FEM multigrid techniques for the simulation of fluid-structure interaction with application to hemodynamics

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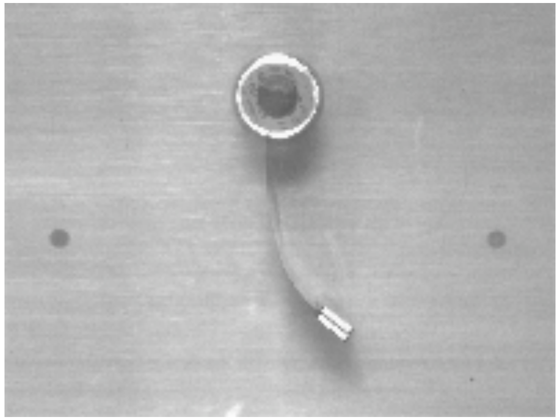
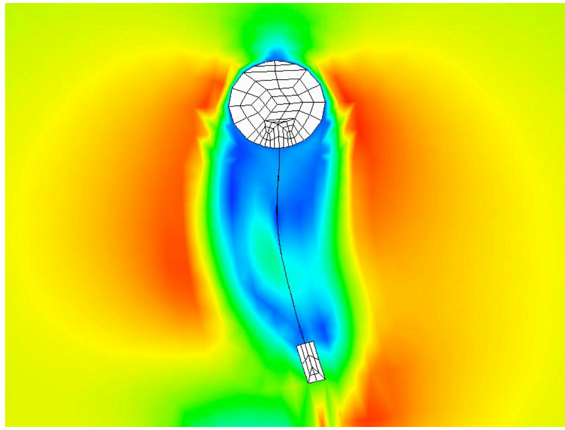
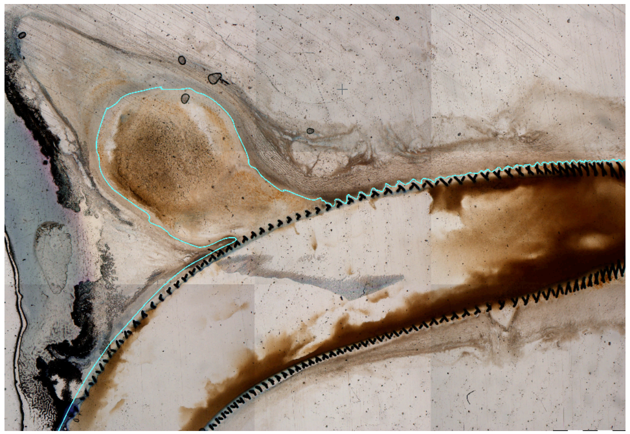
Fluid structure interaction

- Deformation of a structure in internal/external flow
- Model for bioengineering
- Numerical methods
- Benchmarking

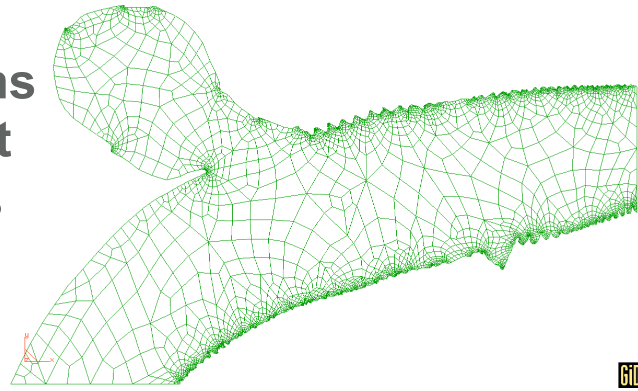


FSI experiment vs numerical simulation

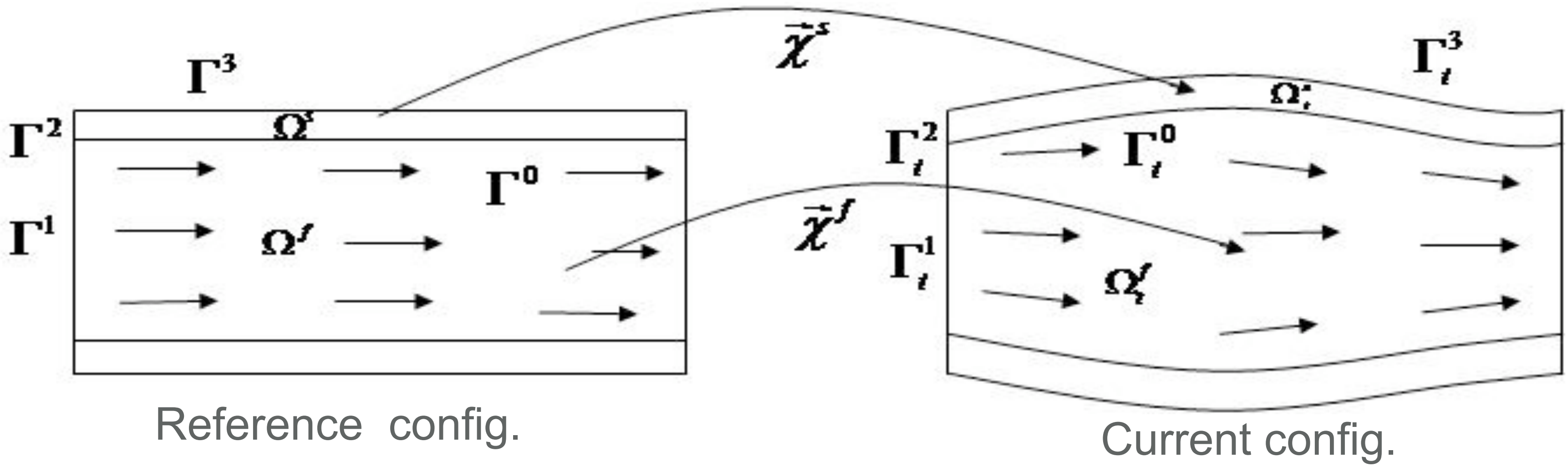




Brain Aneurysms and stent implants



Problem Description



Structure part

$$\chi^s : \Omega^s \times [0,T] \mapsto \Omega_t^s$$

$$u^s = \chi(X,t) - X, \quad v^s = \frac{\partial u^s}{\partial t}$$

$$F = I + \nabla u^s, \quad J = det\,(F)$$

Fluid part

$$\chi^f : \Omega^f \times [0,T] \mapsto \Omega_t^f$$

$$u^f = \chi(X,t) - X$$

$$v^f : \Omega^f \times [0,T] \mapsto \mathbb{R}^n$$

$$\chi : \Omega \times [0,T] \mapsto \Omega_t; \qquad v = \frac{\partial \chi}{\partial t}, \qquad F = \frac{\partial \chi}{\partial X}, \quad J = det\,(F)$$

$$\chi_R : R \times [0,T] \mapsto R_t, \quad R \subset \Omega_t\, \forall t \in [0,T]; \quad v_R = \frac{\partial \chi_R}{\partial t}, \qquad F_R = \frac{\partial \chi_R}{\partial X}, \quad J_R = det\,(F_R)$$

$$\frac{\partial}{\partial t}\int_{R_t} pdv \quad + \quad \int_{\partial R_t} \rho\,(v - v_R) \cdot n_{R_t} da = 0$$

$$\frac{\partial}{\partial t}(\rho J) \quad + \quad \nabla \cdot (\rho J_R\,(\,v - v_R)\,F^{-T}) = 0$$

Lagrangian description

$$\xi_R = \chi \Longrightarrow F_R = F,\, J_R = J,\, v_R = v$$

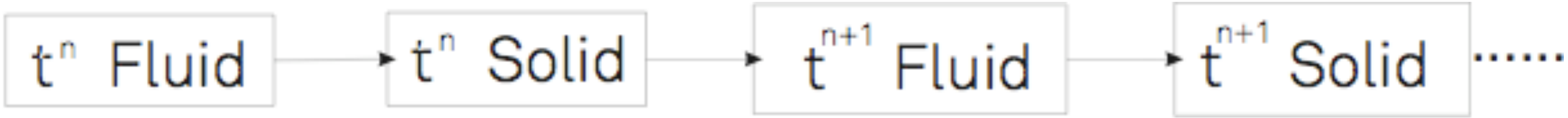
$$\frac{\partial}{\partial t}(\rho\, J) = 0$$

Eulerian description

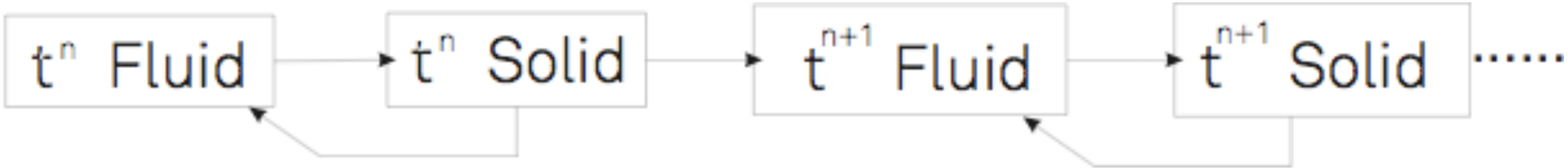
$$\xi_R = Id \Longrightarrow F_R = I,\, J_R = 1,\, v_R = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, v) = 0$$

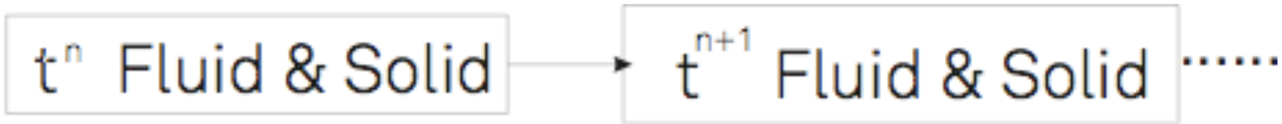
- Weak coupling (separated)



- Strong coupling (separated)



- Monolithic



Discrete Nonlinear System

$$R(X) = 0,\ X = (u_h, v_h, p) \in U_h \times V_h \times P_h$$

$$\frac{\partial R}{\partial X}(X) = \begin{bmatrix} M - \frac{k}{2}L^f & \frac{k}{2}M^s & 0 \\ \frac{1}{2}\frac{\partial\left(N_1+S^s+S^f\right)}{\partial u_h} + k\frac{\partial B}{\partial u_h}p_h & M^s + \beta M^f \frac{k}{2}\frac{\partial N_2}{\partial v_h} + \frac{k}{2}\frac{\partial\left(N_1+S_f^2\right)}{\partial v_h} & k\,B \\ {B^s}^T + \frac{\partial {B^f}^T}{\partial u_h}v_h & {B^f}^T & 0 \end{bmatrix}$$

$$\begin{bmatrix} S_{uu} & S_{uv} & 0 \\ S_{vu} & S_{vv} & k\,B \\ c_u {B^s}^T & c_v {B^f}^T & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ p \end{bmatrix} = \begin{bmatrix} f_u \\ f_v \\ f_p \end{bmatrix}$$

Typical discrete saddle point problem as well known for incompressible NSE

Governing equations

- **Structure part**

$$\frac{\partial v^s}{\partial t} = \nabla \cdot \left(J \sigma^s F^{-T} \right) f, \quad det(F) = 1 \qquad \text{in } \Omega^s$$

$$u^s = 0, \quad \sigma^s n = 0 \qquad \text{on } \Gamma^3$$

- **Interface**

$$v^f = v^s, \quad \sigma^f n = \sigma^s n \quad \text{on } \Gamma_t^0$$

Governing equations

- **Fluid part**
  - **Generalized Navier-Stokes equations**

$$\rho\frac{\partial v}{\partial t} + v\cdot\nabla v - \nabla\cdot\sigma + \nabla p = \rho f \quad , \quad \nabla\cdot v = 0,$$

$$\frac{\partial\Theta}{\partial t} + v\cdot\nabla\Theta - \nabla\cdot(k\nabla\Theta) = D:\sigma,$$

$$\sigma = \sigma^s + \sigma^p \quad , \quad D = \frac{1}{2}\left(\nabla v + (\nabla v)^{\mathrm{T}}\right).$$

  - **Viscous stress** $\sigma^s = 2\eta_s(D_{\mathrm{II}},\Theta)D \quad , \quad D_{\mathrm{II}} = tr(D^2).$

  - **Elastic stress**

$$\sigma^p + \Lambda\frac{\delta_a\sigma^p}{\delta t} = 2\eta_p D,$$

- **FEM-Discretization**
  - **The quadriplex FE $Q_2/Q2/P_1^{disc}/Q2$ for displacement-velocity-pressure-stress**
  - **Edge-oriented stabilization for**
    - **convective dominated problem**
    - **equal interpolation for stress and velocity (special models)**

- **LCR formulation**
  - **Conformation tensor is positive**

$$\tau(X,t) = \int_{\infty}^{t} \frac{\eta_p}{\Lambda} \exp\left(\frac{-(t-s)}{\sqrt{\Lambda}}\right) F(s,t)F(s,t)^{\mathrm{T}}\, ds$$

  - **Positivity preserving via the change of variable**

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right)\tau - (\Omega\tau - \tau\Omega) + 2\,B\tau = \frac{1}{\Lambda}(1-\tau)$$

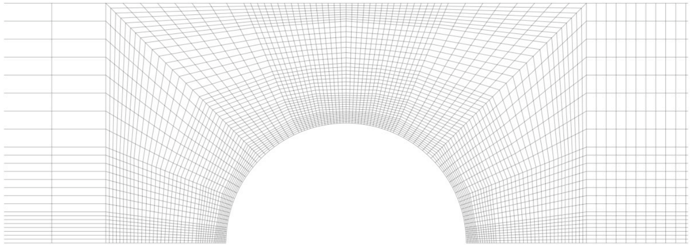
- **Inexact Newton**
  - **The Jacobian matrix is approximated using finite differences**

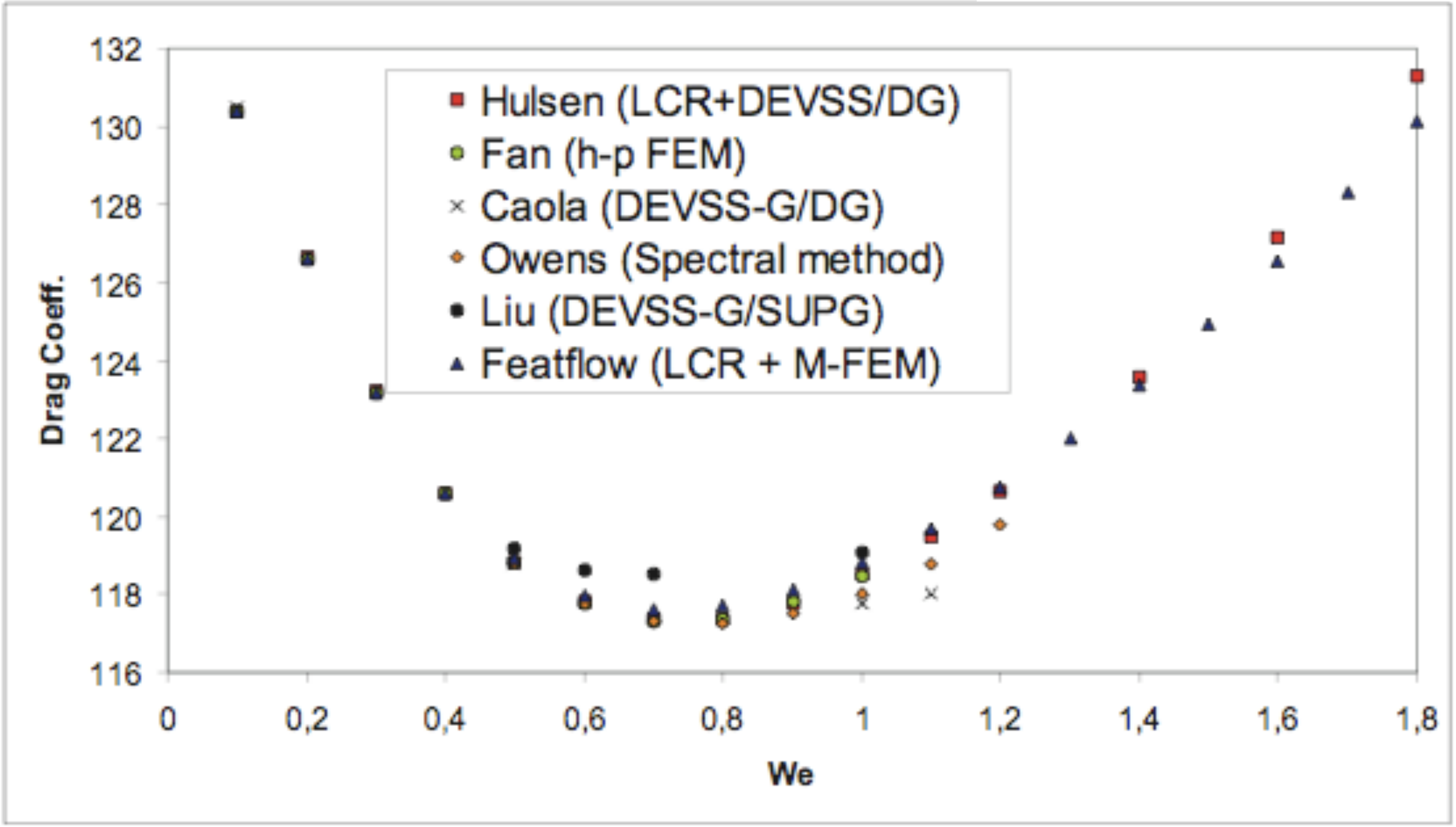
$$\left[\frac{\partial R(\mathbf{x}^n)}{\partial \mathbf{x}}\right]_{ij} \approx \frac{R_j(\mathbf{x}^n + \epsilon e_j) - R_i(\mathbf{x}^n - \epsilon e_i)}{2\epsilon}$$

- **Multgrid Solver**
  - **Local MPSC via Vanka-like smoother**

$$\begin{bmatrix} u^{l+1} \\ v^{l+1} \\ \sigma^{l+1} \\ \Theta^{l+1} \\ p^{l+1} \end{bmatrix} = \begin{bmatrix} u^{l} \\ v^{l} \\ \sigma^{l} \\ \Theta^{l} \\ p^{l} \end{bmatrix} + \omega^l \left[(K+J)_{|T}\right]^{-1} \begin{bmatrix} Res_u \\ Res_v \\ Res_\sigma \\ Res_\Theta \\ Res_p \end{bmatrix}_{|T}$$

- Planar flow around cylinder (Oldroyd-B)

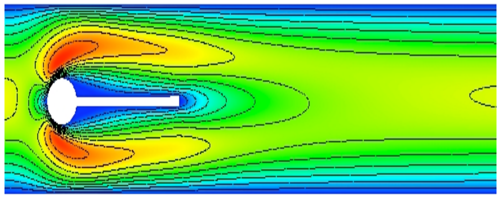
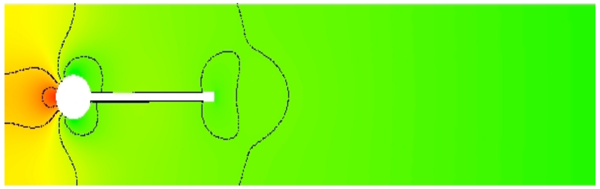
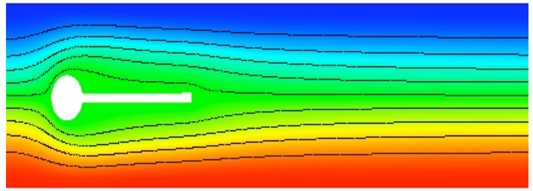




Parameter	FSI1	FSI2	FSI3
$\rho^s[10^3\mathrm{kg/m}^3]$ $\mu^s[10^6\mathrm{kg/ms}^2]$	1 0.4 0.5	1 0.4 0.5	1 0.4 2.0
$\rho^s[10^3\mathrm{kg/m}^3]$ $\nu^s[10^{-3}\mathrm{m}^2/\mathrm{s}]$	1 1	1 1	1 1
$\bar{U}[\mathrm{m/s}]$	0.2	1	2

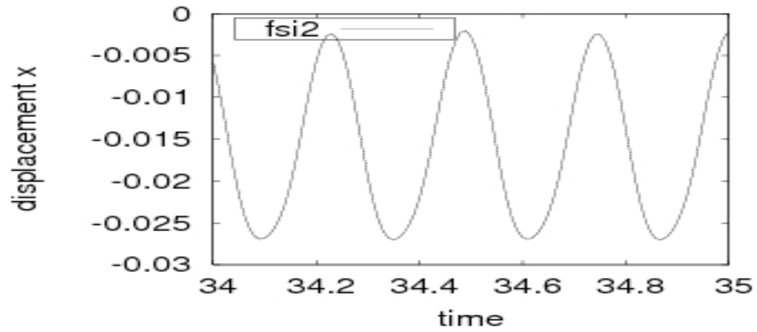
Parameter	FSI1	FSI2	FSI3
$\beta = \frac{\rho^s}{\rho^f}$ $\mathrm{Ae} = \frac{\mathrm{E}^s}{\rho^f \bar{U}^2}$	1 0.4	1 0.4	1 0.4
$\mathrm{Re} = \frac{\bar{U}d}{\nu^f}$ $\bar{U}[\mathrm{m/s}]$	20 0.2	100 1	200 2

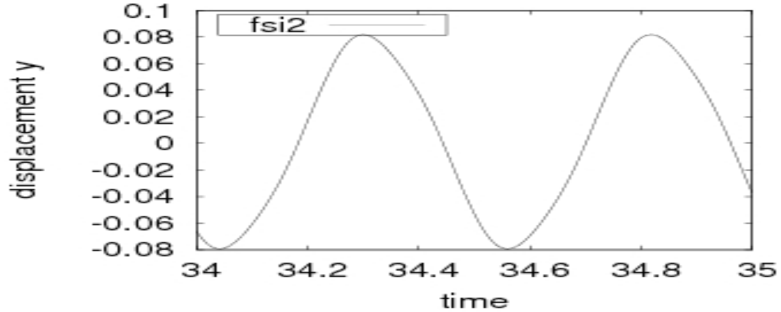
- **FSI1: steady, small deformations**

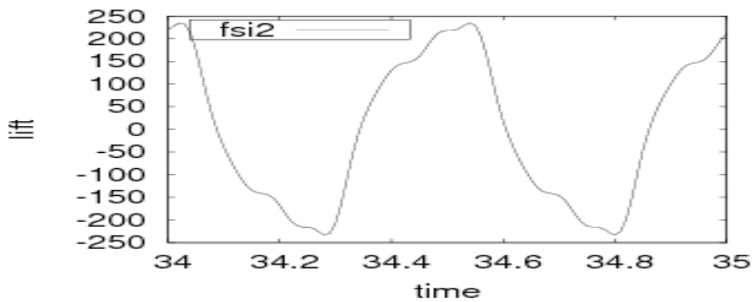




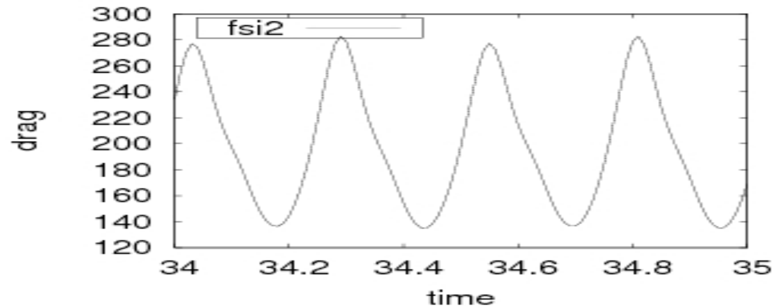
	ux of A $[\times 10^{-3}\mathrm{m}]$	uy of A $[\times 10^{-3}\mathrm{m}]$	drag	lift
FSI1	0.0227	0.8209	14.295	0.7638

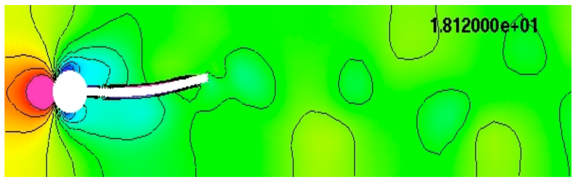
FSI2: large deformations, periodical oscillations

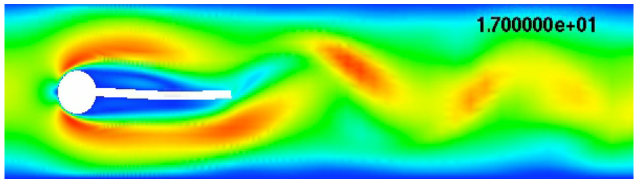


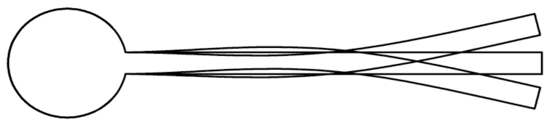






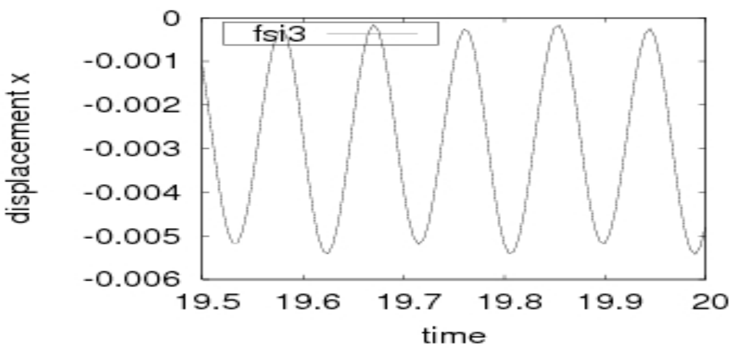


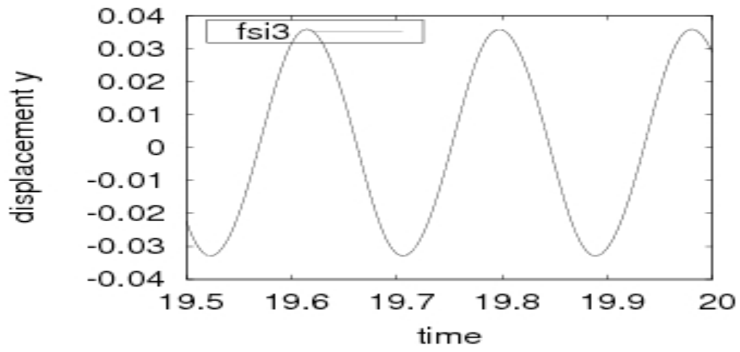


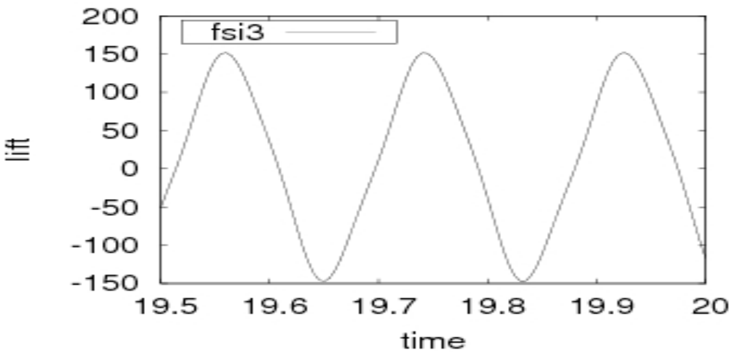


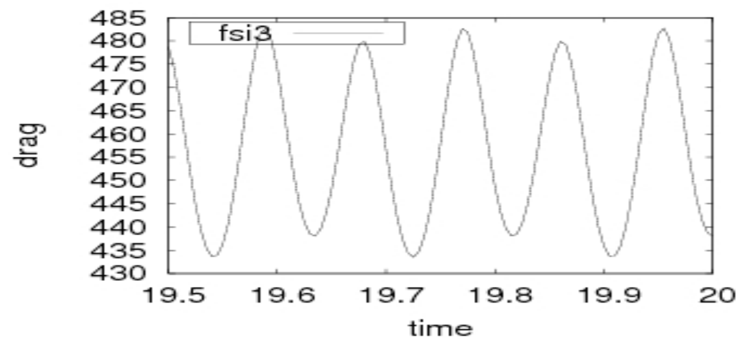
Test	ux of A $[\times 10^{-3} m]$	uy of A $[\times 10^{-3} m]$	drag	lift
FSI2	$-14.58 \pm 12.44[3.8]$	$1.23 \pm 80.6[2.0]$	$208.83 \pm 73.75[3.8]$	$0.88 \pm 234.2[2.0]$

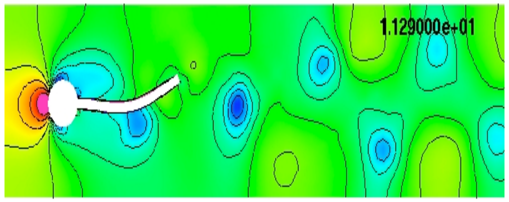
FSI3: large deformations, complex oscillations

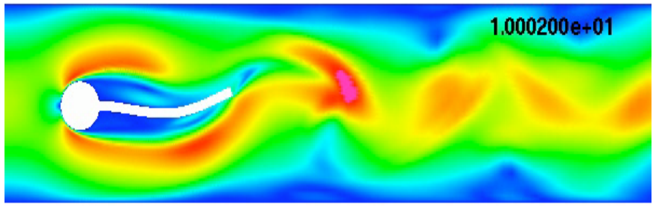


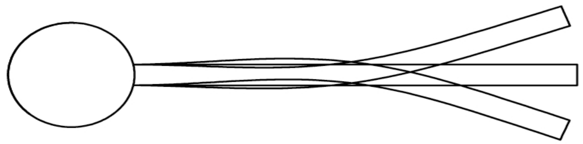








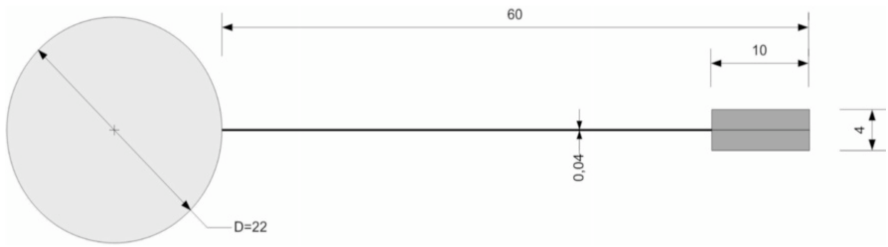


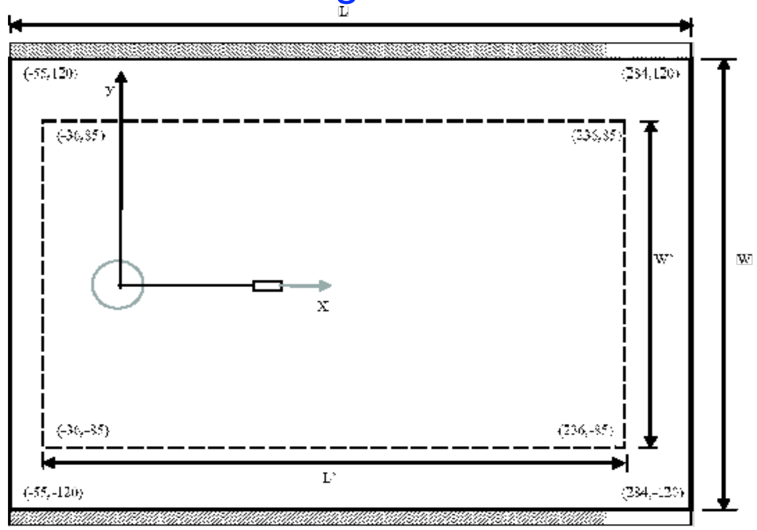


Test	ux of A [m]	ux of A [m]	drag	lift
FSI3	$-2.69 \pm 2.53[10.9]$	$1.48 \pm 34.38[5.3]$	$457.3 \pm 22.66[10.9]$	$2.22 \pm 149.78[5.3]$

FSI4: benchmarking of experimental data

- Flustruc experiment, Erlangen, http://www.lstm.uni-erlangen.de/flustruc





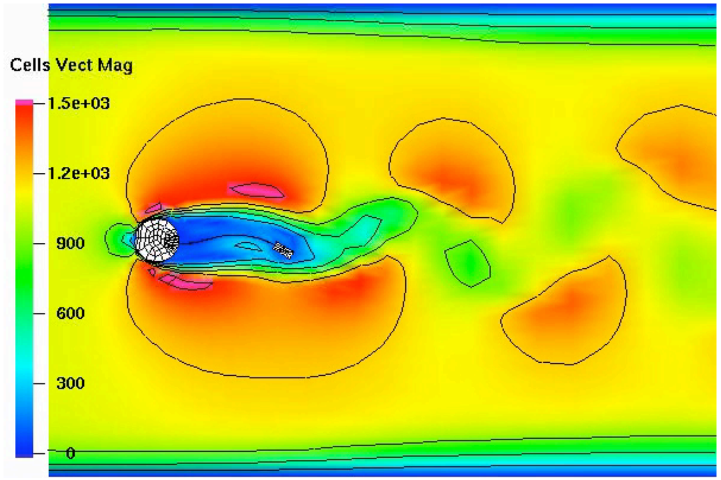
fluid parameters	
density of the fluid kinematic viscosity	1.05e-6 [kg/mm^3] 164.0

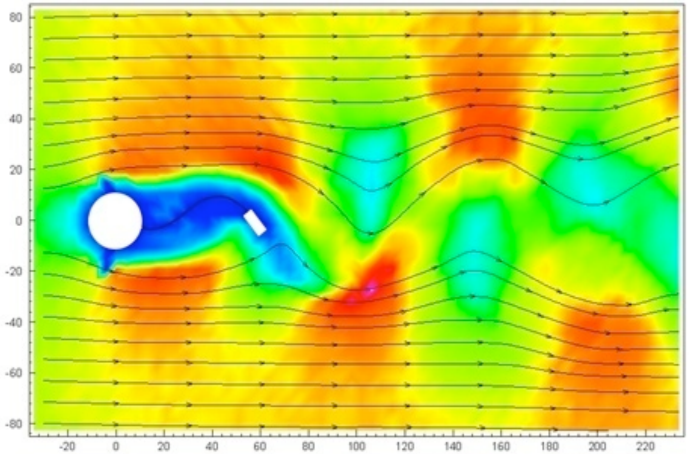
solid parameters	
density of the beam (steel) density of the rear mass shear modulus poisson ratio	7.85e-6[kg/mm^3] 7.8e-6 [kg/mm^3] 7.58e13 0.3

geometry parameter		value [mm]
channel length	*L*	338.0
channel width	*W*	240.0
cylinder center position	*C*	(0.0, 0.0)
cylinder radius	*R*	11.0
elastic structure length	*l*	50
elastic structure thickness	*w*	0.04
rear mass length	*w'*	10.0
rear mass thickness	*h'*	4.0
reference point (at t=0)	*A*	(71.0, 0.0)
reference point	*B*	(11.0, 0.0)

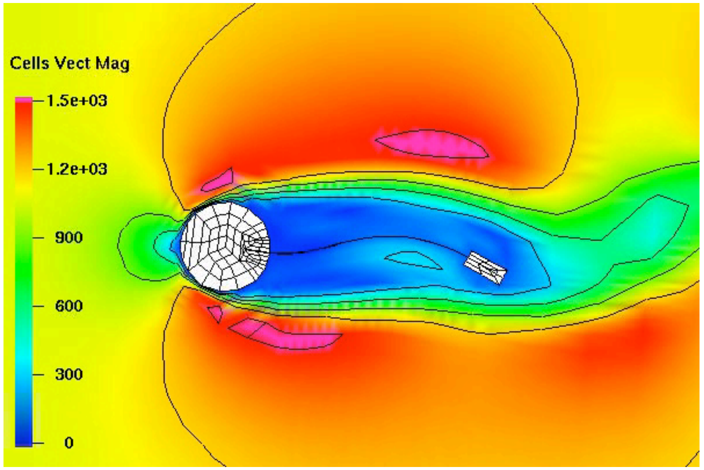
FSI4: benchmarking of experimental data

- + Laminar Flow (glycerine)
- + “2D“ flow and deformation
- - Rotational degree of freedom
- - Large aspect ratio (thin structure)
- - Corners





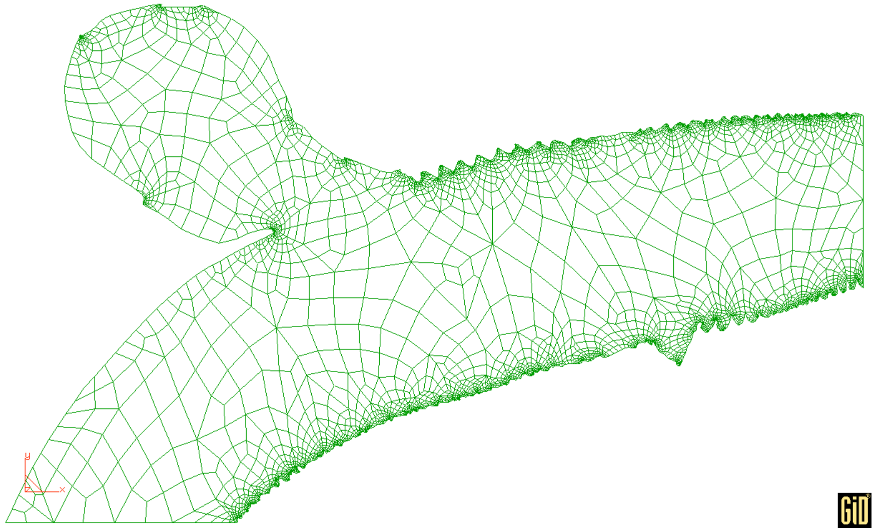
Flustruc experiment,
Erlangen



Computation

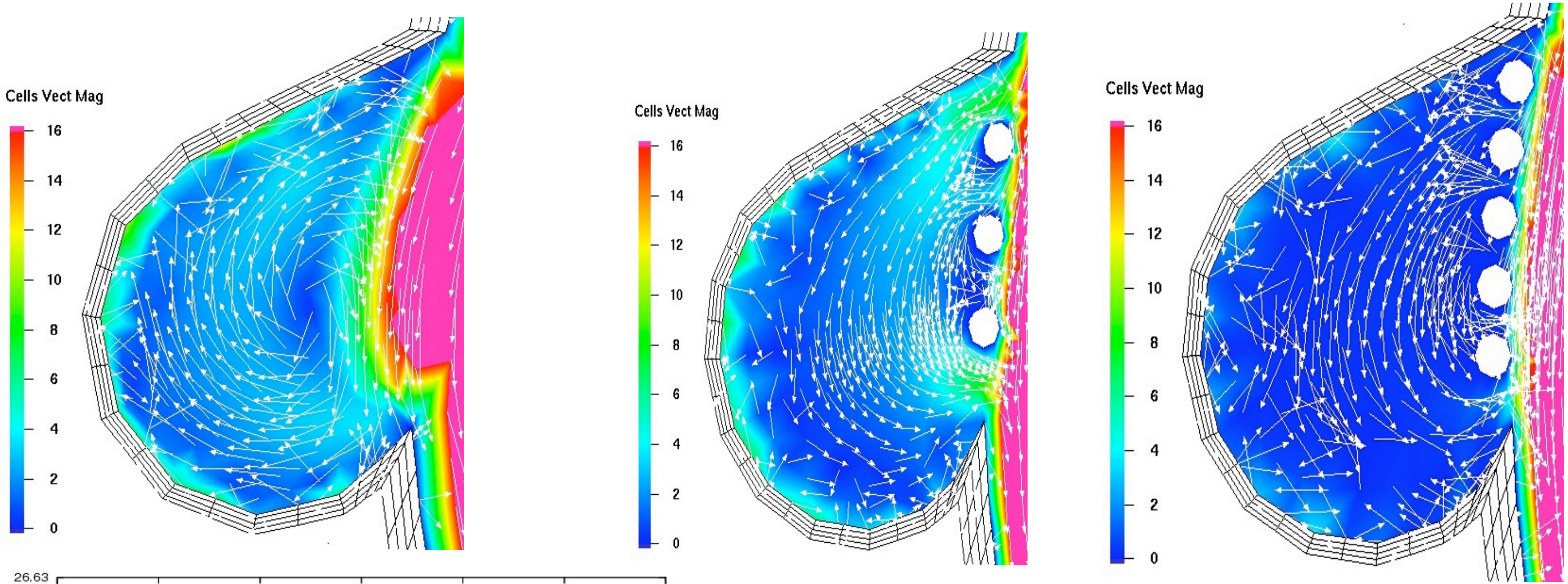
- **Aim:** Numerical study of FSI due to stent geometries **and** elastic wall behaviour

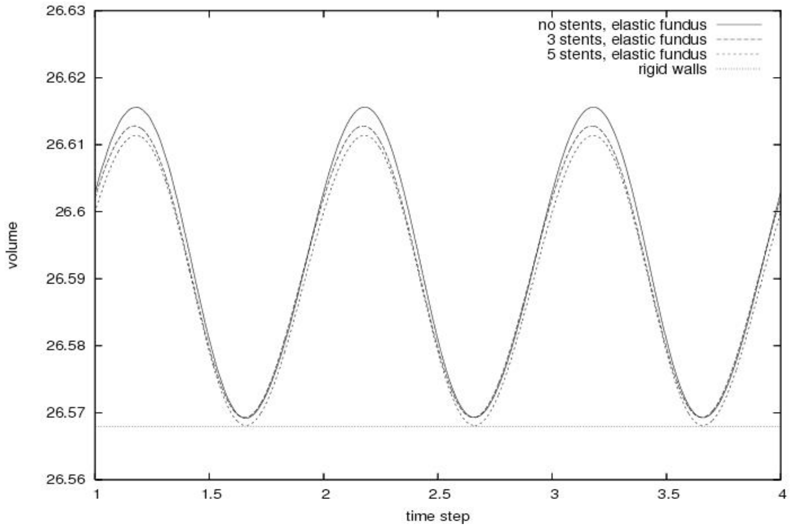
Cerebral aneurysm

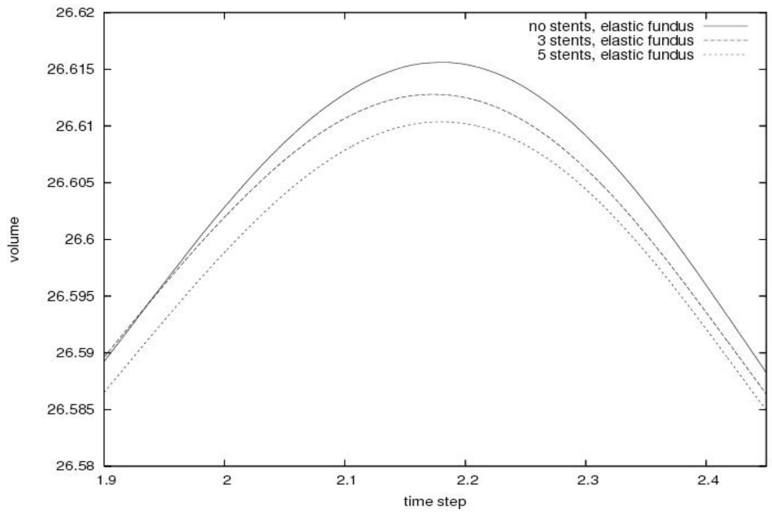


Mesh for numerical Simulation

- Strong influence of stent geometry and elastic wall deformations







New Numerical and Algorithmic tools are availible using

- ✓ Monolithic Finite Element Method
- ✓ Arbitrary Lagrangian-Eulerian Formulation
- ✓ LCR for viscoelastic flow
- ✓ EO-FEM stabilization
- ✓ Fast Multigrid Solver with local MPSC smoother

For the simulation of FSI with application to hemodynamics