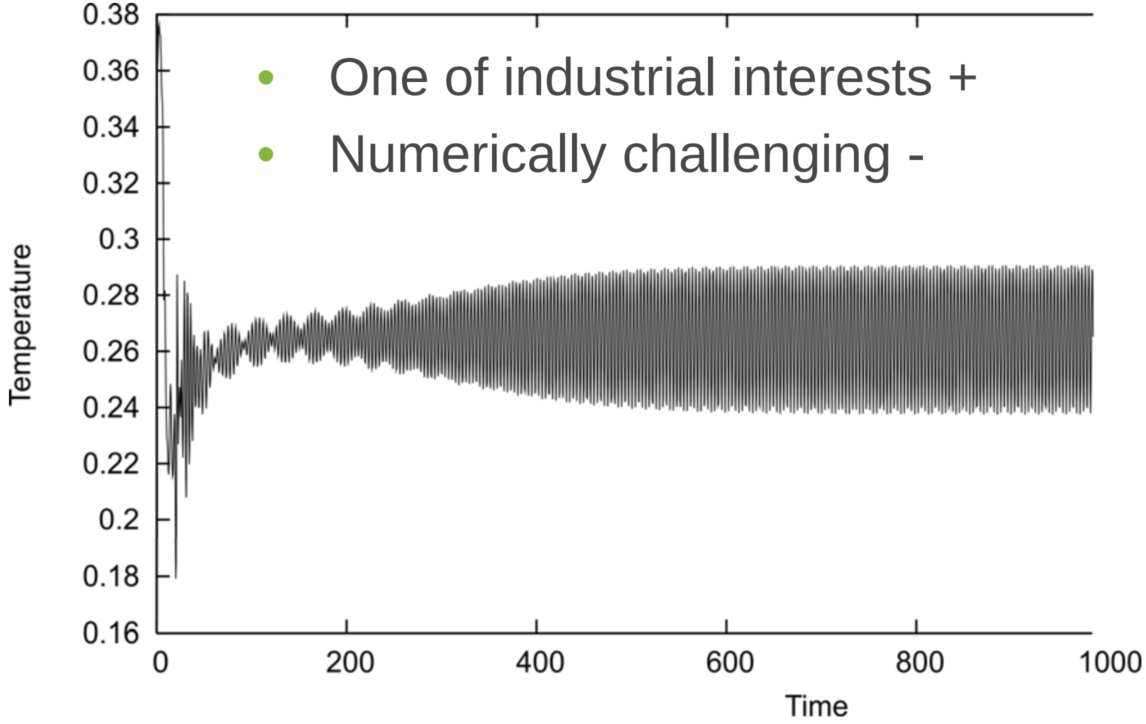
Natural Convection of Incompressible Viscoelastic Fluid Flow

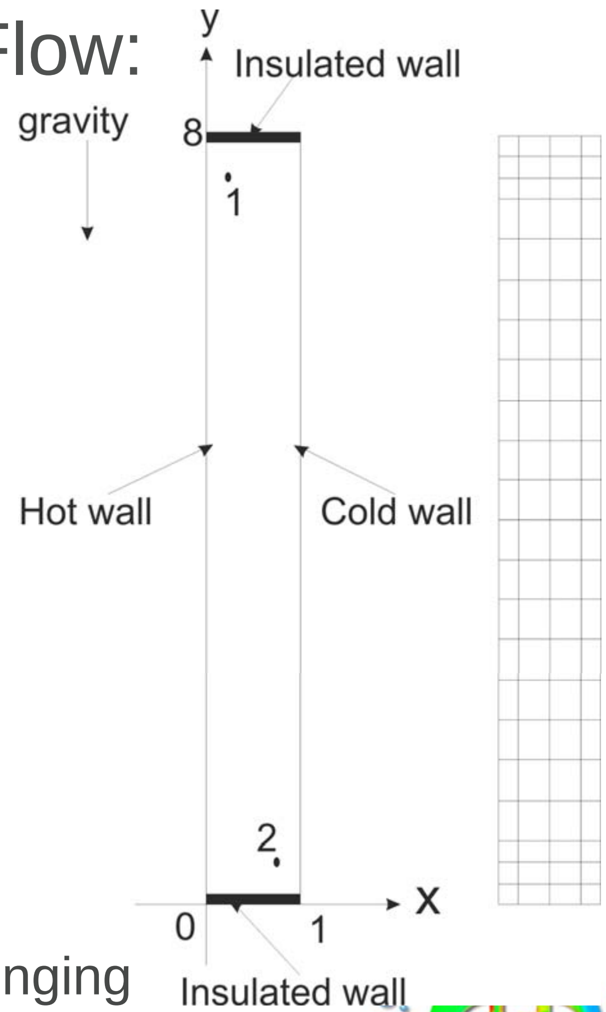
Damanik, H., Turek, S.

(Jakarta, Dec 2017)

Temperature oscillation in Newtonian Flow:



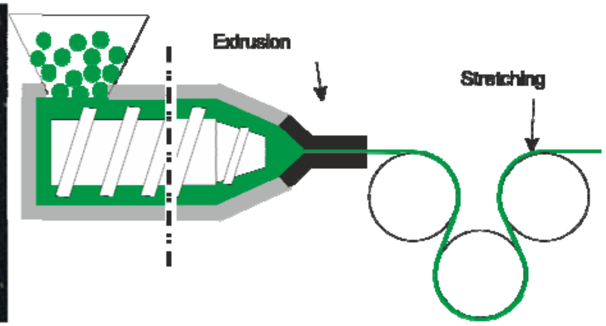
- One of industrial interests +
- Numerically challenging -

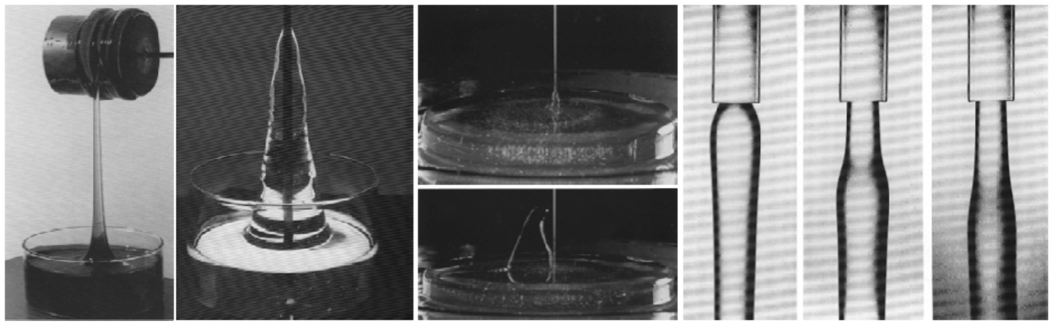


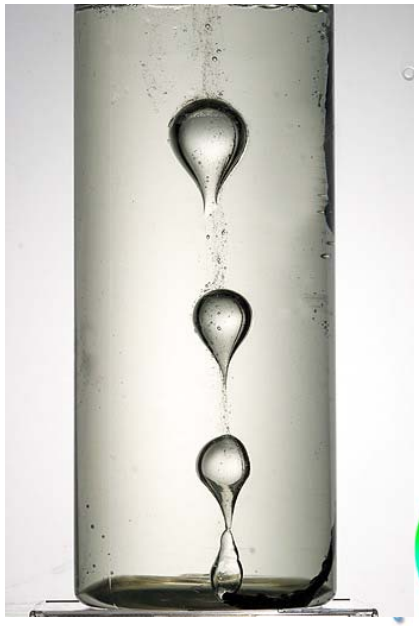
Highly accurate, robust numerical solver which represents the temperature oscillation is challenging

Polymer melts:

- One of industrial interests +
- Physically fascinating +
- Rheologically difficult -
- Numerically challenging -

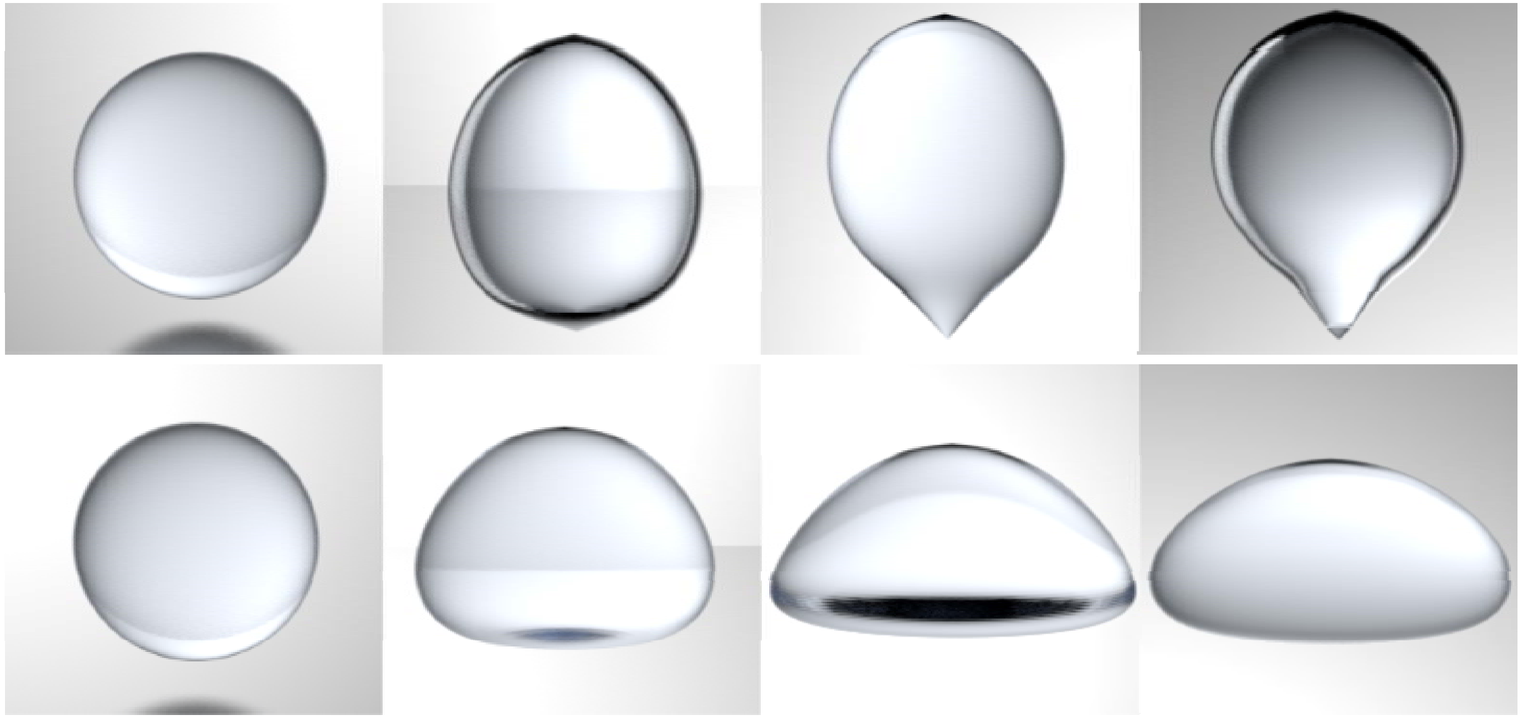






Highly accurate, robust numerical solver which represents the rheological nature is still challenging

Rising bubble in viscoelastic fluid:



A totally different behaviour!

Polymer as Viscoelastic model

Viscoelastic fluid models **(D. D. Joseph)**:

- Integral form

$$\tau(t) = \int_{-\sim}^{t} \frac{1}{We^2} e^{\frac{-(t-s)}{We}} F(s,t)F(s,t)^T \, ds$$

- Differential form: Upper-convected derivative
- More practical to implement than integral form
- Represent many viscoelastic models

$$\frac{\partial \tau}{\partial t} + (u \cdot \nabla)\tau - \nabla u \cdot \tau - \tau \cdot \nabla u^T = f(\tau)$$

- Conformation tensor (τ), velocity (u), source ($f(\tau)$)
- Not able to capture high stress gradient at higher We number
- $f(\tau)$ can be Oldroyd-B, Giesekus, FENE, PTT, WM, Pom-Pom

Experience **(Kupferman et. al)**:

- Stresses grow exponentially
- Conformation tensor looses positive properties during numerics

$$\frac{\partial \tau}{\partial t} + (u \cdot \nabla)\tau - \nabla u \cdot \tau - \tau \cdot \nabla u^T = f(\tau)$$

$$\nabla u = \Omega + B + N\tau^{-1}$$

$$\frac{\partial \tau}{\partial t} + (u \cdot \nabla)\tau - (\Omega\tau - \tau\Omega) - 2B\tau = f(\tau)$$

$$\tau = e^{\psi}$$

$$\frac{\partial \psi}{\partial t} + (u \cdot \nabla)\psi - (\Omega\psi - \psi\Omega) - 2B = g(\psi)$$

LCR based models

technische universität dortmund

LCR based viscoelastic fluid:

- Ability to capture high stress gradients at higher We number
- Positivity preserving by design $\tau = e^{\psi}$
- Numerically more stable with appropriate FEM

$$\frac{\partial \psi}{\partial t} + (u \cdot \nabla)\psi - (\Omega\psi - \psi\Omega) - 2B = g(\psi)$$

- LCR tensor (ψ), velocity (u), source ($g(\psi)$)
- $\nabla u = \Omega + B + N\tau^{-1}$
- $g(\psi)$can be Oldroyd-B, Giesekus, FENE, PTT, WM, Pom-Pom

MIT Benchmark: velocity-temperature-pressure

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + 2\eta\, \nabla \cdot \boldsymbol{D} + (1 - \gamma\theta)\boldsymbol{j} \\ \nabla \cdot \boldsymbol{u} = 0 \\ \dfrac{\partial \theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)\theta = \dfrac{1}{\sqrt{Ra\, Pr}} \Delta\theta \end{cases}$$

With critical values: $Ra = 3.4e + 5, Pr = 0.71$

Total equations

MIT Benchmark + viscoelastic stress

$$
\begin{cases}
\dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + 2\eta_s\,\nabla\cdot\boldsymbol{D} + (1-\gamma\theta)\boldsymbol{j} + \dfrac{\eta_p}{\Lambda}\nabla\cdot\boldsymbol{\tau} \\
\nabla\cdot\boldsymbol{u} = 0 \\
\dfrac{\partial \theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)\theta = \dfrac{1}{\sqrt{Ra\,Pr}}\Delta\theta \\
\dfrac{\partial \boldsymbol{\tau}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\tau} - \nabla\boldsymbol{u}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\nabla\boldsymbol{u}^T + \dfrac{1}{\Lambda}(\boldsymbol{\tau}-\boldsymbol{I}) = \boldsymbol{0}
\end{cases}
$$

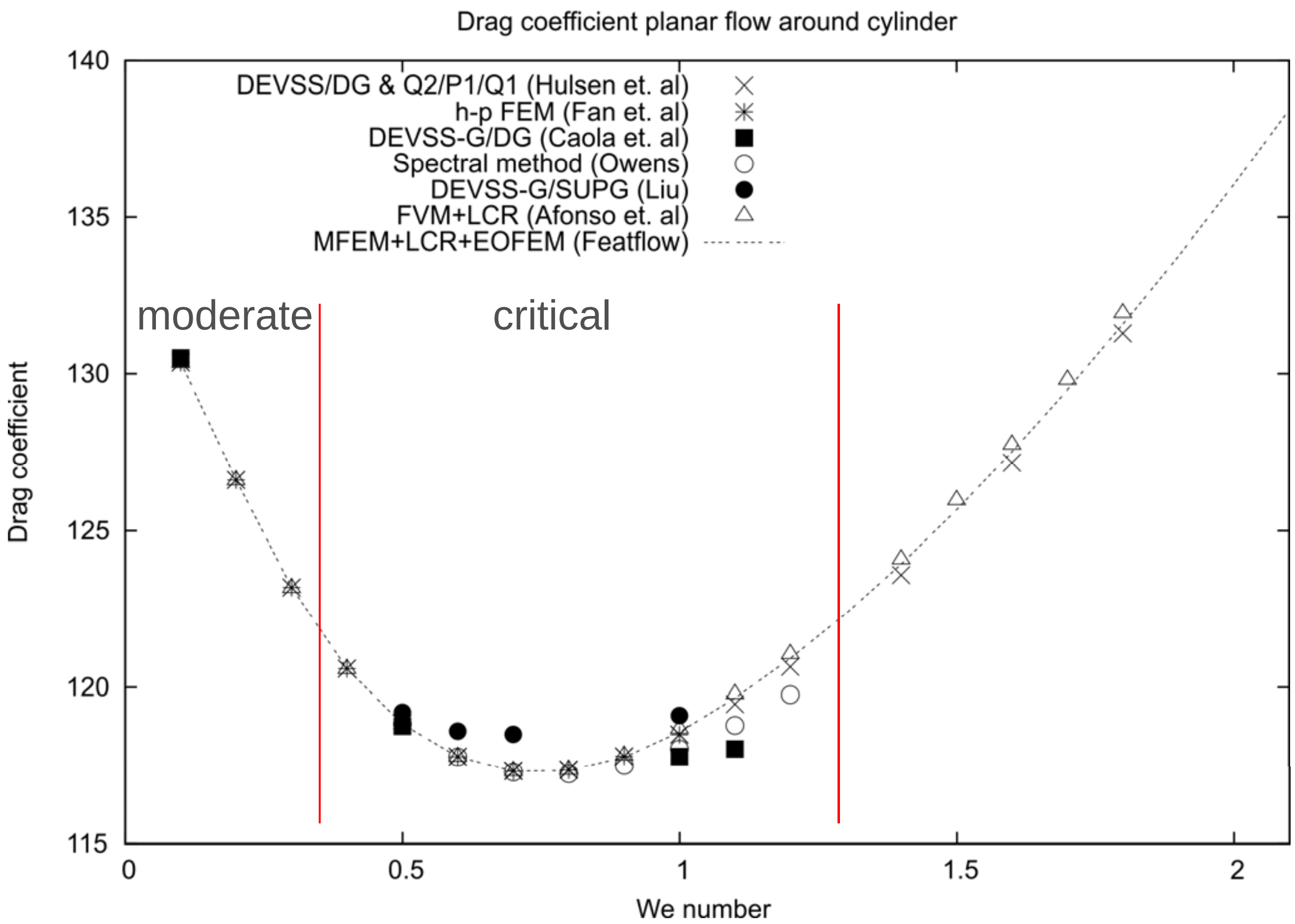
With critical values: $Ra = 3.4e+5, Pr = 0.71$

With moderate: $\Lambda = 0.1$

Why a moderate relaxation time

technische universität dortmund

Flow around cylinder:



A moderate value of relaxation time $\Lambda = 0.1$ has a good agreement between different numerical methods

Discretizations

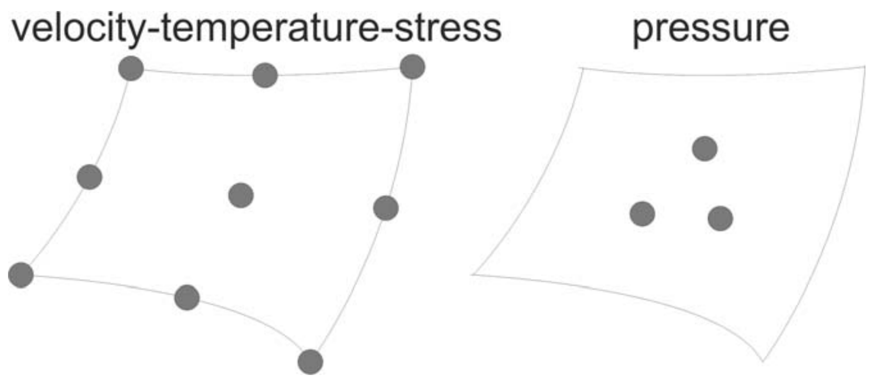
technische universität dortmund

In Time:

- Second order Crank-Nicolson
- Can be adaptively applied

In Space: Higher order finite element **(Arnoldi)**

- Inf-sup stable for velocity and pressure
- High order: good for accuracy
- Discontinuous pressure: good for solver & physics
- Edge oriented FEM for numerical stabilitation **(Burman)**



technische universität dortmund

Saddle point problem:

- $\tilde{u}$ consists of all numerical variables except pressure
- Newton with multigrid as well-known solver
- Monolithic way of solving

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathrm{rhs}_{\tilde{u}} \\ \mathrm{rhs}_p \end{pmatrix}$$

- A consists of differential operators
- B is gradient operator

Newton for nonlinear system:

- Strongly coupled problem
- Automatic damping control ω^n for each nonlinear step
- Black-box for many given viscoelastic models

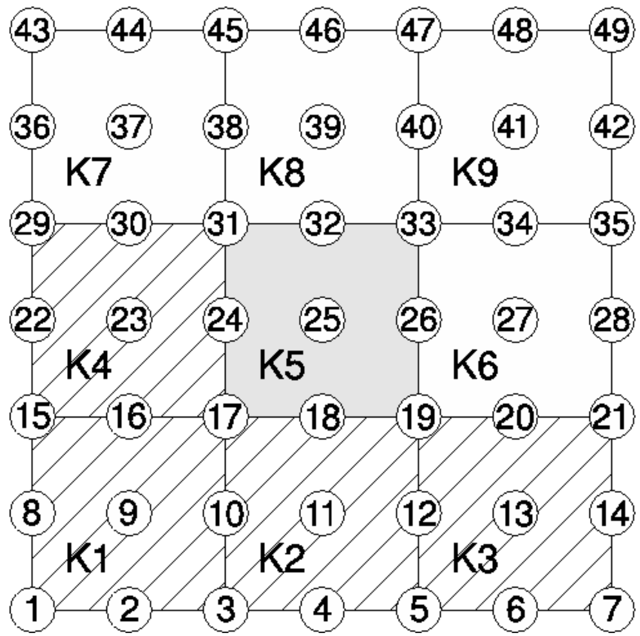
$$x^{n+1} = x^n + \omega^n \left[\frac{\partial \mathcal{R}(x^n)}{\partial x}\right]^{-1} \mathcal{R}(x^n)$$

- Quadratic convergence when iterative solutions are close
- Solution $x^{n+1} = (\tilde{u}, p)$, Residual equation $\mathcal{R}(x^n)$
- Black-box is made possible by divided difference technique

$$\left[\frac{\partial \mathcal{R}(x^n)}{\partial x}\right]_{ij} = \frac{\mathcal{R}_i\left(x^n + \varepsilon e_j\right) - \mathcal{R}_i\left(x^n + \varepsilon e_j\right)}{2\varepsilon}$$

Multigrid for linearized system:

- Full-Vanka for strongly coupled Jacobian in local system
- Full prolongation
- Black-box for many given viscoelastic models

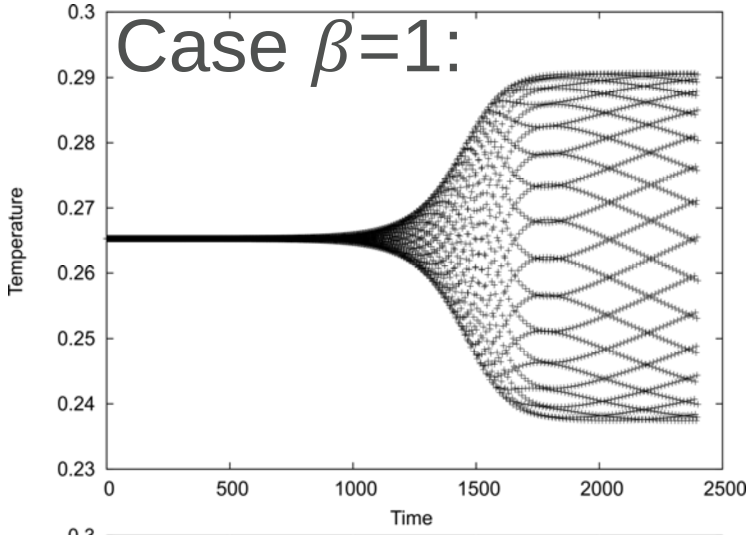


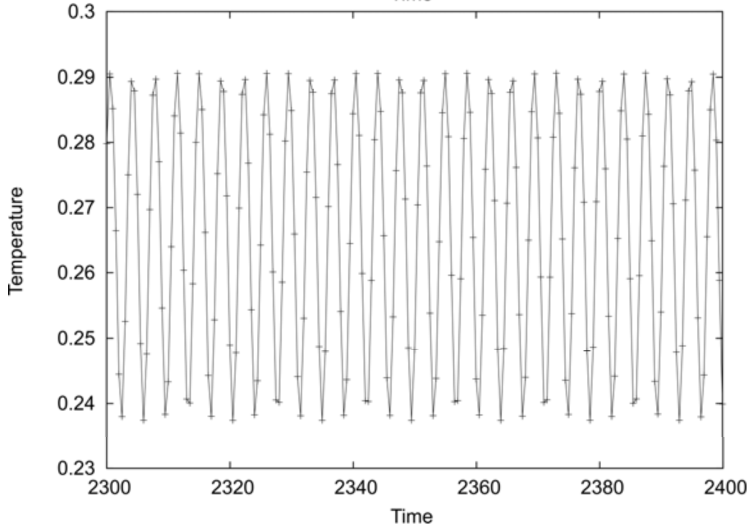
$$\begin{bmatrix}\tilde{u}^{l+1}\\ p^{l+1}\end{bmatrix} = \begin{bmatrix}\tilde{u}^{l}\\ p^{l}\end{bmatrix} + \omega \sum_{\text{patch }\Omega_i} \begin{bmatrix} A_{\tilde{u}|\Omega_i} & kB_{|\Omega_i} \\ B^T{}_{|\Omega_i} & 0 \end{bmatrix}^{-1} \begin{bmatrix}\text{def}_{\tilde{u}|\Omega_i}\\ \text{def}_{p|\Omega_i}\end{bmatrix}$$

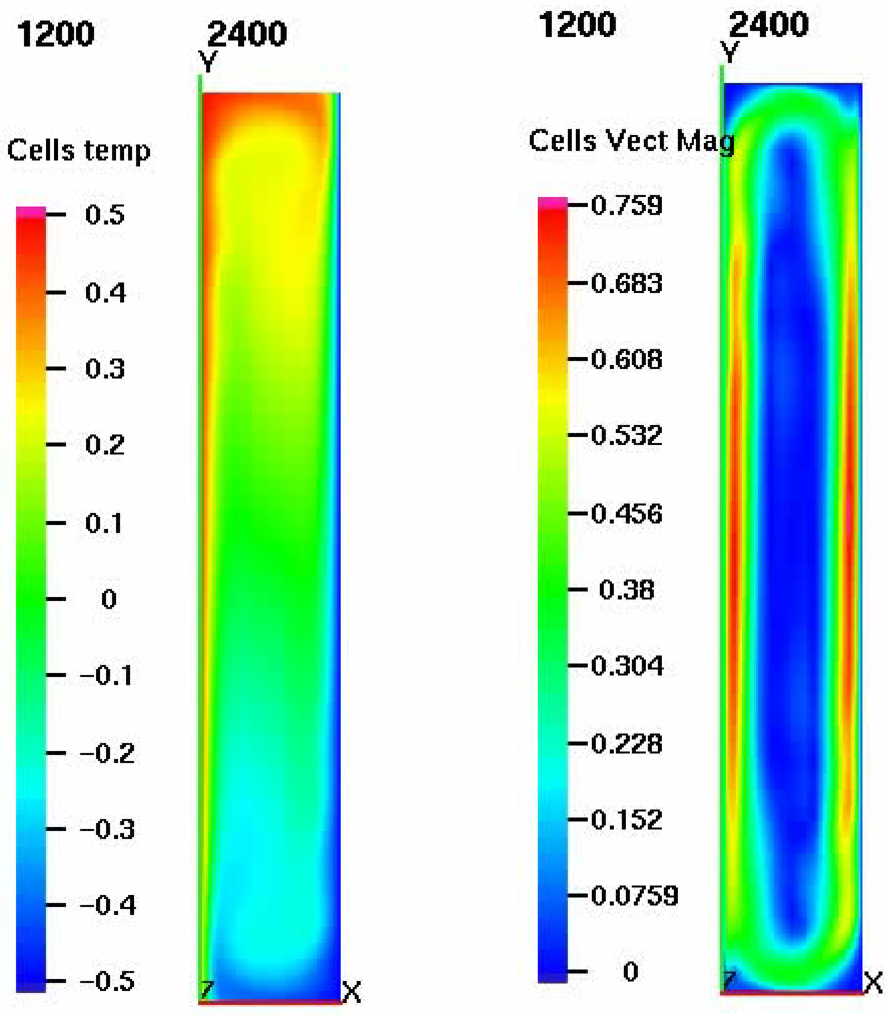
Modified MIT Benchmark

technische universität
dortmund

Case β=1:



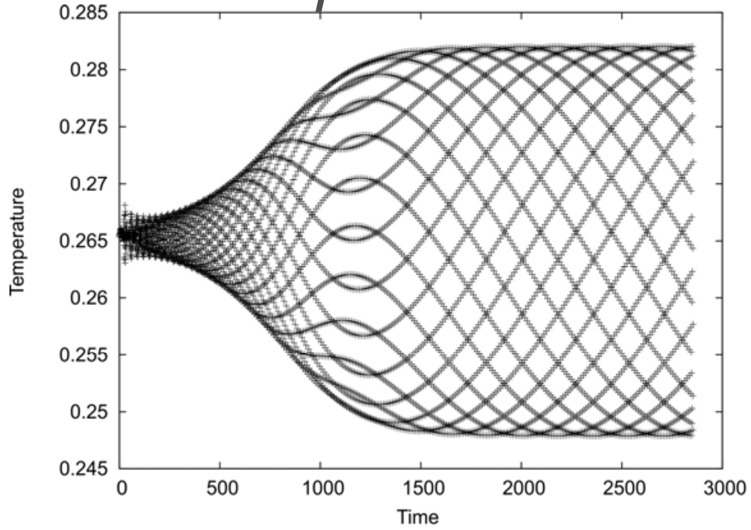


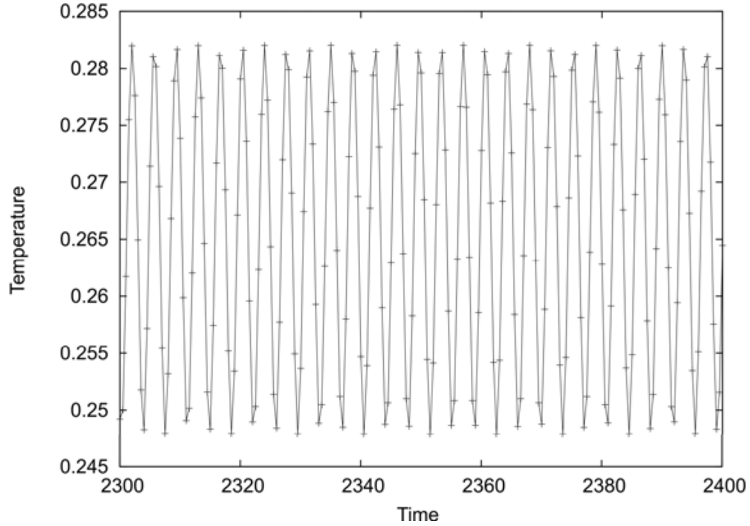


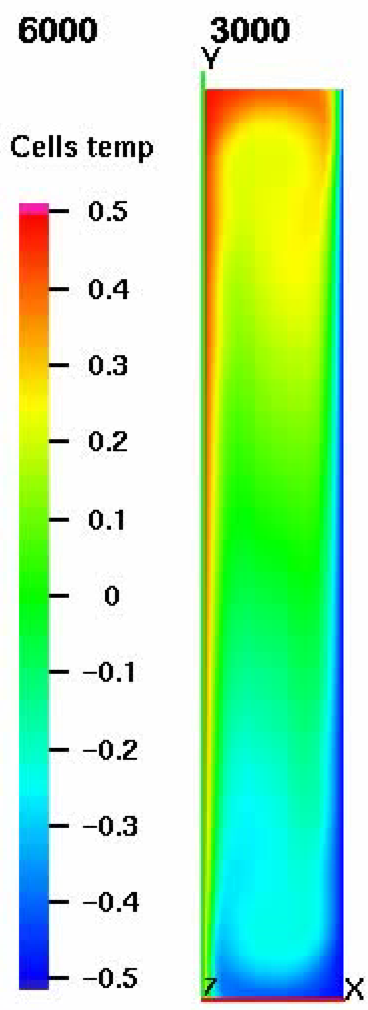
The old MIT enchmark 2001 is recovered

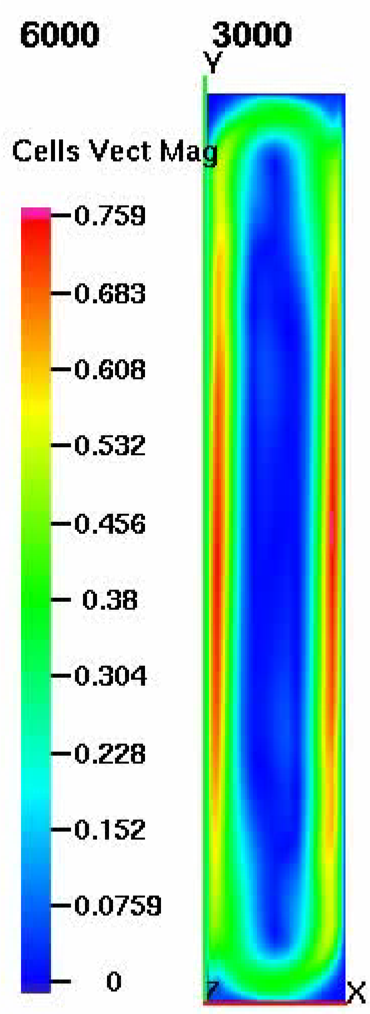
technische universität dortmund

Case β=0.75:



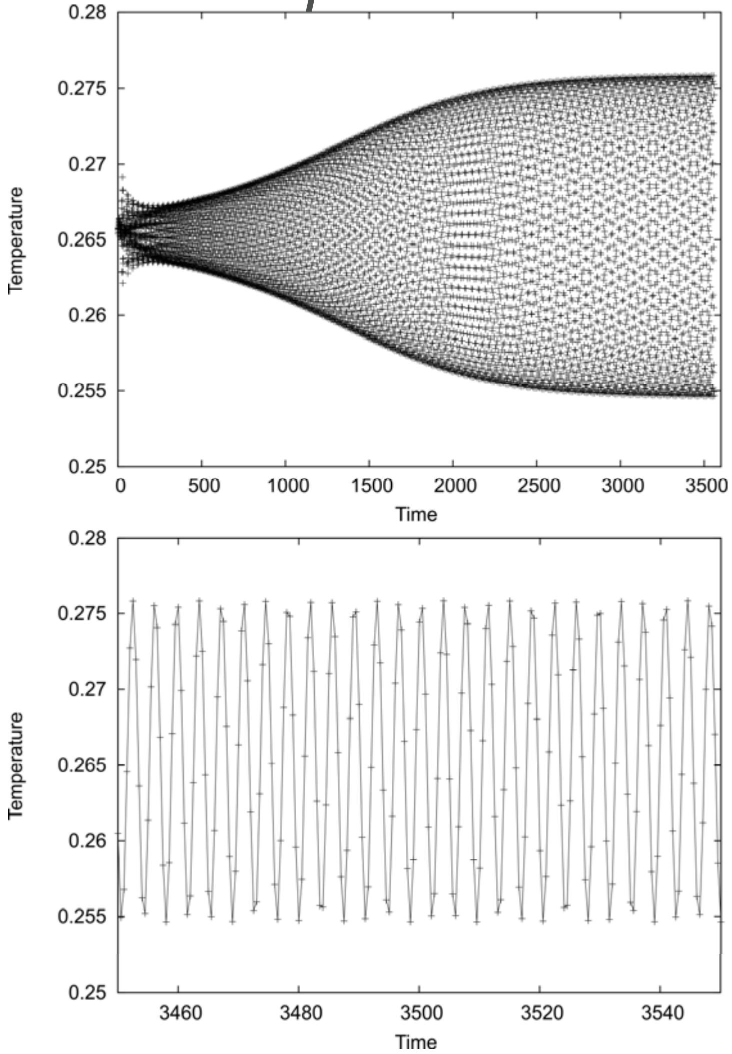


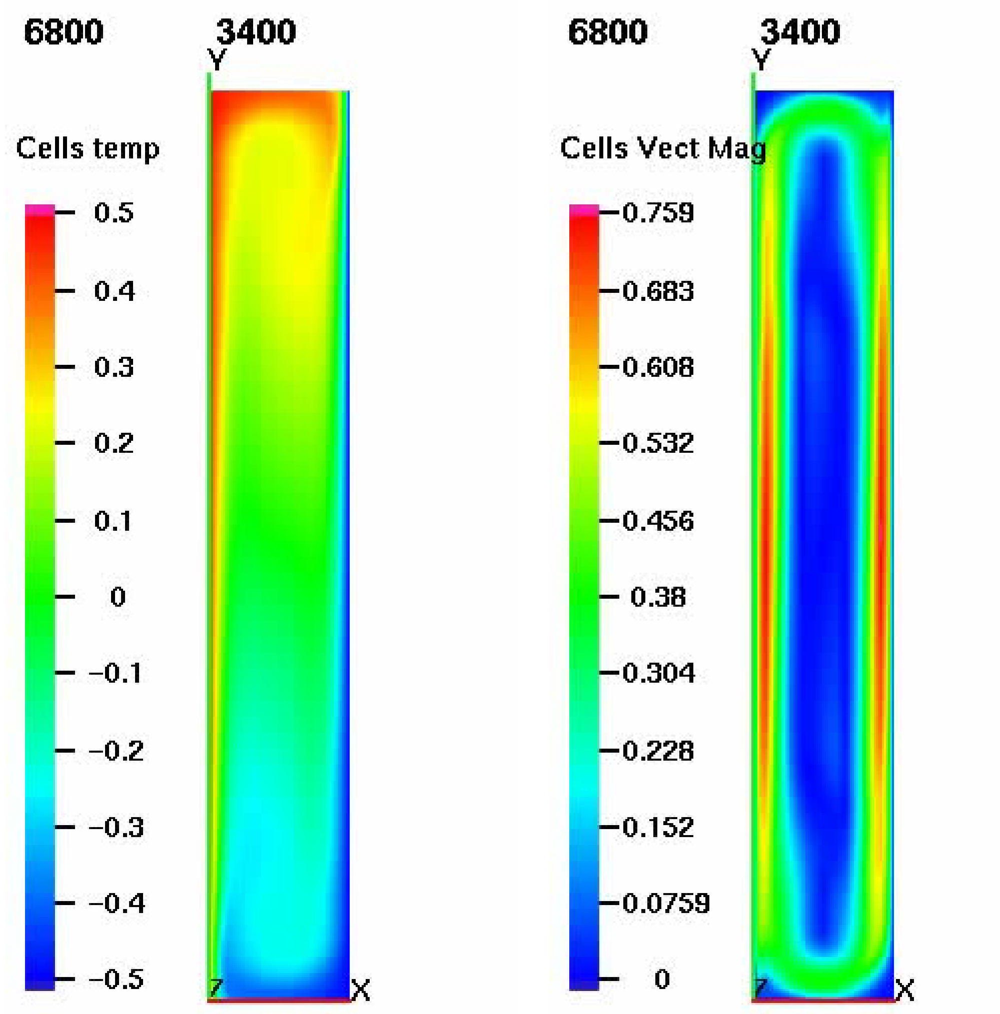




The amplitude is reduced

Case β=0.65:

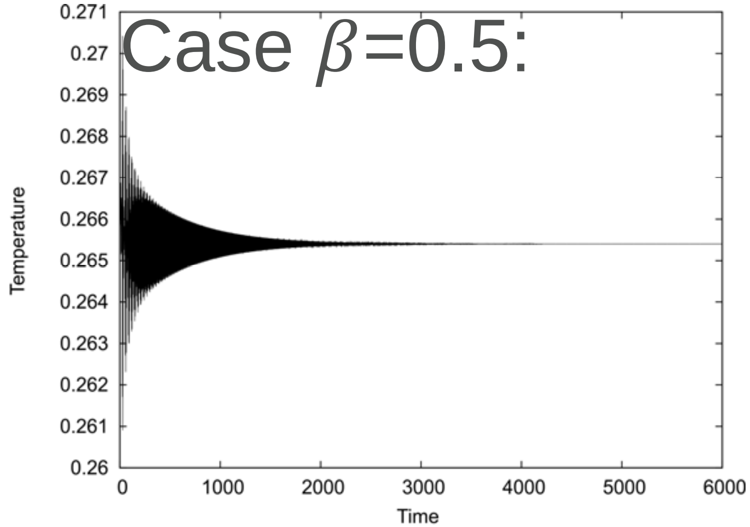


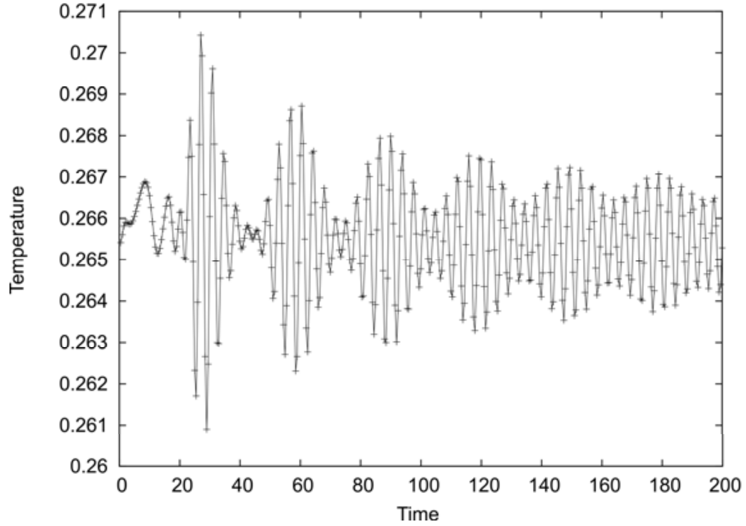


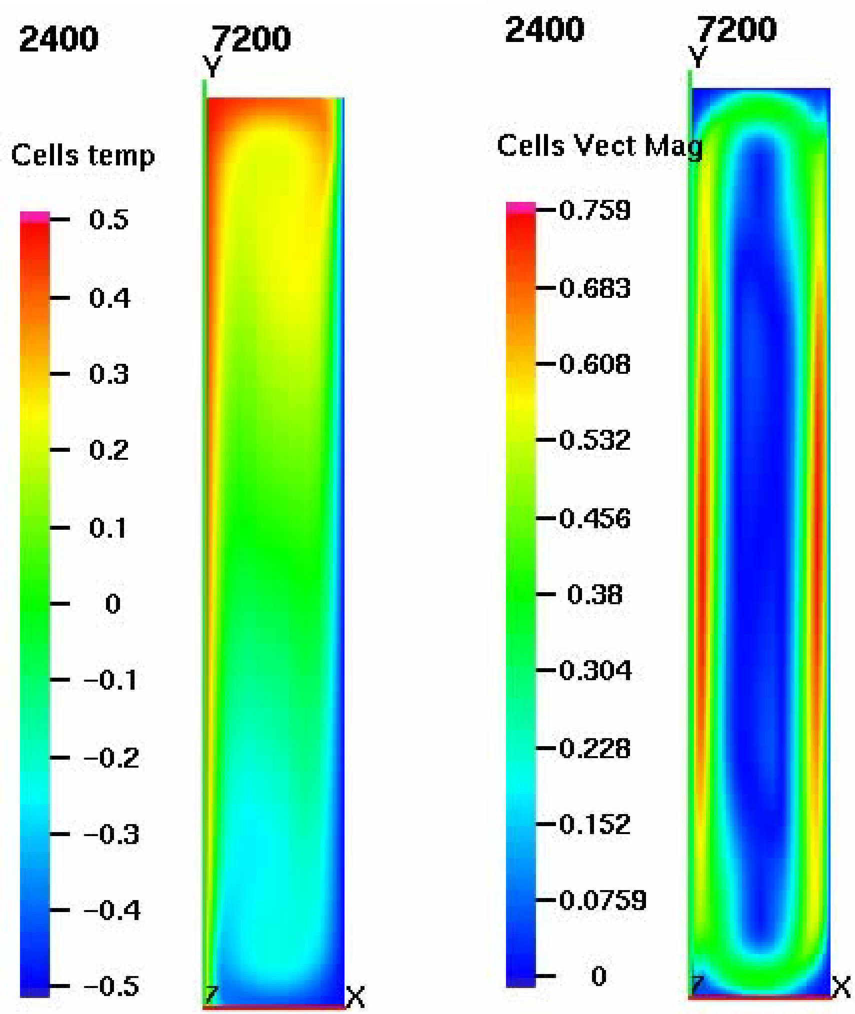
The amplitude is more reduced

technische universität dortmund

Case β=0.5:







The amplitude is completely damped

Case β <0.5:

β	Nonlinear steps	θ_1	Nu
0.5	4	0.2654041	4.668066
0.4	4	0.2654271	4.668867
0.3	5	0.2654530	4.669650
0.2	5	0.2654823	4.670387
0.1	6	0.2655137	4.671005
0	5	0.2655297	4.670980

No oscillation so that Steady data can be obtained! Here with moderate nonlinear steps

We have presented:

- MIT Benhmark with additional viscoelastic model
- Higher order FEM discretizations
- Black-box Newton-multigrid solver
- Numerical examples show that introducing viscoelasticity:
  - Creates more oscillation at the beginning of time iteration
  - In a longer time computation, oscillation goes periodically
  - The amplitude of the oscillation decreases as viscoelasticity increases
  - When viscoelasticity is dominant, the oscillation is completely damped

Future outlook: A comprehensive study on physicall results