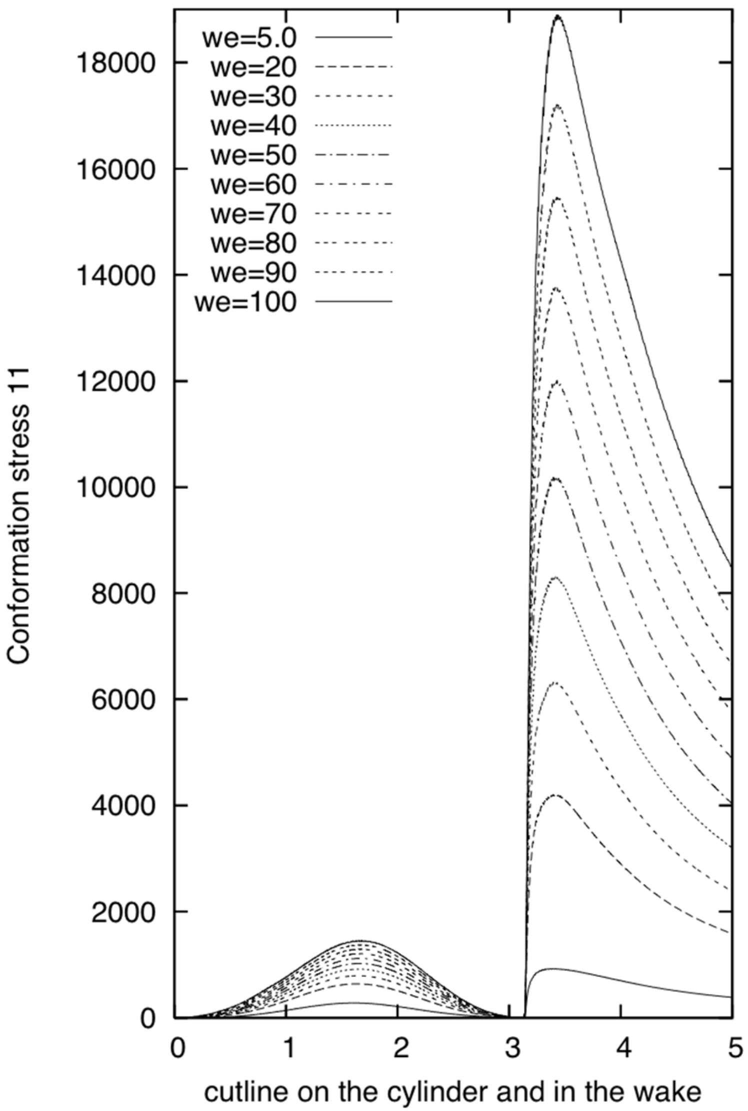
- **Giesekus**

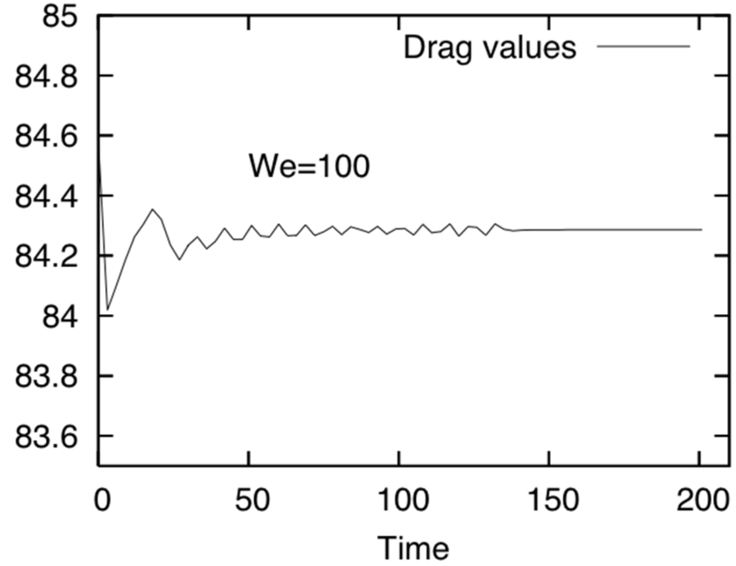
We	Drag	Peak2	NL	We	Drag	Peak2	NL
5	96.943	924.45	14	60	85.859	12010.57	4
20	89.905	4204.51	12	70	85.365	13773.61	4
30	88.304	6318.79	5	80	84.937	15502.45	4
40	87.256	8311.32	5	90	84.585	17207.87	4
50	86.476	10199.1	4	100	84.287	18897.95	4

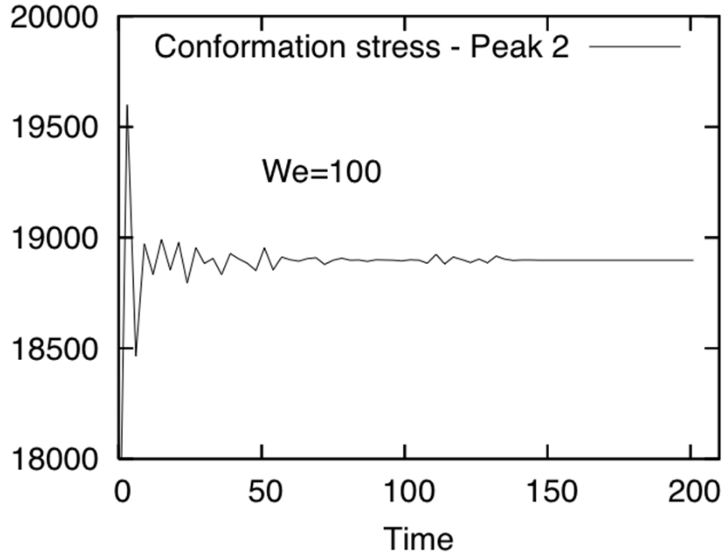
Efficient continuation for increasing We numbers

Flow around Cylinder Benchmark

- **Direct steady vs. non-steady approach for Giesekus**







Flow around Cylinder Benchmark

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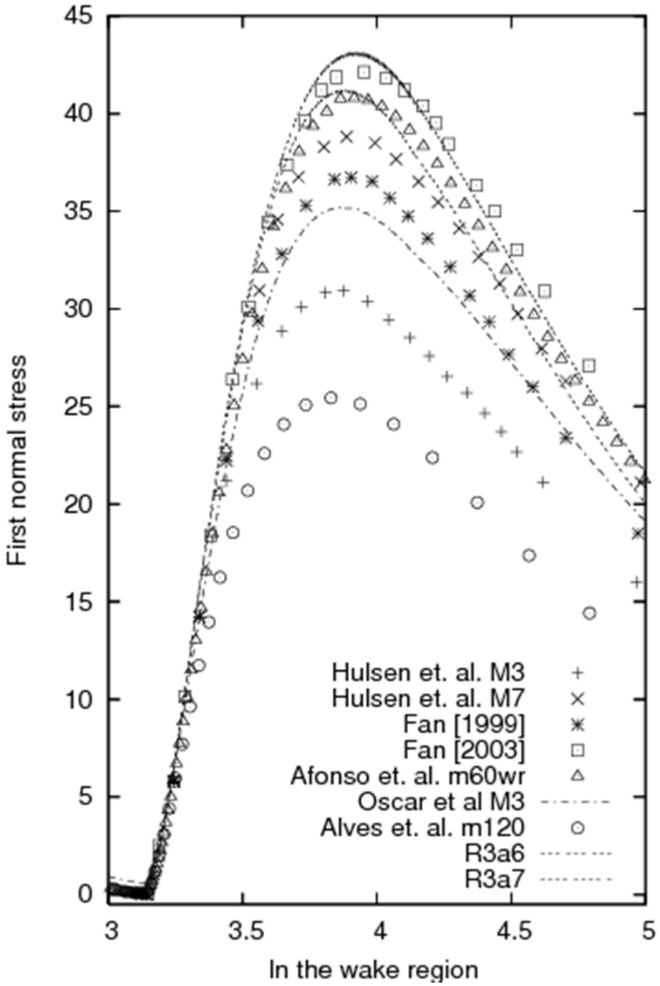
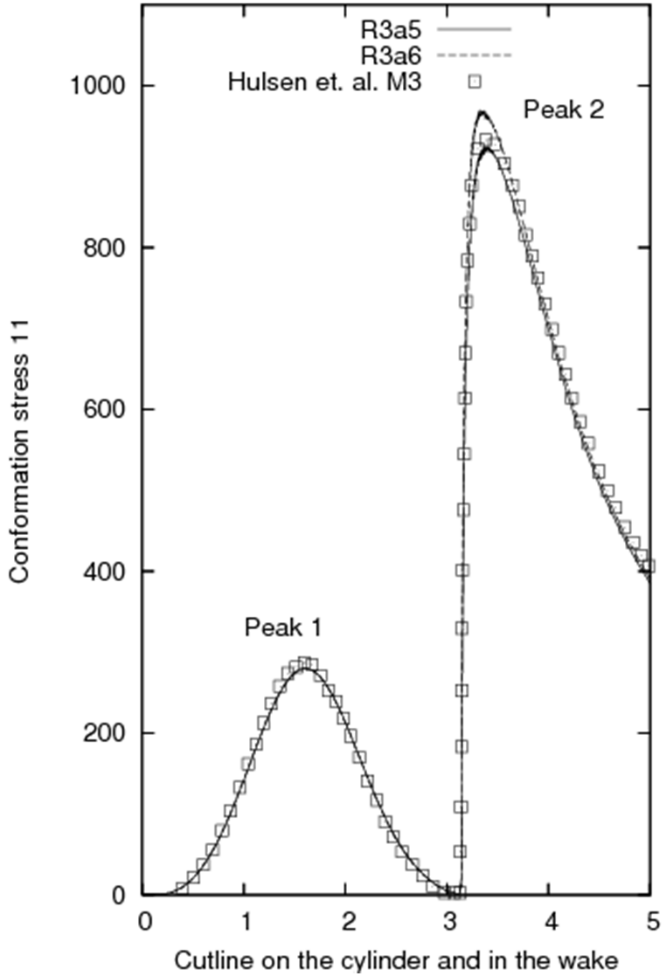
- **Axial stress w.r.t. X-curved: Oldroyd-B vs. Giesekus**

$We = 0.7$



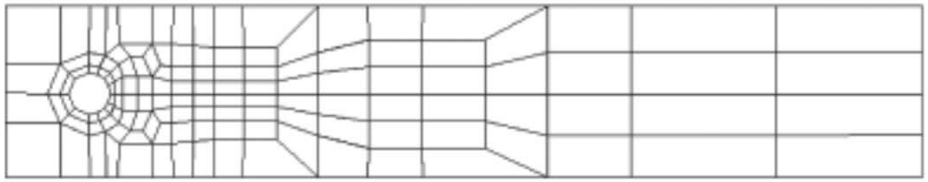
$We = 5.0$



Lack of pointwise mesh convergence due to model

Next Problem

- **Coarse grid and mesh information**



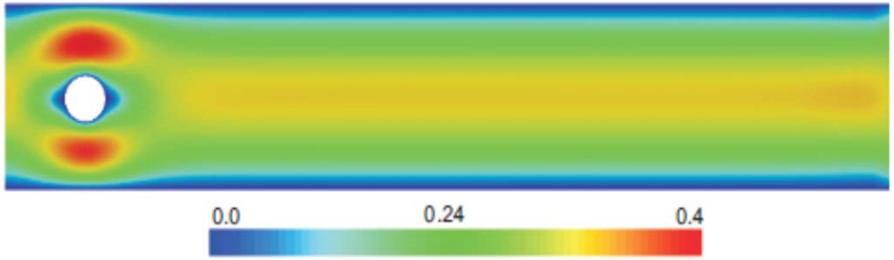
Level	NEL	NMT	NMP
1	520	572	1092
2	2080	2184	4264
3	8320	8528	16848
4	33280	33696	66976
5	133120	133952	267072

- **n.o.f. for different problems**

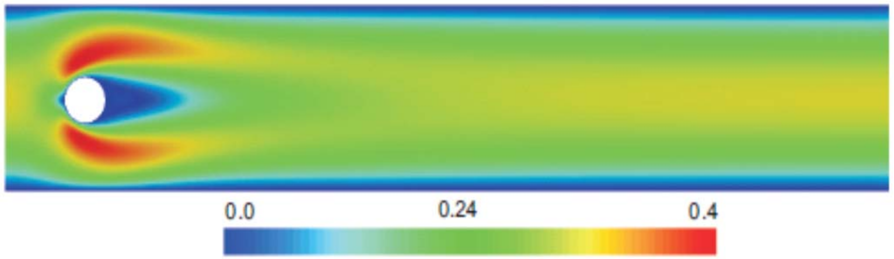
Level	n.o.f. u	n.o.f p	n.o.f. T	n.o.f S	VP	VPT	VSP
0	1144	390	572	1716	1534	2106	3250
1	4368	1560	2184	6552	5928	8112	12489
2	17056	6240	8528	25584	23296	31824	48880
3	67392	24960	33696	101088	92352	126048	193449
4	267904	99840	133952	401856	367744	501696	769600
5	1068288	399360	534144	1602432	1467648	2001792	3070080

Newtonian Problem (VP)

Stokes



Navier-Stokes Re=20



Level	Drag	Lift	N/L	Drag	Lift	N/L
1	3.112646	2.965870e-2	1/6	5.540999	9.447473e-3	5/2
2	3.134342	3.005275e-2	1/7	5.566928	1.046885e-2	5/2
3	3.140327	3.015909e-2	1/7	5.576088	1.056787e-2	5/2
4	3.141893	3.018665e-2	1/7	5.578652	1.060398e-2	5/2
5	3.142292	3.019366e-2	1/7	5.579313	1.061503e-2	5/2

Level independent solver !

Power Law Problem (VP)

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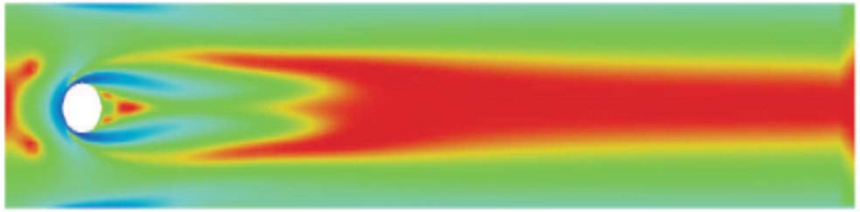
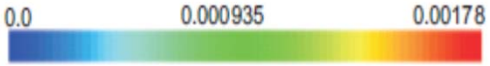


$$\eta_s(\dot{\gamma},\Theta,p)=\eta_0(\varepsilon+\dot{\gamma}^2)^{(\frac{r}{2}-1)},(\eta_0>0,r>1).$$



$\varepsilon=10^{-2}, r=1.5$

Level	Drag	Lift	N/L
2	3.26420	-0.01339	4/2
3	3.27728	-0.01341	3/2
4	3.27956	-0.01338	2/2
5	3.28007	-0.01337	2/2

$\varepsilon=10^{-2}, r=3$

Level	Drag	Lift	N/L
2	13.74280	0.35070	3/2
3	13.77355	0.34963	3/2
4	13.78220	0.35062	3/1
5	13.78445	0.35112	2/2

$\varepsilon=10^{-4}, r=1.5$

Level	Drag	Lift	N/L
2	3.26433	-0.01342	4/2
3	3.27739	-0.01342	3/2
4	3.27968	-0.01339	2/2
5	3.28019	-0.01338	2/2

$\varepsilon=10^{-4}, r=3$

Level	Drag	Lift	N/L
2	13.73800	0.35052	3/2
3	13.76875	0.34941	3/2
4	13.77740	0.35040	3/1
5	13.77970	0.35091	2/2

Level and parameter independent solver !

Cross Model Problem (VP)

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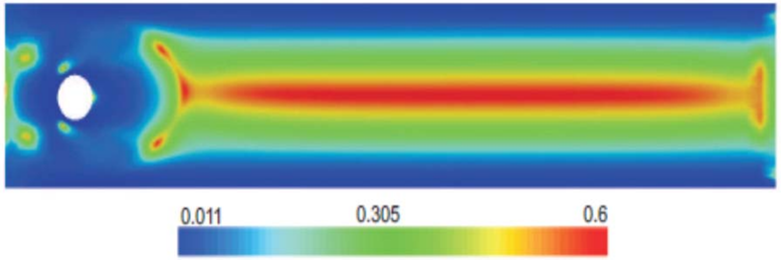


$$\eta_s(\dot{\gamma},\Theta,p)=\eta_\infty+\frac{(\eta_0-\eta_\infty)}{(1+(\lambda\dot{\gamma})^{r_1})^r}\exp(\alpha p+(a_1+\frac{a_2}{a_3+\Theta})),$$

$$\eta_\infty=10^{-3}, a_1=a_2=0,$$

$r=1, r_1=1, \alpha=0, \eta_0=10^{-2}$

Level	Drag	Lift	N/L
2	6.31313	0.02478	3/1
3	6.32337	0.02504	3/2
4	6.32619	0.02509	3/2
5	6.32691	0.02510	2/2

$r=0, r_1=1, \alpha=0.1, \eta_0=10^{-1}$

Level	Drag	Lift	N/L
2	534.29750	6.53247	3/2
3	535.48500	6.55813	3/3
4	535.77950	6.56464	3/3
5	535.84800	6.56621	2/2

$r=0\ r_1=1, \alpha=0.1, \eta_0=10^{-2}$

Level	Drag	Lift	N/L
2	33.22763	0.81901	2/2
3	33.29026	0.82140	2/2
4	33.30657	0.82201	2/2
5	33.31069	0.82217	2/2

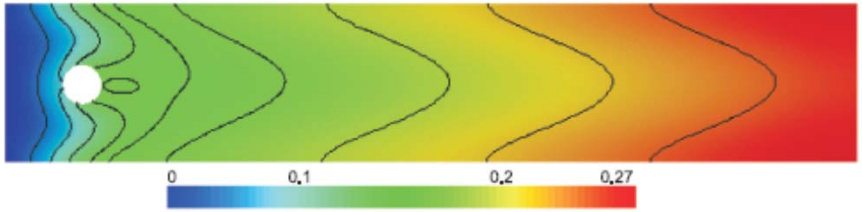
$r=1, r_1=1, \alpha=0.1, \eta_0=10^{-1}$

Level	Drag	Lift	N/L
2	15.16395	0.13886	4/3
3	15.18516	0.13963	4/3
4	15.19108	0.13978	4/3
5	15.19262	0.13982	3/3

Level and model independent solver !

Cross Model problem (VTP)

Non heated cylinder



$$\eta_s(\dot{\gamma}, \Theta, p) = \eta_\infty + \frac{(\eta_0 - \eta_\infty)}{(1+(\lambda\dot{\gamma})^{r_1})^r} \exp(\alpha p + (a_1 + \frac{a_2}{a_3 + \Theta})),$$

$$\eta_\infty = 10^{-3}, a_1 = 0, a_3 = 1, k_1 = k_2 = 10^{-2},$$

$r = 0, r_1 = 1, \ \alpha = 0, \eta_0 = 10^{-2}, a_2 = 0.$

Level	Drag	Lift	N/L
2	74.29465	1.31636	2/2
3	74.43290	1.32009	2/2
4	74.46910	1.32105	2/2
5	74.47830	1.32129	2/2

$r = 0.1, r_1 = 1, \ \alpha = 0, \eta_0 = 10^{-1}, a_2 = 0.$

Level	Drag	Lift	N/L
2	5579.355	54.79350	3/2
3	5589.415	54.96815	3/3
4	5592.050	55.01415	3/3
5	5592.725	55.02585	3/2

$r = 0.1, r_1 = 1, \ \alpha = 0, \eta_0 = 10^{-2}, a_2 = 1.$

Level	Drag	Lift	N/L
2	53.78930	1.05488	2/2
3	53.88590	1.05770	3/2
4	53.91125	1.05844	2/2
5	53.91770	1.05863	2/2

$r = 0.1, r_1 = 1, \ \alpha = 10^{-3}, \eta_0 = 10^{-1}, a_2 = 1.$

Level	Drag	Lift	N/L
2	6005.265	59.75125	3/2
3	6016.220	59.94455	3/2
4	6019.075	59.99535	3/2
5	6019.795	60.00820	3/2

Level and model independent solver !

Cross Model Problem (VTP)

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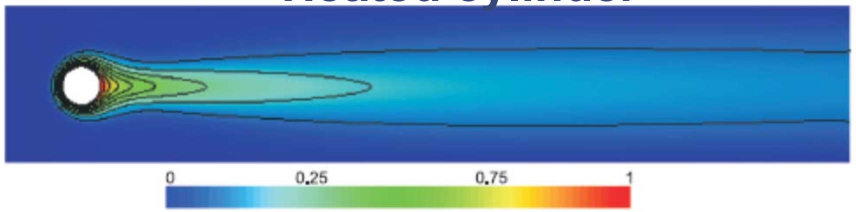
Heated cylinder



$$\eta_s(\dot{\gamma},\Theta,p)=\eta_\infty+\frac{(\eta_0-\eta_\infty)}{(1+(\lambda\dot{\gamma})^{r_1})^r}\exp(\alpha p+(a_1+\frac{a_2}{a_3+\Theta})),$$

$$\eta_\infty=10^{-3}, r=0.1, r_1=1, \alpha=10^{-3}, a_1=0, a_2=1, a_3=1,$$

$\eta_0=10^{-2}, k_1=k_2=10^{-2}.$

Level	Drag	Lift	N/L
2	45.26969	0.90303	3/2
3	45.35251	0.90563	3/2
4	45.37431	0.90632	2/2
5	45.37988	0.90649	2/2

$\eta_0=10^{-1}, k_1=k_2=10^{-3}.$

Level	Drag	Lift	N/L
2	512.7765	5.37640	3/2
3	513.7120	5.39301	3/3
4	513.9585	5.39743	3/3
5	514.0215	5.39856	3/2

$\eta_0=10^{-1}, k_1=k_2=10^{-2}.$

Level	Drag	Lift	N/L
2	464.07865	4.90752	2/2
3	464.93045	4.92313	3/2
4	465.15470	4.92724	2/2
5	465.21195	4.92828	2/2

$\eta_0=1.0, k_1=k_2=10^{-3}.$

Level	Drag	Lift	N/L
2	5528.860	53.11685	3/2
3	5539.025	53.28790	3/2
4	5541.690	53.33335	3/2
5	5542.365	53.34490	3/2

Level and model independent solver !

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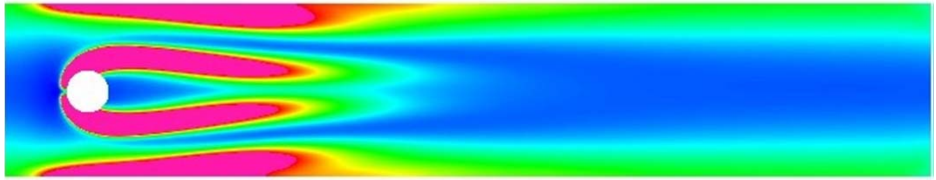


$We = 0.002$ **Oldroyd-B**

Level	Drag	Lift	N/L
2	5.57150	0.01031	2/2
3	5.58032	0.01047	3/2
4	5.58285	0.01051	2/2
5	5.58351	0.01052	2/2

$We = 0.002$ **Giesekus**

Level	Drag	Lift	N/L
2	5.56474	0.01053	2/2
3	5.57511	0.01064	2/2
4	5.57936	0.01062	2/2
5	5.58131	0.01059	2/2

$We = 1.0$ **Oldroyd-B**

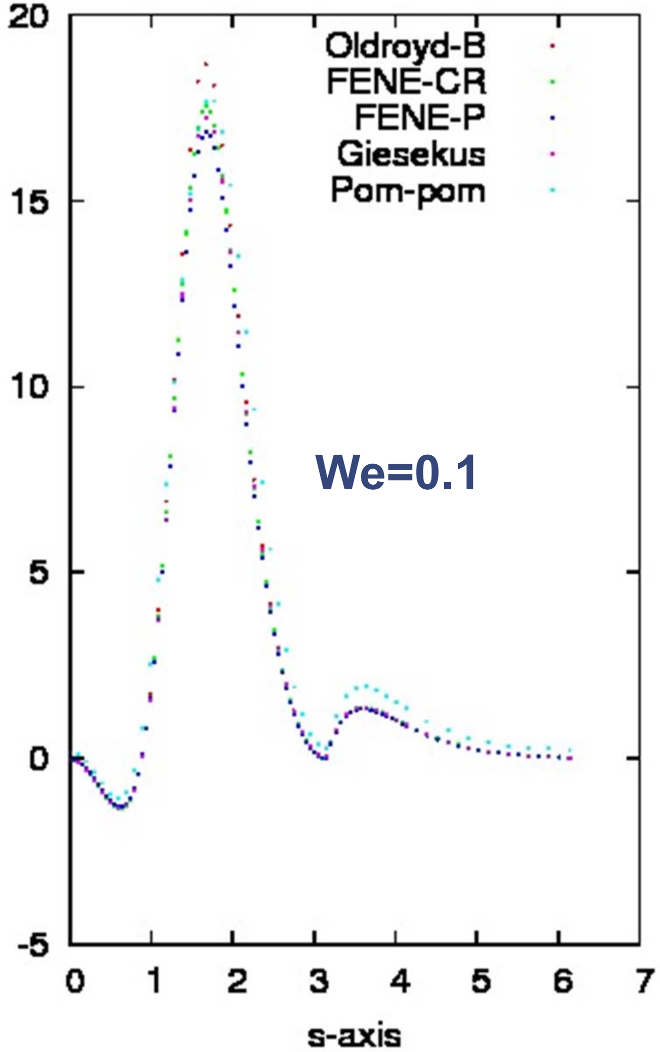
Level	Drag	Lift	N/L
2	20.8412	0.32761	7/7
3	17.7123	0.22910	6/8
4	15.0096	0.14311	6/9
5	12.58895	0.07002	6/9

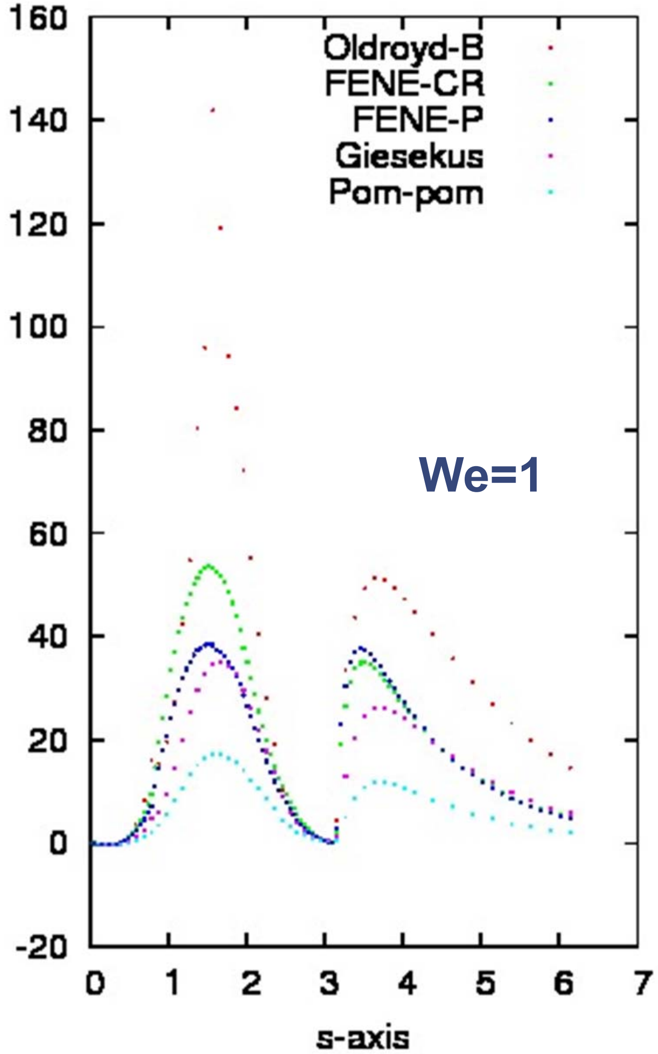
$We = 1.0$ **Giesekus**

Level	Drag	Lift	N/L
2	5.03961	-0.00172	4/2
3	4.93834	-0.00210	4/3
4	4.84483	-0.00252	3/3
5	4.77541	-0.00276	3/4

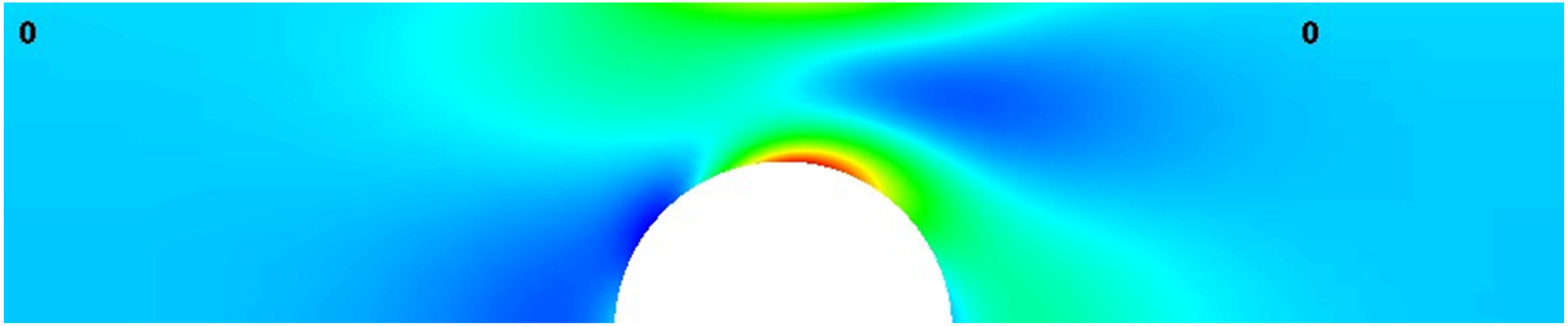
Level independent solver !

Viscoelastic Fluids (VSP)





Lower We vs. higher We !



We	0.1	0.2	0.3	0.4	0.5	0.6	…	1.0
Oldroyd-B	5 [130.06]	4 [130.06]	4 [130.06]	5 [120.40]	5 [118.67]	5 [117.66]	…	7 [119.33]
Giesekus	4 [129.37]	4 [124.41]	4 [119.86]	3 [116.31]	3 [113.67]	3 [111.71]	…	3 [107.29]
Fene-P	4 [128.91]	4 [124.62]	4 [120.70]	3 [117.67]	3 [115.51]	4 [114.02]	…	3 [111.84]
Fene-CR	4 [130.15]	4 [126.72]	4 [123.68]	3 [121.51]	3 [120.14]	3 [119.42]	…	3 [120.13]
Pom-pom	3 [137.17]	4 [127.23]	4 [119.78]	4 [114.51]	4 [110.66]	5 [107.76]	…	3 [101.00]

Lower We vs higher We !

Outlook: Multiphase Flow

- **Distanced based FEM Level Set formulation**

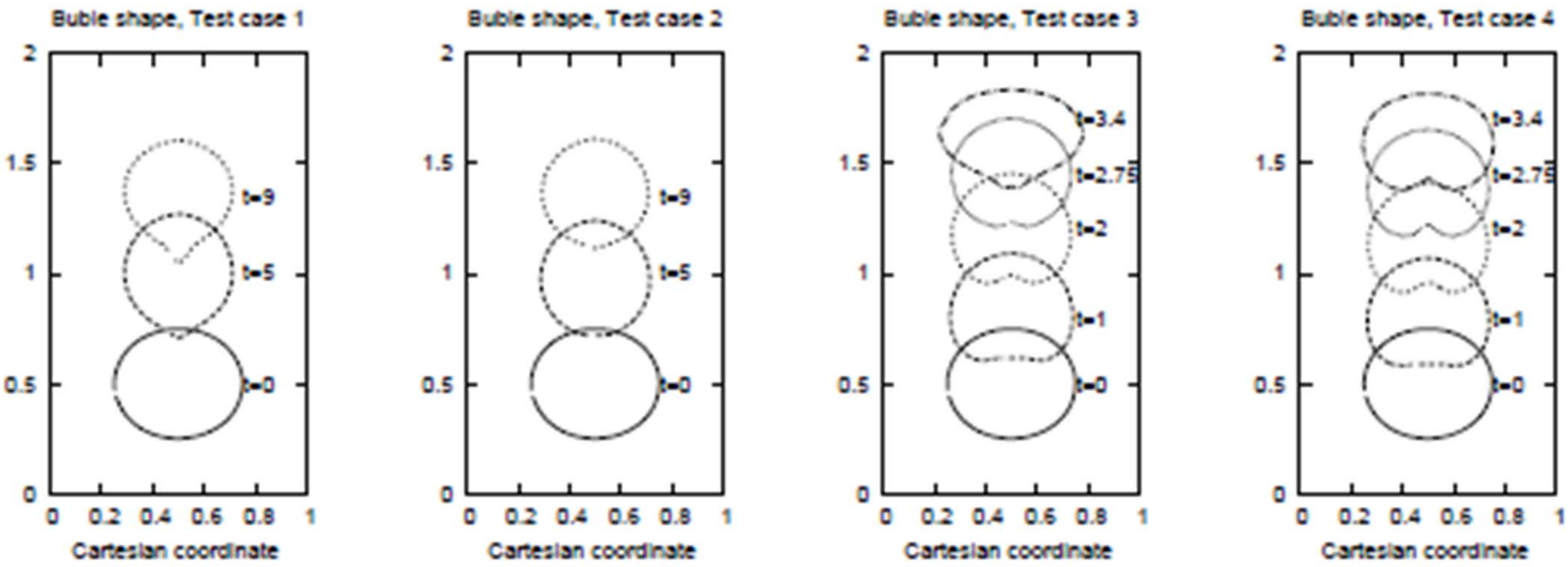
$$\left(\frac{\partial}{\partial t}+u\cdot\nabla\right)\varphi=0,\quad \left\|\nabla\varphi\right\|=1,\quad F_{CSF}=\sigma\kappa n\Big|_{surface}$$

- **Test cases**

Test case	ρ_1	ρ_2	η_1	η_2	g	σ
1. **Viscoelastic** $\Lambda=10$	10	0.1	10	1	9.8	0.245
2. **Newtonian** $\Lambda=0$	10	0.1	10	1	9.8	0.245
3. **Viscoelastic** $\Lambda=10$	10	0.1	2	1	9.8	0.245
4. **Newtonian** $\Lambda=0$	10	0.1	2	1	9.8	0.245

Preliminary Results

- **Rising bubble surrounded by viscoelastic fluids**



Typical cusp formation

Summary

Robust numerical and algorithmic tools are available using

- ✓ **Monolithic Finite Element Method (M-FEM)**
- ✓ **Classical and Log Conformation Reformulation (LCR)**
- ✓ **Edge Oriented stabilization (EO-FEM)**
- ✓ **Fast Newton-Multigrid Solver with local MPSC smoother**

for the simulation of nonlinear flow with viscoelastic behaviour

Advantages

- ✓ **No CFL-condition restriction due to the fully implicit coupling**
- ✓ **Positivity preserving**
- ✓ **Higher order and local adaptivity**