Monolithic Finite Element Method for Three Fields Formulation of Yield Stress Fluids

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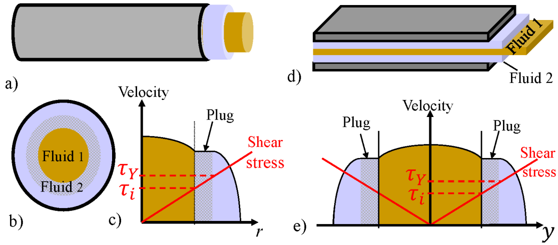
Viscoplastic Lubrication

(a) Viscoplastically lubricated flow (b) Cross-section of flow with Newtonian Ω_1 lubricated by Bingham fluid Ω_2 (c) Velocity and Stress profiles along radial section

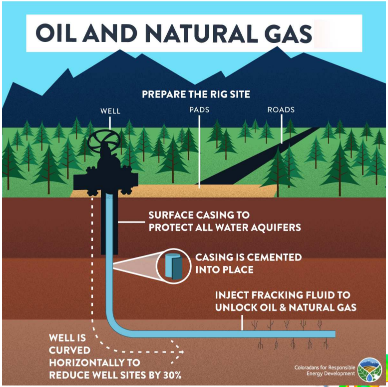


- Examples: Heavy crude oil transpotation along pipelines, coal-water slurry transpotation and co-extrusion operations

VPL Application

- Stabilization of interfaces in multi-layer flows by means of viscoplastic fluids
- Fluid encapsulation by entrapping one substance within another
- Oil/gas fracking, site-specific drug delivery, medical imaging, food, cosmetic, and pharmaceutical product manufacturing, ...

Process of fracking of oil/gas



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Classification of Fluids

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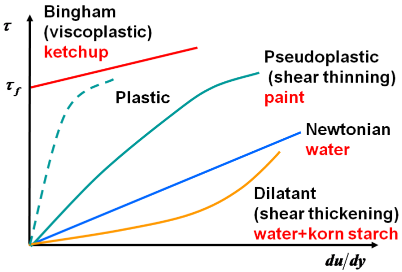
- Classification according to stress τ and deformation rate $\frac{d\boldsymbol{u}}{dy}$
- Linear relation $\rightarrow$ Newtonian
- Otherwise $\rightarrow$ Non-Newtonian



Bingham Constitutive Law

$$\begin{cases} \boldsymbol{\tau} = 2\eta \mathbf{D}(\boldsymbol{u}) + \tau_s \dfrac{\mathbf{D}(\boldsymbol{u})}{\|\mathbf{D}(\boldsymbol{u})\|} & \text{if } \|\mathbf{D}(\boldsymbol{u})\| \neq 0 \\ \|\boldsymbol{\tau}\| \leq \tau_s & \text{if } \|\mathbf{D}(\boldsymbol{u})\| = 0 \end{cases} \tag{1}$$

- Applied stress $\geq$ critical value of $\tau_s \rightarrow$ Shear region
- Applied stress $\leq$ critical value of $\tau_s \rightarrow$ Rigid or plug region

Primitive Variable Formulation

- Viscosity model for Bingham flow

$$\eta(\|\mathbf{D}(\boldsymbol{u})\|_{\epsilon}) = 2\eta + \tau_s \frac{1}{\|\mathbf{D}(\boldsymbol{u})\|_{\epsilon}} \tag{2}$$

<u>Generalized N-S equation</u>

$$\begin{cases} -\nabla \cdot \eta\left((\|\mathbf{D}(\boldsymbol{u})\|_{\epsilon})\mathbf{D}(\boldsymbol{u})\right) + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g}_D & \text{on } \Gamma_D \end{cases} \tag{3}$$

$$\text{Bercovier-Engelman} \rightarrow \|\mathbf{D}(\boldsymbol{u})\|_{\epsilon} = \sqrt{\mathbf{D} : \mathbf{D} + \epsilon^2} \tag{4}$$

Three Fields Formulation

- Bingham model with additional symmetric viscoplastic stress tensor

$$\boldsymbol{\sigma} = \frac{\mathbf{D}(\boldsymbol{u})}{\|\mathbf{D}(\boldsymbol{u})\|_{\epsilon}}$$

$$\begin{cases} \|\mathbf{D}(\boldsymbol{u})\|_{\epsilon}\,\boldsymbol{\sigma} - \mathbf{D}(\boldsymbol{u}) = 0 & \text{in } \Omega \\ -\nabla \cdot (2\eta \mathbf{D}(\boldsymbol{u}) + \tau_s \boldsymbol{\sigma}) + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g}_D & \text{on } \Gamma_D \end{cases} \tag{5}$$

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Newton Method

First, the system (5) is linearized using Newton method

- $\mathcal{U} = (\boldsymbol{\sigma}, \boldsymbol{u}, p)$ and $\mathcal{R}_{\mathcal{U}}$ denote the discrete residuals
- Nonlinear iteration, updated with the correction $\rightarrow \delta\mathcal{U}$,
  $\mathcal{U}^{n+1} = \mathcal{U}^n + \delta\mathcal{U}$
- Approximation for the residuals:$\rightarrow \mathcal{R}(\mathcal{U}^{n+1}) = \mathcal{R}(\mathcal{U}^n) + \left[\frac{\partial\mathcal{R}(\mathcal{U}^n)}{\partial\mathcal{U}}\right]\delta\mathcal{U}$

$$\begin{bmatrix} \boldsymbol{\sigma}^{n+1} \\ \boldsymbol{u}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\sigma}^{n} \\ \boldsymbol{u}^{n} \\ p^{n} \end{bmatrix} - \omega_n \begin{bmatrix} \frac{\partial R_{\boldsymbol{\sigma}}(\mathcal{U}^n)}{\partial\boldsymbol{\sigma}} & \frac{\partial R_{\boldsymbol{\sigma}}(\mathcal{U}^n)}{\partial\boldsymbol{u}} & \frac{\partial R_{\boldsymbol{\sigma}}(\mathcal{U}^n))}{\partial p} \\ \frac{\partial R_{\boldsymbol{u}}(\mathcal{U}^n)}{\partial\boldsymbol{\sigma}} & \frac{\partial R_{\boldsymbol{u}}(\mathcal{U}^n)}{\partial\boldsymbol{u}} & \frac{\partial R_{\boldsymbol{u}}(\mathcal{U}^n))}{\partial p} \\ \frac{\partial R_{p}(\mathcal{U}^n)}{\partial\boldsymbol{\sigma}} & \frac{\partial R_{p}(\mathcal{U}^n)}{\partial\boldsymbol{u}} & \frac{\partial R_{p}(\mathcal{U}^n)}{\partial p} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{R}_{\boldsymbol{\sigma}}(\mathcal{U}^n) \\ \mathcal{R}_{\boldsymbol{u}}(\mathcal{U}^n) \\ \mathcal{R}_{p}(\mathcal{U}^n) \end{bmatrix} \quad (6)$$

Newton Method

First, the system (3) is linearized using Newton method

- $\mathcal{U} = (\boldsymbol{u}, p)$ and $\mathcal{R}_{\mathcal{U}}$ denote the discrete residuals
- Nonlinear iteration, updated with the correction $\rightarrow \delta\mathcal{U}$, $\mathcal{U}^{n+1} = \mathcal{U}^n + \delta\mathcal{U}$
- Approximation for the residuals:$\rightarrow \mathcal{R}(\mathcal{U}^{n+1}) = \mathcal{R}(\mathcal{U}^n) + \left[\frac{\partial\mathcal{R}(\mathcal{U}^n)}{\partial\mathcal{U}}\right]\delta\mathcal{U}$

$$\begin{bmatrix} \boldsymbol{u}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{n} \\ p^{n} \end{bmatrix} - \omega_n \begin{bmatrix} \frac{\partial R_{\boldsymbol{u}}(\mathcal{U}^n)}{\partial \boldsymbol{u}} & \frac{\partial R_{\boldsymbol{u}}(\mathcal{U}^n))}{\partial p} \\ \frac{\partial R_{p}(\mathcal{U}^n)}{\partial \boldsymbol{u}} & \frac{\partial R_{p}(\mathcal{U}^n)}{\partial p} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{R}_{\boldsymbol{u}}(\mathcal{U}^n) \\ \mathcal{R}_{p}(\mathcal{U}^n) \end{bmatrix} \tag{7}$$

Jacobian Calculation

- In primitive variable

$$\left[\frac{\partial \mathcal{R}_{\boldsymbol{u}}(\mathcal{U}^n)}{\partial \boldsymbol{u}}\right] \boldsymbol{v} = -\nabla \cdot \left[2\eta D(\boldsymbol{v}) - 2\left[\frac{D(\boldsymbol{u}^n)}{\|\mathbf{D}(\boldsymbol{u}^n)\|_\epsilon} : D(\boldsymbol{v})\right] \frac{D(\boldsymbol{u}^n)}{\|\mathbf{D}(\boldsymbol{u}^n)\|_\epsilon}\right]$$

$$\left[\frac{\partial \mathcal{R}_{\boldsymbol{u}}(\mathcal{U}^n)}{\partial p}\right] q = \nabla q \tag{8}$$

$$\left[\frac{\partial \mathcal{R}_p(\mathcal{U}^n)}{\partial \boldsymbol{u}}\right] \boldsymbol{v} = -\nabla \cdot \boldsymbol{v}$$

$$\left[\frac{\partial \mathcal{R}_p(\mathcal{U}^n)}{\partial p}\right] q = 0 \tag{9}$$

Jacobian Calculation

- In three fields formulation

$$
\begin{aligned}
\left[\frac{\partial \mathcal{R}_{\boldsymbol{\sigma}}(\mathcal{U}^n)}{\partial \boldsymbol{\sigma}}\right] \boldsymbol{\tau} &= \|(\mathbf{D}(\boldsymbol{u}^n))\|_{\epsilon}\, \boldsymbol{\tau} \\
\left[\frac{\partial \mathcal{R}_{\boldsymbol{\sigma}}(\mathcal{U}^n)}{\partial \boldsymbol{u}}\right] \boldsymbol{v} &= \left(\boldsymbol{\sigma}^n : \mathbf{D}(\boldsymbol{v})\right) \boldsymbol{\sigma}^n - \mathbf{D}(\boldsymbol{v}) \\
\left[\frac{\partial \mathcal{R}_{\boldsymbol{\sigma}}(\mathcal{U}^n)}{\partial p}\right] q &= 0
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\left[\frac{\partial \mathcal{R}_{\boldsymbol{u}}(\mathcal{U}^n)}{\partial \boldsymbol{\sigma}}\right] \boldsymbol{\tau} &= -\tau_s \nabla \cdot \boldsymbol{\tau} \\
\left[\frac{\partial \mathcal{R}_{\boldsymbol{u}}(\mathcal{U}^n)}{\partial \boldsymbol{u}}\right] \boldsymbol{v} &= -2\eta \nabla \cdot \mathbf{D}(\boldsymbol{v}) \\
\left[\frac{\partial \mathcal{R}_{\boldsymbol{u}}(\mathcal{U}^n)}{\partial p}\right] q &= \nabla q
\end{aligned}
\tag{11}
$$

Jacobian Calculation

- In three fields formulation

$$
\begin{aligned}
\left[\frac{\partial \mathcal{R}_p(\mathcal{U}^n)}{\partial \boldsymbol{\sigma}}\right] \boldsymbol{\tau} &= 0 \\
\left[\frac{\partial \mathcal{R}_p(\mathcal{U}^n)}{\partial \boldsymbol{u}}\right] \boldsymbol{v} &= -\nabla \cdot \boldsymbol{v} \\
\left[\frac{\partial \mathcal{R}_p(\mathcal{U}^n)}{\partial p}\right] q &= 0
\end{aligned}
\tag{12}
$$

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Variational Formulation

- $\mathbb{V} = \boldsymbol{H}_0^1(\Omega) := \left(H_0^1(\Omega)\right)^2 \rightarrow$ velocity and dual space $\rightarrow \mathbb{V}'$
- $\mathbb{Q} = L_0^2(\Omega) \rightarrow$ pressure and dual space $\rightarrow \mathbb{Q}'$
- $\mathbb{M} = \left(L^2(\Omega)\right)^4_{\text{sym}} \rightarrow$ stress and dual space $\rightarrow \mathbb{M}'$

$$\langle \mathcal{A}_1 \boldsymbol{u}, \boldsymbol{v} \rangle := 2\eta \int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})\, dx \tag{13}$$

$$\langle \mathcal{A}_2 \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle = \tau_s \|(\mathbf{D}(\boldsymbol{u}^n)\|_\epsilon \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\, dx \tag{14}$$

Variational Formulation

- The associated bilinear forms

$$a_1(\boldsymbol{u},\boldsymbol{v}) = \langle \mathcal{A}_1 \boldsymbol{u},\boldsymbol{v}\rangle, \quad a_2(\boldsymbol{\sigma},\boldsymbol{\tau}) = \langle \mathcal{A}_2 \boldsymbol{\sigma},\boldsymbol{\tau}\rangle \tag{15}$$

$$\langle \mathcal{B}_1 \boldsymbol{v}, q\rangle := -\int_\Omega \nabla\cdot\boldsymbol{v}\; q\, dx \tag{16}$$

$$\langle \mathcal{B}_2 \boldsymbol{v},\boldsymbol{\tau}\rangle := \tau_s \int_\Omega \boldsymbol{\tau} : \mathbf{D}(\boldsymbol{v})\, dx$$

$$\langle \tilde{\mathcal{B}}_2 \boldsymbol{v},\boldsymbol{\tau}\rangle := \tau_s \int_\Omega \left(\boldsymbol{\sigma}^n : \mathbf{D}(\boldsymbol{v})\right)\left(\boldsymbol{\sigma}^n : \boldsymbol{\tau}\right) dx \tag{17}$$

Variational Formulation

- The associated bilinear forms

$$b_1(\boldsymbol{v}, q) := \left\langle \mathcal{B}_1 \boldsymbol{v}, q \right\rangle, \quad c_2(\boldsymbol{v}, \boldsymbol{\tau}) := \left\langle \mathcal{C}_2 \boldsymbol{v}, \boldsymbol{\tau} \right\rangle \tag{18}$$

$$b_2(\boldsymbol{v}, \boldsymbol{\tau}) := \left\langle \mathcal{B}_2 \boldsymbol{v}, \boldsymbol{\tau} \right\rangle, \quad \tilde{b}_2(\boldsymbol{v}, q) := \left\langle \tilde{\mathcal{B}}_2 \boldsymbol{v}, \boldsymbol{\tau} \right\rangle \tag{19}$$

$$\mathcal{C}_2 = \mathcal{B}_2 + \tilde{\mathcal{B}}_2 \tag{20}$$

- Newton iteration (6) becomes:

$$\begin{bmatrix} \boldsymbol{u}^{n+1} \\ \boldsymbol{\sigma}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^{n} \\ \boldsymbol{\sigma}^{n} \\ p^{n} \end{bmatrix} - \omega_n \begin{bmatrix} \mathcal{A}_1 & \mathcal{C}_2^{\mathsf{T}} & \mathcal{B}_1^{\mathsf{T}} \\ \mathcal{B}_2 & -\mathcal{A}_2 & \mathbf{0} \\ \mathcal{B}_1 & \mathbf{0} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{R}_{\boldsymbol{u}}(\boldsymbol{\sigma}^n, \boldsymbol{u}^n, p^n) \\ \mathcal{R}_{\boldsymbol{\sigma}}(\boldsymbol{\sigma}^n, \boldsymbol{u}^n, p^n) \\ \mathcal{R}_{p}(\boldsymbol{\sigma}^n, \boldsymbol{u}^n, p^n) \end{bmatrix} \tag{21}$$

Saddle Point Problem

$$\langle \mathcal{A}(\boldsymbol{u},\boldsymbol{\sigma}),(\boldsymbol{v},\boldsymbol{\tau})\rangle = \langle \mathcal{A}_1\boldsymbol{u},\boldsymbol{v}\rangle + \langle \mathcal{A}_2\boldsymbol{\sigma},\boldsymbol{\tau}\rangle + \langle \mathcal{C}_2\boldsymbol{v},\boldsymbol{\sigma}\rangle - \langle \mathcal{B}_2\boldsymbol{u},\boldsymbol{\tau}\rangle \quad (22)$$

$$a(\mathcal{U},\mathcal{V}) = a_1(\boldsymbol{u},\boldsymbol{v}) + a_2(\boldsymbol{\sigma},\boldsymbol{\tau}) + c_2(\boldsymbol{v},\boldsymbol{\sigma}) - b_2(\boldsymbol{u},\boldsymbol{\tau}) \quad (23)$$

$$\langle \mathcal{B}\mathcal{V}, q\rangle := \langle \mathcal{B}_1\boldsymbol{v}, q\rangle \quad (24)$$

- The Jacobian has a saddle point structure

$$\mathbf{J} = \begin{bmatrix} \mathcal{A} & \mathcal{B}^{\mathsf{T}} \\ \mathcal{B} & 0 \end{bmatrix} \quad (25)$$

- Find $\mathcal{U} \in \text{Ker}\mathcal{B}$ such that:

$$a(\mathcal{U}, \mathcal{V}) = \langle \boldsymbol{f}, \mathcal{V} \rangle \qquad \forall \mathcal{V} \in \text{Ker}\mathcal{B} \tag{26}$$

Theorem

Let $\mathbb{X} = \mathbb{V} \times \mathbb{M}$ be a Hilbert space and $\boldsymbol{f} \in \mathbb{X}'$, topological dual space of $\mathbb{X}$, and let $a(.,.)$ be a bilinear form on $\mathbb{X}$ satisfying the following three hypothesis:

(H1) There exists a constant $\alpha > 0$ such that :

$$a(\mathcal{U}, \mathcal{V}) \leq \alpha \, \|\mathcal{U}\| \, \|\mathcal{V}\| \qquad \forall \mathcal{U}, \mathcal{V} \in \mathbb{X} \tag{27}$$

Solvability of Problem

Theorem (cont...)

(H2) There exists a constant $\beta > 0$ *such that :*

$$\sup_{\mathcal{U}\in\mathbb{X}} \frac{a(\mathcal{U},\mathcal{V})}{\|\mathcal{U}\|} \geq \beta \|\mathcal{V}\| \qquad \forall \mathcal{V} \in \mathbb{X} \tag{28}$$

(H3) There exists a constant $\beta' > 0$ *such that :*

$$\sup_{\mathcal{V}\in\mathbb{X}} \frac{a(\mathcal{U},\mathcal{V})}{\|\mathcal{V}\|} \geq \beta' \|\mathcal{U}\| \qquad \forall \mathcal{U} \in \mathbb{X} \tag{29}$$

then problem has a unique solution $\mathcal{U} \in \mathbb{X}$ *such that* $\|\mathcal{U}\| \leq \frac{1}{\beta'} \|\boldsymbol{f}\|_{\mathbb{X}'}$

J. BARANGER,D. S ANDRI,"A formulation of Stokes problem and the linear elasticity equations suggested by the Oldroyd model for viscoelastic flow"Mathematical Modeling and Numerical Aanalysis

Solution of the problem

$$a(\mathcal{U},\mathcal{V}) = a_1(\boldsymbol{u},\boldsymbol{v}) + a_2(\boldsymbol{\sigma},\boldsymbol{\tau}) + c_2(\boldsymbol{v},\boldsymbol{\sigma}) - b_2(\boldsymbol{u},\boldsymbol{\tau})$$

$$\tilde{b}_2(\boldsymbol{v},\boldsymbol{\tau}) := \tau_s \int_\Omega \Big(\boldsymbol{\sigma}^n : \mathbf{D}(\boldsymbol{v})\Big)\Big(\boldsymbol{\sigma}^n : \boldsymbol{\tau}\Big)\, dx$$

With the extension of $\|.\|_{L^2}$ using the second order symmetric tensor $\boldsymbol{\sigma}^n$

$$\tilde{b}_2(\boldsymbol{v},\boldsymbol{\tau}) \equiv \tau_s \|\boldsymbol{\sigma}^n\|^2 \int_\Omega \Big(\mathbf{D}(\boldsymbol{v}) : \boldsymbol{\tau}\Big)$$

$$\sup_{\boldsymbol{\tau}\in\mathcal{M}} \frac{\tilde{b}_2(\boldsymbol{v},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|} \geq \beta \|\boldsymbol{\sigma}^n\|^2 \|\boldsymbol{v}\| \quad \forall \boldsymbol{v} \in \mathcal{V}$$

and

$$\sup_{\boldsymbol{v}\in\mathcal{V}} \frac{\tilde{b}_2(\boldsymbol{v},\boldsymbol{\tau})}{\|\boldsymbol{v}\|} \geq \beta \|\boldsymbol{\sigma}^n\|^2 \|\boldsymbol{\tau}\| \quad \forall \boldsymbol{\tau} \in \mathcal{M}$$

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Finite Element Discretization

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- Domain $\Omega \subset \mathbb{R}^d \longrightarrow$ *grid* $\mathcal{T}_h$ consisting of elements $K \in \mathcal{T}_h$
- Approximation spaces

$$\begin{aligned} \mathbb{V}^h &= \left\{ \boldsymbol{v}_h \in \mathbb{V}, \boldsymbol{v}_{h|K} \in Q_2(K) \right\} \\ \mathbb{M}^h &= \left\{ \boldsymbol{\tau}_h \in \mathbb{M}, \boldsymbol{\sigma}_{h|K} \in Q_2(K) \right\} \\ \mathbb{Q}^h &= \left\{ q_h \in \mathbb{Q}, q_{h|K} \in P_1^{\text{disc}}(K) \right\} \end{aligned} \tag{30}$$

- $\mathbb{X}^h = \mathbb{V}^h \times \mathbb{M}^h$. Find $(\mathcal{U}_h, p_h) \in \mathbb{X}^h \times \mathbb{Q}^h$ such that:

$$\begin{cases} a(\mathcal{U}_h, \mathcal{V}_h) + b(\mathcal{V}_h, p_h) = \langle \boldsymbol{f}, \mathcal{V}_h \rangle & \forall \mathcal{V}_h \in \mathbb{X}^h \\ b(\mathcal{U}_h, q_h) = 0 & \forall q_h \in \mathbb{Q}^h \end{cases} \tag{31}$$

- $\mathbb{V}^h$ and $\mathbb{Q}^h$ satisfy the inf-sup condition

Discrete saddle point problem

- $\mathbb{V}^h$ and $\mathbb{M}^h$ do not satisfy the inf-sup condition
- Find $\mathcal{U}_h \in \mathrm{Ker}\mathcal{B}_h$ such that:

$$a(\mathcal{U}_h, \mathcal{V}_h) + j(\mathcal{U}_h, \mathcal{V}_h) = \langle \boldsymbol{f}, \mathcal{V}_h \rangle \qquad \forall \mathcal{V}_h \in \mathrm{Ker}\mathcal{B}_h \tag{32}$$

$$j(\mathcal{U}_h, \mathcal{V}_h) = \gamma \sum_{e \in \mathcal{E}_h} h\tau_s \left(1 + \|\boldsymbol{\sigma}_h^n\|^2\right) \int_e [\nabla \boldsymbol{u}_h] : [\nabla \boldsymbol{v}_h] \, d\Omega \tag{33}$$

$$|\!|\!|\mathcal{V}_h|\!|\!|^2 = \|\mathcal{V}_h\|^2 + j(\mathcal{V}_h, \mathcal{V}_h) \tag{34}$$

Solvability of Problem

Theorem

Let $\mathbb{X}^h = \mathbb{V}^h \times \mathbb{M}^h$ be a Hilbert space and $f_h \in \mathbb{X}^{h\prime}$, topological dual space of $\mathbb{X}^h$, and let $a(.,.)$ be a bilinear form on $\mathbb{X}^h$ satisfying the following three hypothesis:

(H1) There exists a constant $\alpha > 0$ such that :

$$a(\mathcal{U}_h, \mathcal{V}_h) \leq \alpha \, \|\mathcal{U}_h\| \, \|\mathcal{V}_h\| \qquad \forall \mathcal{U}_h, \mathcal{V}_h \in \mathbb{X}^h \tag{35}$$

Solvability of Problem

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Theorem

(H2) There exists a constant $\beta > 0$ *such that :*

$$\sup_{\mathcal{U}_h \in \mathbb{X}^h} \frac{a(\mathcal{U}_h, \mathcal{V}_h)}{\|\mathcal{U}_h\|} \geq \beta \|\mathcal{V}_h\| \qquad \forall \mathcal{V}_h \in \mathbb{X}^h \tag{36}$$

(H3) There exists a constant $\beta' > 0$ *such that :*

$$\sup_{\mathcal{V}_h \in \mathbb{X}^h} \frac{a(\mathcal{U}_h, \mathcal{V}_h)}{\|\mathcal{V}_h\|} \geq \beta' \|\mathcal{U}_h\| \qquad \forall \mathcal{U}_h \in \mathbb{X}^h \tag{37}$$

then problem has a unique solution $\mathcal{U}_h \in \mathbb{X}^h$ *such that* $\|\mathcal{U}_h\| \leq \frac{1}{\beta'} \|f_h\|_{\mathbb{X}^{h\prime}}$

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Bingham fluid flow in channel

- Channel domain: two parallel plates with h length apart and long
- Dirichlet boundary conditions
- $u_2 = 0$, $p = -x$, $\eta = 1$, $\boldsymbol{f} = 0$ and yield stress τ_s
- Analytical solution for velocity

$$u_1 = \begin{cases} \frac{1}{8}\left[(h-2\tau_s)^2 - (h-2\tau_s-2y)^2\right], & 0 \le y < \frac{h}{2} - \tau_s \\ \frac{1}{8}(h-2\tau_s)^2, & \frac{h}{2} - \tau_s \le y \le \frac{h}{2} + \tau_s, \\ \frac{1}{8}\left[(h-2\tau_s)^2 - (2y-2\tau_s-h)^2\right], & \frac{h}{2} + \tau_s < y \le h \end{cases} \quad (38)$$

Numerical Results

Table: Number of Newton iterations for fitted mesh at yield stress $\tau_s = 0.25$

Level	ϵ=10^{-1}	ϵ=10^{-2}	ϵ=10^{-3}	ϵ=10^{-4}	ϵ=10^{-5}	ϵ=0
3	6	45	14	49	39	18
4	3	4	6	5	13	4
5	2	3	4	4	5	3

Numerical Results

Table: L^2 **errors of velocity for fitted mesh:** $\|\boldsymbol{u} - \boldsymbol{u}_{ex}\|$ at yield stress $\tau_s = 0.25$

Level	ϵ=10^{-1}	ϵ=10^{-2}	ϵ=10^{-3}
3	2.598×10^{-3}	5.873×10^{-4}	6.257×10^{-5}
4	2.597×10^{-3}	5.818×10^{-4}	6.415×10^{-5}
5	2.597×10^{-3}	5.815×10^{-4}	6.416×10^{-5}
	ϵ=10^{-4}	ϵ=10^{-5}	ϵ=0
3	6.407×10^{-6}	6.788×10^{-7}	2×10^{-11}
4	6.262×10^{-6}	6.378×10^{-7}	7×10^{-12}
5	6.298×10^{-6}	6.297×10^{-7}	4×10^{-12}

Numerical Results

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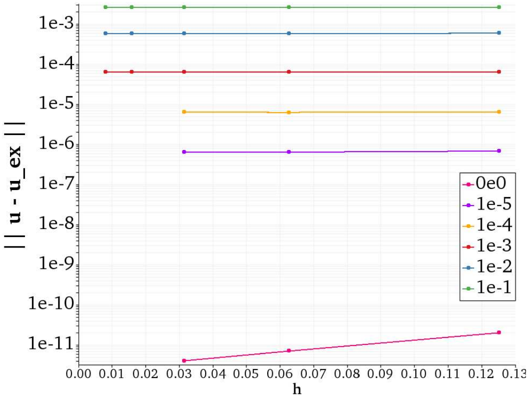

Figure: $\|\boldsymbol{u} - \boldsymbol{u}_{ex}\|$ at yield stress i.e. $\tau_s = 0.25$ with step size h

Table: Number of Newton Iterations for fitted mesh at different values of yield stress i.e. $\tau_s = 0.125, 0.25, 0.75$ and 0.875

Level	τ_s=0.125	τ_s=0.25	τ_s=0.75	τ_s=0.875
3	3	5	4	11
4	4	3	2	5
5	5	2	2	4

Table: L^2 **errors of velocity for fitted mesh:** $\|\boldsymbol{u} - \boldsymbol{u}_{ex}\|$ at different values of yield stress i.e. $\tau_s = 0.125, 0.25, 0.75$ and 0.875

Level	τ_s=0.125	τ_s=0.25	τ_s=0.75	τ_s=0.875
3	4.0257×10^{-3}	6.5687×10^{-4}	3.3499×10^{-4}	1.5016×10^{-3}
4	2.0836×10^{-3}	1.4250×10^{-4}	1.5318×10^{-4}	4.6894×10^{-5}
5	1.9920×10^{-3}	3.6533×10^{-5}	4.6423×10^{-5}	2.4105×10^{-5}

Numerical Results

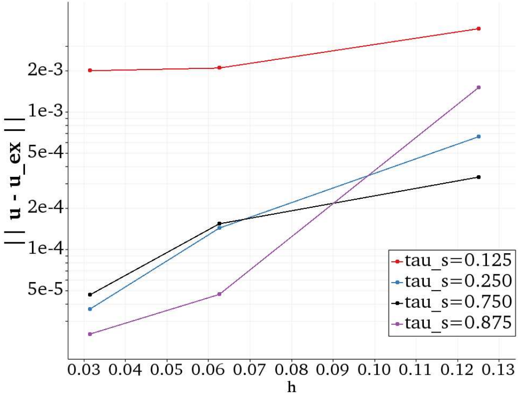

Figure: $\|\boldsymbol{u} - \boldsymbol{u}_{ex}\|$ at different values of yield stress i.e. $\tau_s = 0.125, 0.25, 0.75$ and 0.875 with step size h

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A new regularization free solver for yield stress fluid is under development

- By introducing a new auxiliary stress in three fields formulation
- Resulting saddle-point problem with monolithic finite element method

to simulate viscoplastic lubricated flows for stabilization of the interfaces in multi-layer shear flows

Advantages

- Solves efficiently and accurately
- Method does not effect the shape of the yield surfaces
- The formulation does not need any regularization