A time-parallel Navier-Stokes solver using Augmented Lagrangian acceleration and space-time multigrid methods

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Motivation

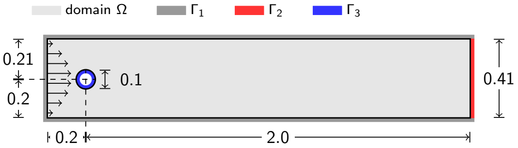
Incompressible Navier-Stokes equation

$$\begin{aligned} \partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla p = \boldsymbol{f} \quad & (x,t) \in \Omega \times (0,T) \\ \nabla\cdot\boldsymbol{u} = 0 \quad & (x,t) \in \Omega \times (0,T) \end{aligned} \tag{1}$$

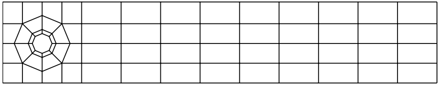
with $\boldsymbol{u}(x,t) \in \mathbb{R}^d$, $p(x,t) \in \mathbb{R}$.

- FEAT3:
  - FE software package with geometric multigrid solvers (monolithic and splitting approaches)
  - Spatial parallelization by domain decomposition (data parallel)
  - DNS for low and moderate Reynolds numbers
  - Non-Newtonian fluids
  - Only sequential time stepping
- This presentation:
  - Additional parallelism (in space or time) by solving multiple time steps at once
  - LSC preconditioner, Augmented Lagrangian
  - Space-time multigrid for velocity problems
  - Decoupled pressure problems

Test case: DFG-Benchmark 3



(a) Flow-around-cylinder domain



(b) Mesh on level 0; 64 elements

Figure: Test domain

- Instationary parabolic inflow, final time $T = 8$
- Parabolic inflow $\boldsymbol{v}(x,t) = v_{max} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(\frac{1}{8}\pi t)\frac{4x_2(0.41-x_2)}{0.41^2}$ on Γ_1
- Standard test case: $Re = 100$, $\nu = 10^{-3}$, $v_{max} = 1.5$
- Modified test case: $Re = 10$, $\nu = 10^{-2}$, $v_{max} = 1.5$
- Modified test case: $Re = 2$, $\nu = 10^{-2}$, $v_{max} = 0.3$

Benchmark 3 - Velocity fields

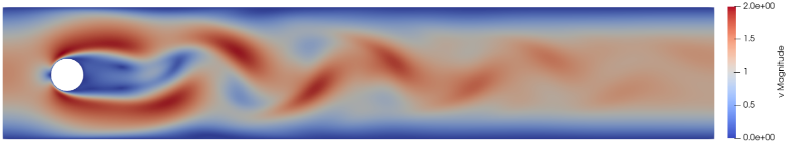


Figure: Bench 3, $\nu = 10^{-3}$, $v_{max} = 1.5$: velocity v at $t = 5$

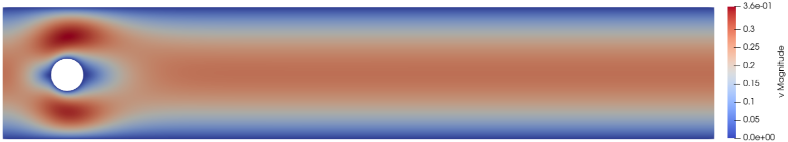


Figure: Bench 3, $\nu = 10^{-2}$, $v_{max} = 0.3$: velocity v at $t = 5$

Outline

1. Motivation
2. Time simultaneous Multigrid methods for linear parabolic problems
   - Multigrid waveform relaxation
   - Space-time multigrid for linear problems
3. Navier-Stokes solver
   - All-at-once LSC preconditioner
   - Augmented Lagrangian acceleration
4. Conclusion

Linear Multigrid methods

Linear parabolic evolution equation

$$
\begin{aligned}
\partial_t u(x,t) - \mathcal{L}u(x,t) &= f(x,t) & (x,t) &\in \Omega_T := \Omega \times (0,T) \\
u(x,t) &= g(x,t) & (x,t) &\in \partial\Omega \times (0,T) \\
u(x,0) &= u_0(x) & x &\in \Omega
\end{aligned}
$$

with $T > 0$, $\Omega \subset \mathbb{R}^d$ and an elliptic differential operator $\mathcal{L}(t)$.

'All-at-once' system with a linear one-step method:

$$
\underbrace{\begin{bmatrix}
\boldsymbol{A}_{i,1} & & & & \\
\boldsymbol{A}_{e,1} & \boldsymbol{A}_{i,2} & & & \\
 & \boldsymbol{A}_{e,2} & \boldsymbol{A}_{i,3} & & \\
 & & \ddots & \ddots & \\
 & & & \boldsymbol{A}_{e,K-1} & \boldsymbol{A}_{i,K}
\end{bmatrix}}_{=:\boldsymbol{A}_K \in \mathbb{R}^{NK \times NK}}
\underbrace{\begin{bmatrix}
\boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \vdots \\ \boldsymbol{u}_K
\end{bmatrix}}_{=:\boldsymbol{u} \in \mathbb{R}^{NK}}
=
\underbrace{\begin{bmatrix}
\boldsymbol{f}_1 - \boldsymbol{A}_{e,0}\boldsymbol{u}_0 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \\ \vdots \\ \boldsymbol{f}_K
\end{bmatrix}}_{=:\boldsymbol{f} \in \mathbb{R}^{NK}}
\tag{2}
$$

Multigrid waveform relaxation

- Multigrid waveform relaxation, Multigrid dynamic iteration [Lubich, Ostermann (1987)]
- Transfer operators (*semi coarsening in space*):
  - Standard coarsening for each time step
  - Prolongation:

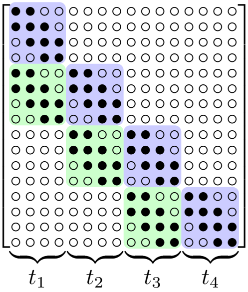
$$\boldsymbol{P}_{\tau,2h}^{\tau,h} = \boldsymbol{I}_K \otimes \boldsymbol{P}_{2h}^{h}$$

  - Restriction:

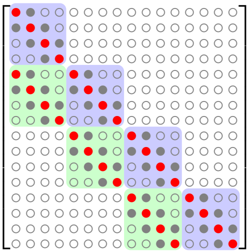
$$\boldsymbol{R}_{\tau,h}^{\tau,2h} = \boldsymbol{I}_K \otimes \boldsymbol{R}_{h}^{2h} = \left(\boldsymbol{P}_{\tau,2h}^{\tau,h}\right)^T$$

- Line smoothing in time direction (block preconditioned GMRES smoother)
  - Preconditionier $\boldsymbol{C}^{-1}$

$\boldsymbol{A} =$



$\boldsymbol{C} =$



STMG: Multigrid applied to the all-at-once system

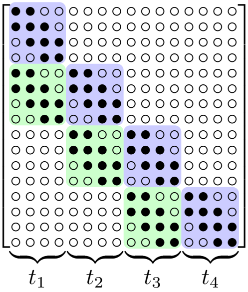
- Space-time multigrid [Gander, Neumüller (2016)]
- Transfer operators (*semi coarsening in time*):
  - Coarsening in time direction (restriction forward in time)

$$P_{2\tau,h}^{\tau,h} = P_{2\tau}^{\tau} \otimes I_N$$

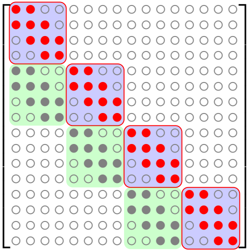
$$R_{\tau,h}^{2\tau,h} = R_{\tau}^{2\tau} \otimes I_N = \left(P_{2\tau,h}^{\tau,h}\right)^T$$

- Block-Jacobi smoother, $\omega = 0.5$ (/block preconditioned GMRES)
  - Preconditionier C^{-1}
  - Approximation of $\bar{C}^{-1}$ by a single V-cycle multigrid step in space
- particularly suited for DG(p) time discretizazion schemes

$A =$



$C =$



Linearized test problem

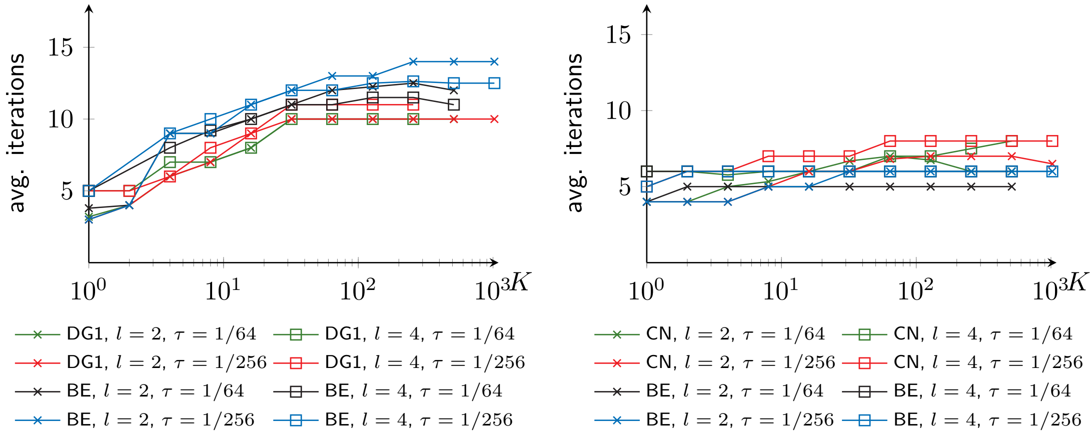
Convection-diffusion-reaction equation

$$\begin{aligned} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{v} &= f && (x,t) \in \Omega \times (0,T) \\ \boldsymbol{u}(x,t) &= \boldsymbol{g}_D && (x,t) \in \Gamma_D \times (0,T) \\ \partial_n \boldsymbol{u} &= \boldsymbol{g}_N && (x,t) \in \Gamma_N \times (0,T) \\ \boldsymbol{u}(x,0) &= \boldsymbol{u}_0(x) && x \in \Omega \end{aligned} \tag{3}$$

with $T > 0$, $\Omega \subset \mathbb{R}^d$ and $\Gamma_D \,\dot{\cup}\, \Gamma_N = \partial\Omega$.

- Convection-diffusion(-reaction) equation
- Reactive term from linearization by Newton's method
- Velocity field $\boldsymbol{v}$ is the solution of the non-linear Benchmark

Bench3 - Newton linearization - $\nu = 0.01$, $u_{max} = 0.3$

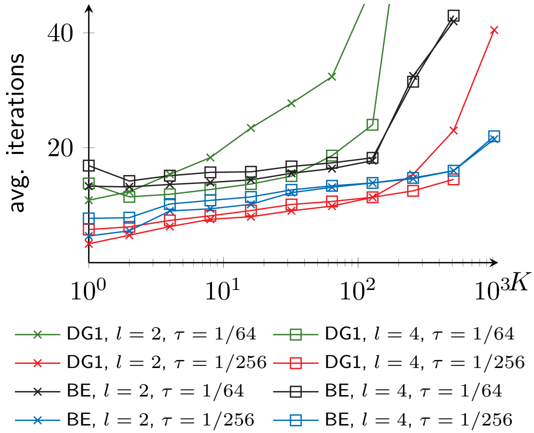


(a) STMG: 2+2 FGMRES smoothing steps, V-cycle

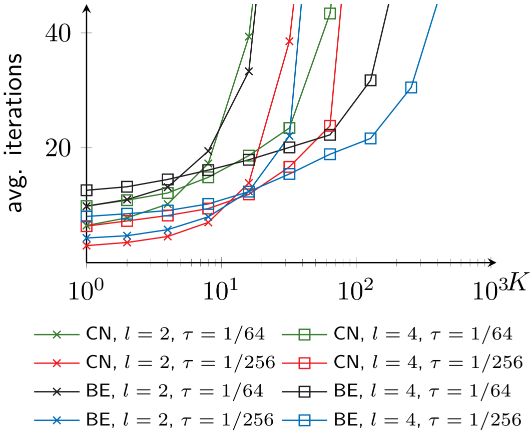
(b) WRMG: V-cycle, 4+4 FGMRES smoothing steps

Figure: Linearized test problem: number of iterations

Bench3 - Newton linearization - $\nu = 0.001$, $u_{max} = 1.5$



(a) STMG: F-cycle, 2+2 FGMRES smoothing steps



(b) WRMG: F-cycle, 4+4 FGMRES smoothing steps

Figure: Linearized test problem: number of iterations

WRMG: Strong scaling test

- Fritz@FAU (2x Intel Xeon Platinum 8360Y “Ice Lake” and 256GB DDR4 RAM per node, Blocking HDR 100 Infiniband interconnect)
- 2 FGMRES pre- and post-smoothing steps, F-cycle
- Level 5, 2048 total time steps
- Mesh with 48 coarse grid elements

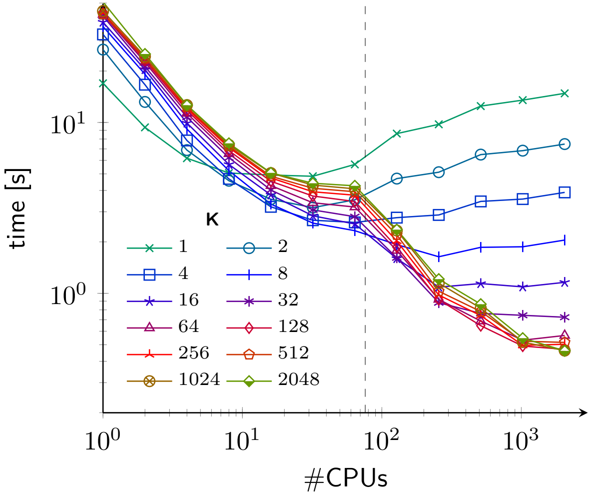


Figure: Solver time per iteration

Navier-stokes solver - Discretization

Incompressible Navier-Stokes equation

$$\begin{aligned} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla p = \boldsymbol{f} &\quad (x,t) \in \Omega \times (0,T) \\ \nabla \cdot \boldsymbol{u} = 0 &\quad (x,t) \in \Omega \times (0,T) \end{aligned}$$

with $\boldsymbol{u}(x,t) \in \mathbb{R}^d$, $p(x,t) \in \mathbb{R}$.

- Linearization by Netwon's method
- Space discretization: LBB-stable FE-pair Q2-P1(disc)
- Quadrature-based mass lumping
- For illustration: Backward Euler discretization in time

Linear system in each time step

$$\begin{aligned} \left(\boldsymbol{M} + \tau \boldsymbol{K} + \tau \boldsymbol{C}(v_k) + \tau \boldsymbol{R}(v_k)\right) \boldsymbol{u}_k + \tau \boldsymbol{B} \boldsymbol{p}_k &= \boldsymbol{f}_k + \boldsymbol{M} \boldsymbol{u}_{k-1} \\ \boldsymbol{B}^T \boldsymbol{u}_k &= \boldsymbol{g}_k \end{aligned} \qquad k = 1, \ldots, K$$

All at once system

$$\boldsymbol{A}_{i,m} = \boldsymbol{M} + \tau\boldsymbol{K} + \tau\boldsymbol{C}(v_m) + \tau\boldsymbol{R}(v_m) \qquad \boldsymbol{A}_{e,m} = -\boldsymbol{M}$$

$$\left(\begin{array}{ccccc|ccccc}
\boldsymbol{A}_{i,1} & & & & & \tau\boldsymbol{B} & & & & \\
\boldsymbol{A}_{e,1} & \boldsymbol{A}_{i,2} & & & & & \tau\boldsymbol{B} & & & \\
 & \boldsymbol{A}_{e,2} & \boldsymbol{A}_{i,3} & & & & & \tau\boldsymbol{B} & & \\
 & & \ddots & \ddots & & & & & \ddots & \\
 & & & \boldsymbol{A}_{e,K-1} & \boldsymbol{A}_{i,K} & & & & & \tau\boldsymbol{B} \\
\hline
\tau\boldsymbol{B}^T & & & & & & & & & \\
 & \tau\boldsymbol{B}^T & & & & & & & & \\
 & & \tau\boldsymbol{B}^T & & & & & & & \\
 & & & \tau\boldsymbol{B}^T & & & & & & \\
 & & & & \tau\boldsymbol{B}^T & & & & &
\end{array}\right)
\left(\begin{array}{c}
\boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \\ \vdots \\ \boldsymbol{u}_K \\ \hline \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \\ \vdots \\ \boldsymbol{p}_K
\end{array}\right)
=
\left(\begin{array}{c}
\boldsymbol{f}_1 - \boldsymbol{A}_{e,0}\boldsymbol{u}_0 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \\ \vdots \\ \boldsymbol{f}_K \\ \hline \tau\boldsymbol{g}_1 \\ \tau\boldsymbol{g}_2 \\ \tau\boldsymbol{g}_3 \\ \vdots \\ \tau\boldsymbol{g}_K
\end{array}\right)$$

$$\Leftrightarrow \quad \begin{bmatrix} \boldsymbol{A}_K & \tilde{\boldsymbol{B}}_K \\ \tilde{\boldsymbol{B}}_K^T & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \tilde{\boldsymbol{g}} \end{bmatrix}, \qquad \tilde{\boldsymbol{B}}_K := \boldsymbol{I}_K \otimes \tilde{\boldsymbol{B}} := \boldsymbol{I}_K \otimes \tau\boldsymbol{B}$$

- Similar structure for different time discretizations

Preconditioning

- Based on decomposition

$$\begin{bmatrix} \boldsymbol{A}_K & \tilde{\boldsymbol{B}}_K \\ \tilde{\boldsymbol{B}}_K^T & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_K & \boldsymbol{0} \\ \tilde{\boldsymbol{B}}_K^T & -\boldsymbol{S}_K \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_K & \boldsymbol{A}_K^{-1}\tilde{\boldsymbol{B}}_K \\ \boldsymbol{0} & \boldsymbol{I}_K \end{bmatrix} \quad \text{with} \quad \boldsymbol{S}_K = \tilde{\boldsymbol{B}}_K^T \boldsymbol{A}_K^{-1} \tilde{\boldsymbol{B}}_K$$

- Use FGMRES solver with block preconditioner $\boldsymbol{P}^{-1}$

$$\boldsymbol{P}^{-1} = \begin{bmatrix} \boldsymbol{A}_K & \boldsymbol{0} \\ \tilde{\boldsymbol{B}}_K^T & -\boldsymbol{C}_K \end{bmatrix}^{-1}$$

Preconditioning

- Based on decomposition

$$\begin{bmatrix} \boldsymbol{A}_K & \tilde{\boldsymbol{B}}_K \\ \tilde{\boldsymbol{B}}_K^T & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_K & \boldsymbol{0} \\ \tilde{\boldsymbol{B}}_K^T & -\boldsymbol{S}_K \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_K & \boldsymbol{A}_K^{-1}\tilde{\boldsymbol{B}}_K \\ \boldsymbol{0} & \boldsymbol{I}_K \end{bmatrix} \quad \text{with} \quad \boldsymbol{S}_K = \tilde{\boldsymbol{B}}_K^T \boldsymbol{A}_K^{-1} \tilde{\boldsymbol{B}}_K$$

- Use FGMRES solver with block preconditioner $\boldsymbol{P}^{-1}$

$$\boldsymbol{P}^{-1} = \begin{bmatrix} \boldsymbol{A}_K & \boldsymbol{0} \\ \tilde{\boldsymbol{B}}_K^T & -\boldsymbol{C}_K \end{bmatrix}^{-1}$$

- approximate Schur complement by least squares commutator (LSC) preconditioner

$$\boldsymbol{C}_K^{-1} := \tilde{\boldsymbol{D}}_{p,K}^{-1} \tilde{\boldsymbol{B}}_K^T \boldsymbol{M}_{u,K}^{-1} \tilde{\boldsymbol{A}}_K \boldsymbol{M}_{u,K}^{-1} \tilde{\boldsymbol{B}}_K^T \tilde{\boldsymbol{D}}_{p,K}^{-1} \approx \boldsymbol{S}_K^{-1}$$

- With Pressure Laplace matrix $\tilde{\boldsymbol{D}}_{p,K} = \tilde{\boldsymbol{B}}_K^T \boldsymbol{M}_{u,K} \tilde{\boldsymbol{B}}_K$ and $\boldsymbol{M}_{u,K} = \boldsymbol{I}_K \otimes \boldsymbol{M}_u$

$$\boldsymbol{C}_K^{-1} := \left(\boldsymbol{I}_K \otimes (\tilde{\boldsymbol{D}}_p^{-1} \tilde{\boldsymbol{B}}^T \boldsymbol{M}_u^{-1})\right) \tilde{\boldsymbol{A}}_K \left(\boldsymbol{I}_K \otimes (\boldsymbol{M}_u^{-1} \tilde{\boldsymbol{B}}^T \tilde{\boldsymbol{D}}_p^{-1})\right) \approx \boldsymbol{S}_K^{-1}$$

$\Rightarrow$ 2x K independent pressure Laplace solves in space, 1 space-time velocity problem

- $\boldsymbol{D}_p^{-1}$, $\boldsymbol{A}_K^{-1}$ are approximated by a few (1-4) multigrid steps

Results

- Modified DFG95-Benchmark 3: $Re = 10$
- Linear solver stagnates:

K	2	5	10	20	50	100	200	400	800
nonlinear iters.	1.97	1.99	2	2.23	2.69	4.88	7.75	-	-
linear iters.	25.2	34.5	54.9	78.3	132	396	699	-	-
linear per nonlin.	12.8	17.3	27.5	35.2	49.1	81.2	90.2	-	.

Table: Number of iterations (averaged over imte domain); level 3

- Stokes test case

K	2	5	10	20	50	100	200	400	800
$\nu = 0.01$	14.58	17.78	26.75	39.5	66.5	94.5	100	158	184
$\nu = 10$	77.96	80.9	76.1	84.2	89.75	105	118	126	132

Table: Number of linear iterations (averaged over imte domain); level 3

Augmented Lagrangian - single time step

$$\begin{bmatrix} \boldsymbol{A}_{i,k} & \tilde{\boldsymbol{B}} \\ \tilde{\boldsymbol{B}}^T & \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_k \\ \boldsymbol{p}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_k - \boldsymbol{A}_{e,0}\boldsymbol{u}_{k-1} \\ \tilde{\boldsymbol{g}}_k \end{bmatrix}$$

- Augmented Lagrangian (AL) modification [Benzi, Olshanskii (2006)]
- Modify the linear system

$$\begin{bmatrix} \boldsymbol{A}_{i,k} + \gamma\tilde{\boldsymbol{B}}\boldsymbol{W}\tilde{\boldsymbol{B}}^T & \tilde{\boldsymbol{B}} \\ \tilde{\boldsymbol{B}}^T & \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_k \\ \boldsymbol{p}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_k - \boldsymbol{A}_{e,k-1}\boldsymbol{u}_{k-1} + \gamma\tilde{\boldsymbol{B}}\boldsymbol{W}\tilde{\boldsymbol{g}}_k \\ \tilde{\boldsymbol{g}}_k \end{bmatrix} \quad \text{with} \quad \boldsymbol{W} = \frac{1}{\tau}\boldsymbol{M}_p^{-1}$$

- Notation: $\boldsymbol{A}_{i,\gamma} = \boldsymbol{A}_{i,k} + \gamma\tilde{\boldsymbol{B}}\boldsymbol{W}\tilde{\boldsymbol{B}}^T$
- Does not change the solution

Augmented Lagrangian - single time step

$$\begin{bmatrix} \boldsymbol{A}_{i,k} & \tilde{\boldsymbol{B}} \\ \tilde{\boldsymbol{B}}^T & \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_k \\ \boldsymbol{p}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_k - \boldsymbol{A}_{e,0}\boldsymbol{u}_{k-1} \\ \tilde{\boldsymbol{g}}_k \end{bmatrix}$$

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$$\begin{bmatrix} \boldsymbol{A}_{i,k} + \gamma\tilde{\boldsymbol{B}}\boldsymbol{W}\tilde{\boldsymbol{B}}^T & \tilde{\boldsymbol{B}} \\ \tilde{\boldsymbol{B}}^T & \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_k \\ \boldsymbol{p}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_k - \boldsymbol{A}_{e,k-1}\boldsymbol{u}_{k-1} + \gamma\tilde{\boldsymbol{B}}\boldsymbol{W}\tilde{\boldsymbol{g}}_k \\ \tilde{\boldsymbol{g}}_k \end{bmatrix} \quad \text{with} \quad \boldsymbol{W} = \frac{1}{\tau}\boldsymbol{M}_p^{-1}$$

- Notation: $\boldsymbol{A}_{i,\gamma} = \boldsymbol{A}_{i,k} + \gamma\tilde{\boldsymbol{B}}\boldsymbol{W}\tilde{\boldsymbol{B}}^T$
- Does not change the solution
- Sherman-Morrison-Woodbury identity yields:

$$\boldsymbol{S}_\gamma^{-1} = \tilde{\boldsymbol{B}}^T\boldsymbol{A}_{i,\gamma}\tilde{\boldsymbol{B}}^{-1} = \left(\tilde{\boldsymbol{B}}^T\boldsymbol{A}_i\tilde{\boldsymbol{B}}\right)^{-1} + \frac{\gamma}{\tau}\boldsymbol{M}_p^{-1}$$

- LSC preconditioner:

$$\boldsymbol{C}_\gamma^{-1} := \boldsymbol{C}^{-1} + \frac{\gamma}{\tau}\boldsymbol{M}_p^{-1} = \left(\tilde{\boldsymbol{D}}_p^{-1}\tilde{\boldsymbol{B}}^T\boldsymbol{M}_u^{-1}\right)\boldsymbol{A}_{i,\gamma}\left(\boldsymbol{M}_u^{-1}\tilde{\boldsymbol{B}}^T\tilde{\boldsymbol{D}}_p^{-1}\right) \approx \boldsymbol{S}_\gamma^{-1}$$

Augmented Lagrangian - all-at-once system

- Make the same modification to the time diagonal

$$\begin{bmatrix} \boldsymbol{A}_K + \gamma \tilde{\boldsymbol{B}}_K \boldsymbol{W}_K \tilde{\boldsymbol{B}}_K^T & \tilde{\boldsymbol{B}}_K \\ \tilde{\boldsymbol{B}}_K^T & \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} + \gamma \tilde{\boldsymbol{B}}_K \boldsymbol{W}_K \tilde{\boldsymbol{g}} \\ \tilde{\boldsymbol{g}} \end{bmatrix} \quad \text{with} \quad \boldsymbol{W}_K = \frac{1}{\tau} \boldsymbol{I}_K \otimes \boldsymbol{M}_p^{-1}$$

- Sherman-Morrison-Woodbury identity holds:

$$\boldsymbol{S}_{K,\gamma}^{-1} = \boldsymbol{S}_K^{-1} + \frac{\gamma}{\tau} \boldsymbol{I}_K \otimes \boldsymbol{M}_p^{-1}$$

- LSC preconditioner:

$$\boldsymbol{C}_{K,\gamma}^{-1} := \boldsymbol{C}_K^{-1} + \frac{\gamma}{\tau} \boldsymbol{I}_K \otimes \boldsymbol{M}_p^{-1} = \left(\boldsymbol{I}_K \otimes (\tilde{\boldsymbol{D}}_p^{-1} \tilde{\boldsymbol{B}}^T \boldsymbol{M}_u^{-1})\right) \boldsymbol{A}_{K,\gamma} \left(\boldsymbol{I}_K \otimes (\boldsymbol{M}_u^{-1} \tilde{\boldsymbol{B}}^T \tilde{\boldsymbol{D}}_p^{-1})\right) \approx \boldsymbol{S}_{K,\gamma}^{-1}$$

Velocity solver for Augmented Lagrangian ($Q_2 - P_1$)

- Discrete equation (single time step):

$$\boldsymbol{A}_{i,k}\boldsymbol{u}_k + \gamma\tau\,\boldsymbol{B}\boldsymbol{M}_p^{-1}\boldsymbol{B}^T\boldsymbol{u}_k = \boldsymbol{f}_k - \boldsymbol{A}_{e,k-1}\boldsymbol{u}_{k-1}$$

- $\boldsymbol{B}\boldsymbol{M}_p^{-1}\boldsymbol{B}^T$ is singular, system ill-conditioned for large γ
- Tailored multigrid methods necessary [Wechsung (2019)]
- Domain decomposition into local patches $\bigcup_i V_{h,i} = V_h$
- kernel of the AL operator:

$$\mathcal{N}_h = \mathrm{kern}(\boldsymbol{B}\boldsymbol{M}_p^{-1}\boldsymbol{B}^T) = \{v \in V_h : \nabla \cdot v = 0\}$$

- kernel capturing property:

$$\mathcal{N}_h = \bigcup_i (V_{h,i} \cap \mathcal{N}_h)$$

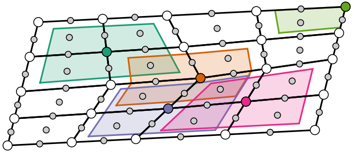


Figure: Decomposition of $V_h = \bigcup_i V_{h,i}$

Stationary AL multigrid solver

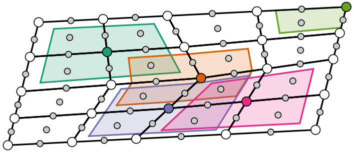


Figure: Smoother

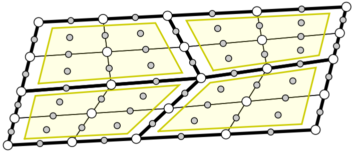


Figure: Prolongation

- Smoother
  - GMRES with overlapping additive schwarz preconditioner
  - Patches around each mesh node

$$\boldsymbol{C} = \sum_j \mathcal{I}_j \tilde{\boldsymbol{A}}_j^{-1} \mathcal{I}_j^T$$

- Transfers
  - Correction by solving local problems on each coarse cell

$$\boldsymbol{P}_\gamma = \left(\boldsymbol{I} - \left(\sum_r \mathcal{I}_r \tilde{\boldsymbol{A}}_r^{-1} \mathcal{I}_r^T \right) \gamma\tau \boldsymbol{B}\boldsymbol{M}_p^{-1}\boldsymbol{B}^T \right) \boldsymbol{P}$$

$$\boldsymbol{R}_\gamma = \boldsymbol{P}_\gamma^T$$

MG and AL

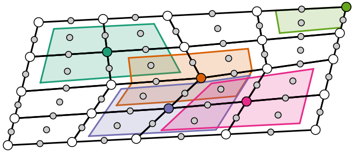


Figure: Smoother

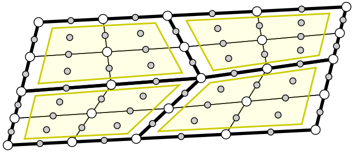
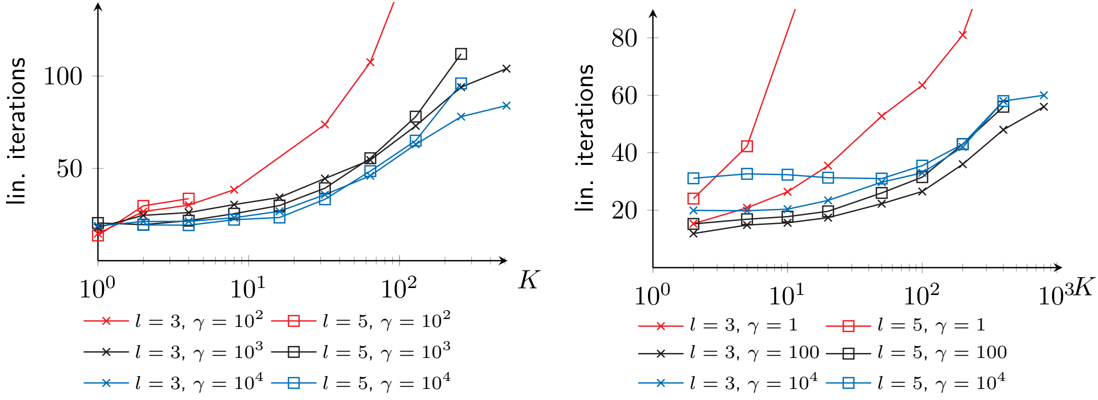


Figure: Prolongation

- STMG: apply specialized multigrid to each time step
  - Apply smoother to all stages in each inner spatial block
  - Larger local solves with increasing polynomial degree
  - Apply space transfers to each stage and time step
- WRMG:
  - apply specialized MG in a waveform relaxation manner
  - smoother: preconditioned GMRES
  - PC: for each patch solve the evolution equation
  - apply specialized transfers in each time step

Convergence results w.r.t γ, $Re = 10$

FGMRES(4), 1 velocity mg iteration, 4 pressure mg iterations



(a) STMG, $\tau = 1/64$, DG(0) / BE

(b) WRMG, $\tau = 0.01$, BDF2

Figure: Number of linear iterations: avg. over time domain

(Preliminary) scaling behavior of WRMG solver, $Re = 10$

- LiDO3 (2x Intel Xeon E5-2640v4 and 64GB memory per node, Infiniband QDR interconnect (40Gbps))
- 16 processes per node

- Level 4, CN, $\tau = 0.01$, 800 total time steps
- Unoptimized implementation
- Scaling bottleneck for large K: pressure coarse grid solver due to unsuited implementation

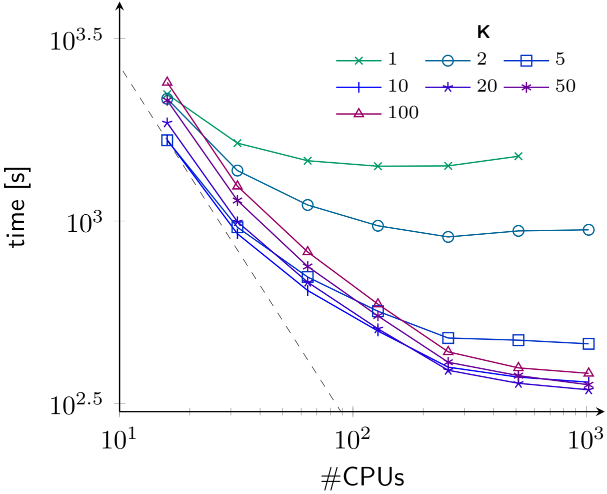


Figure: Total solver time

Convergence results w.r.t γ, $Re = 100$

- FGMRES(4), 1 velocity mg iteration, 4 pressure mg iterations
- Alpine: alternating between Picard and Newton
- Number of nonlinear iterations increases slightly
- Velocity solver convergence deteriorates

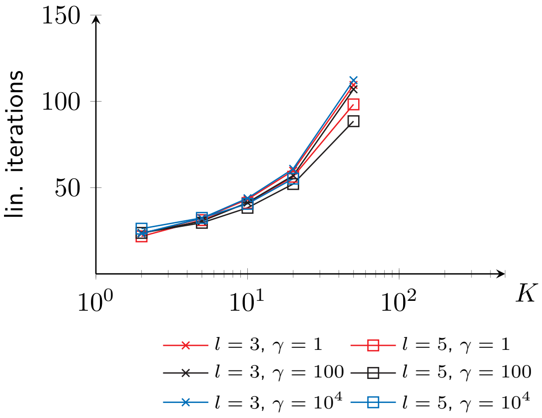


Figure: WRMG, $\tau = 0.01$, BDF2: Number of linear iterations, avg. over time domain

Conclusion

- LSC preconditioner with AL acceleration can lead to robust convergence behavior
- Low diffusion is problematic
- WRMG methods demonstrates speedup in a like-for-like comparison

Future research:

- Improve convergence behavior of STMG approach
- Time parallelization in the STMG method
- Efficient implementation of the pressure solver (multiple r.h.s)
- Block Krylov methods in the pressure solver

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