Numerical analysis and a-posteriori error control for a new nonconforming quadrilateral linear finite element

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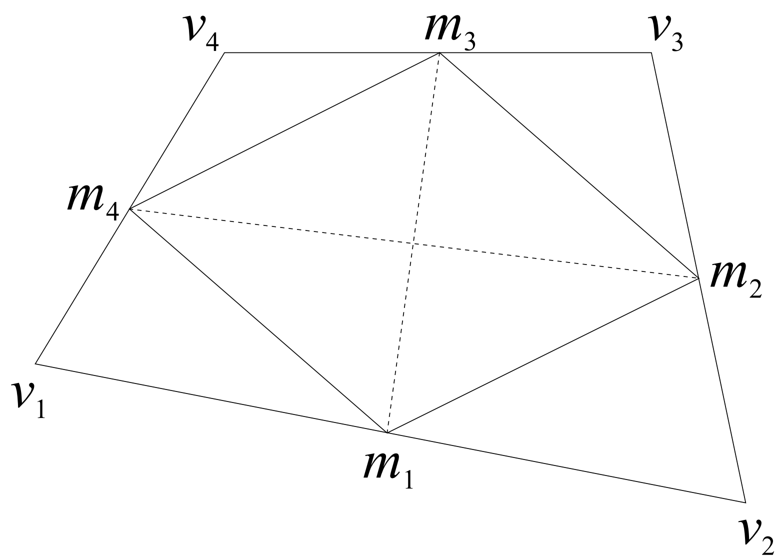
structure of the talk

1. Introduction of the $\tilde{P}_1$-element
2. Numerical analysis of $\tilde{P}_1$
3. Dual-weighted a-posteriori error-control

basic observations

By connecting the midpoints of an arbitrary quadrilateral, one gets a parallelogram.



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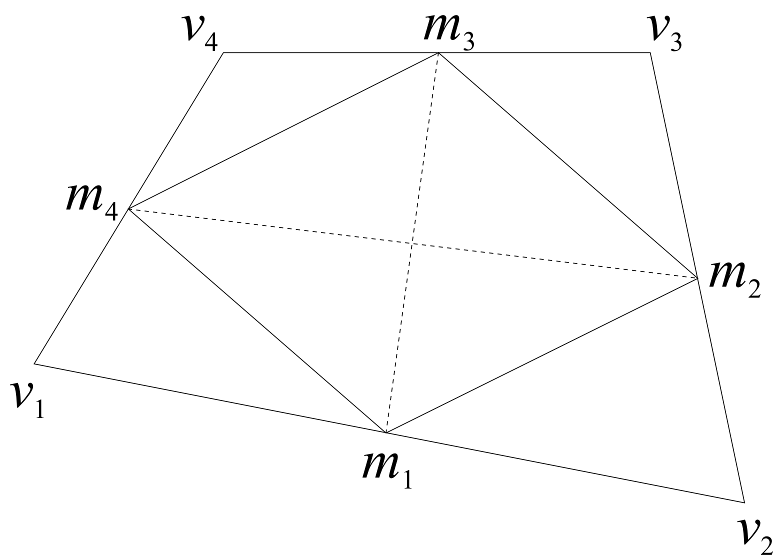


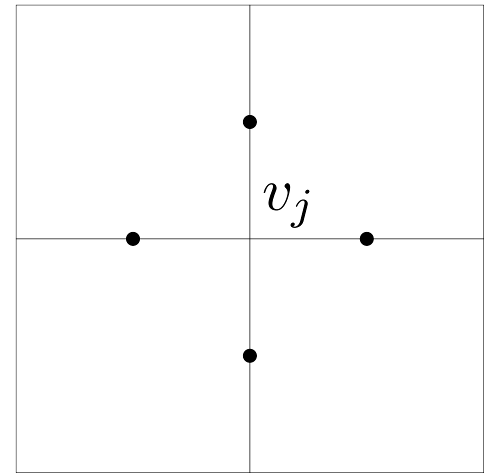
$$u \in \mathcal{P}_1 \quad \Rightarrow \quad u(m_1) + u(m_3) = u(m_2) + u(m_4)$$

$$u_{m_1} + u_{m_3} = u_{m_2} + u_{m_4} \quad \Rightarrow \quad \exists! u \in \mathcal{P}_1 : u(m_i) = u_{m_i}$$

definition of $\tilde{P}_1$

v_j: vertex, m_i: edge midpoint

$$\mathcal{M}(j) := \{ i \in \mathbb{N} \,\big|\, \exists \Gamma \in \partial\mathbb{T} : m_i \in \Gamma \wedge v_j \in \Gamma \}$$



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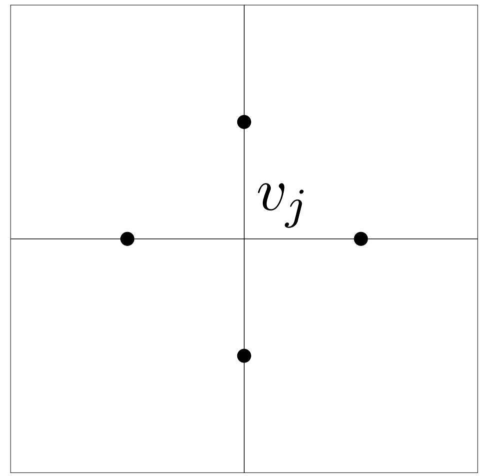

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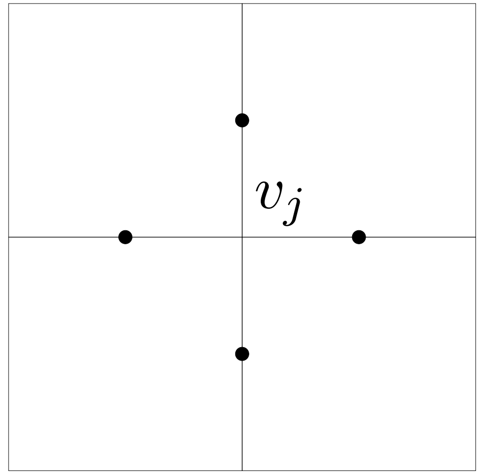
$$\Phi_j(m_i) = \begin{cases} 1 & , \quad i \in \mathcal{M}(j) \\ 0 & , \quad \text{else} \end{cases} \quad , \Phi_j \in \mathcal{P}_1(T)$$

definition of $\tilde{P}_1$



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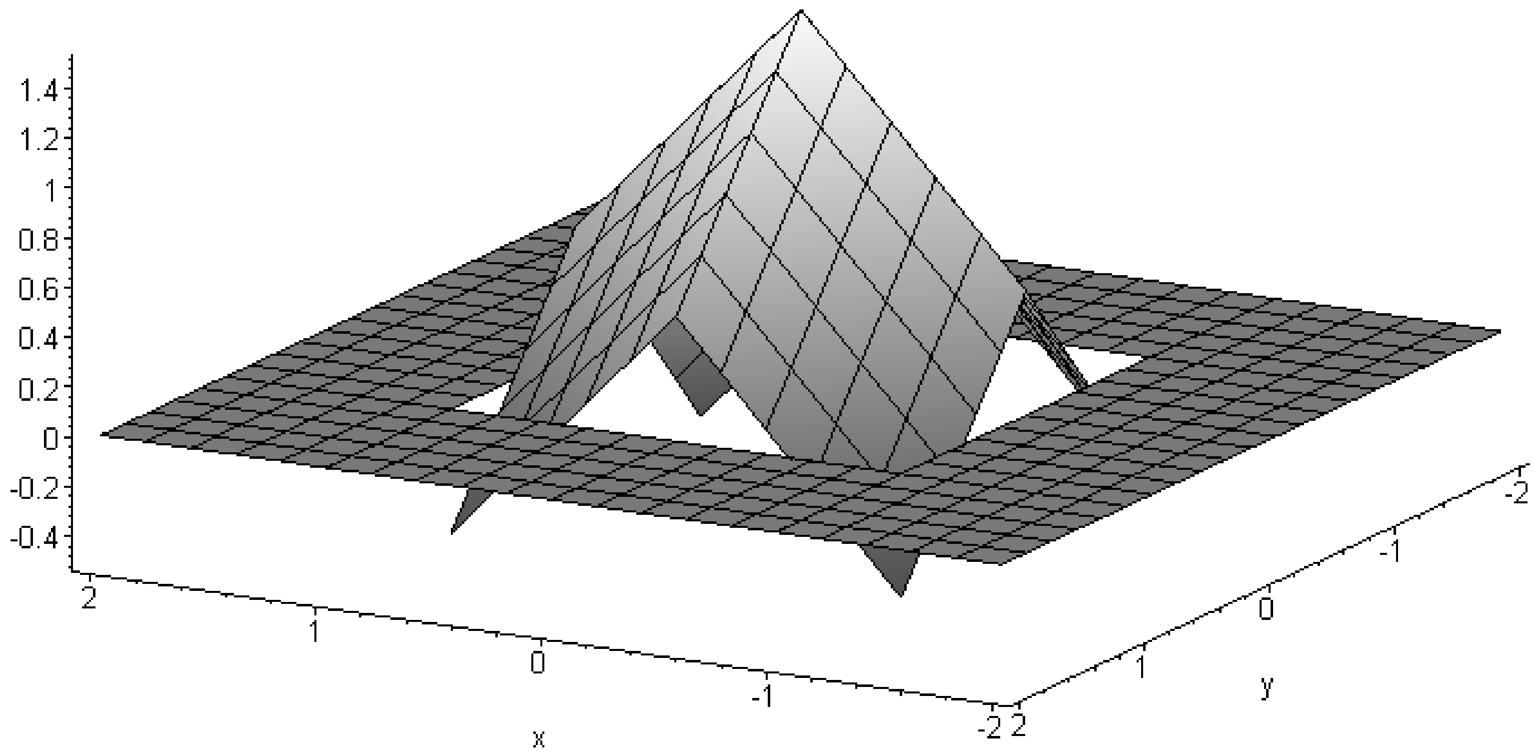
$$\begin{aligned} \tilde{P}_1(\mathbb{T}) &:= \left\{ v : \Omega \to \mathbb{R} \,\middle|\, v|_T \in \mathcal{P}_1(T) \,\forall\, T \in \mathbb{T} \wedge v \text{cont. w.r.t. } F_\Gamma \right\} \\ F_\Gamma(v) &= v(m_\Gamma) \text{ (continuity constraint)} \end{aligned}$$

$\tilde{P}_1$ basis function $\Phi_{(0,0)}$



theorem (Park)

Ω **simply** connected with polygonal boundary, $\mathbb{T}$ triangulation of Ω with N vertices, $0 < \hat{\jmath} \leq N$

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$$(\nabla u, \nabla \varphi)_\Omega + (cu, \varphi)_\Omega + (u, \varphi)_{\partial\Omega} = (f, \varphi)_\Omega + (g, \varphi)_{\partial\Omega} \quad \forall\, \varphi \in H_1(\Omega).$$

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$||u - u_h||_0 \leq Ch^2 ||u||_2$, if $g \in H_{\frac{1}{2}}(\partial\Omega)$, $f \in L_2(\Omega)$

Dirichlet boundary cond. I (Park)

$$u_h(m_j) = c_j + c_{j+1} \quad (*)$$

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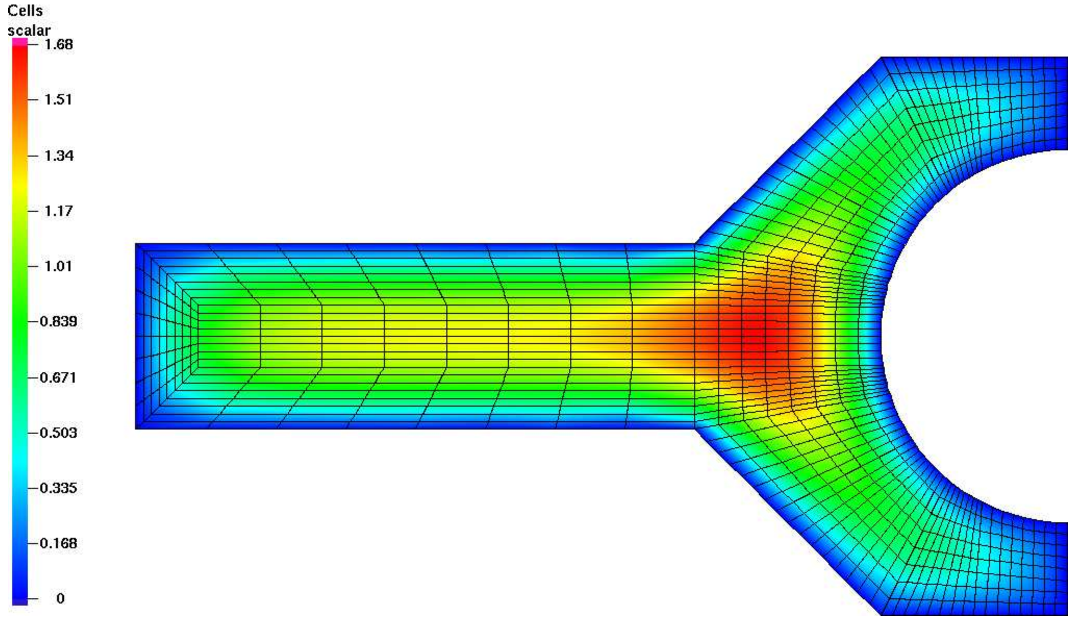
$\Rightarrow$ “explicit boundary treatment”:

- fix one arbitrary coefficient per boundary component
- compute recursively the coefficients on the boundary according to (*)

Dirichlet boundary cond. II



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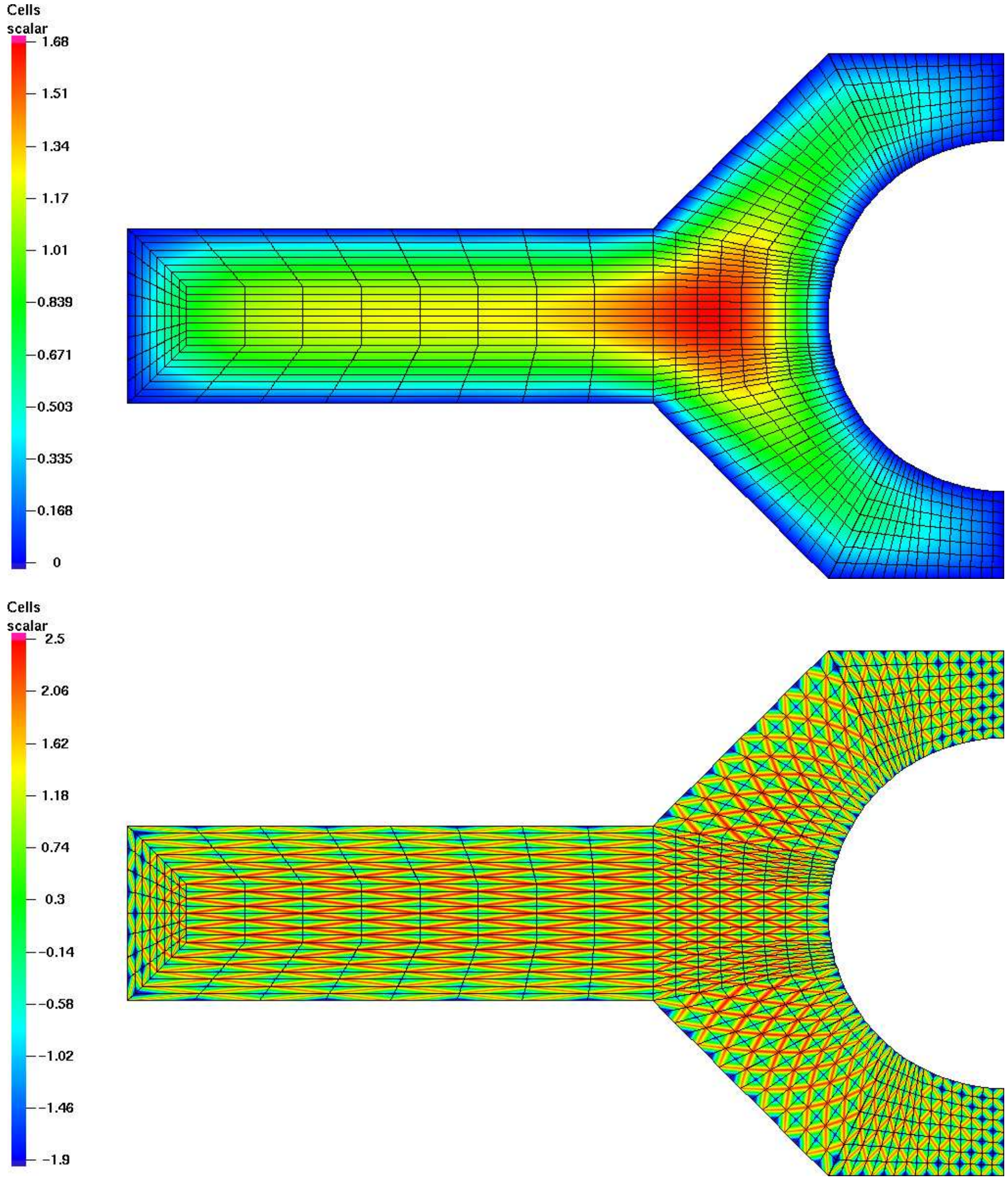

u_h and coefficient vector of u_h

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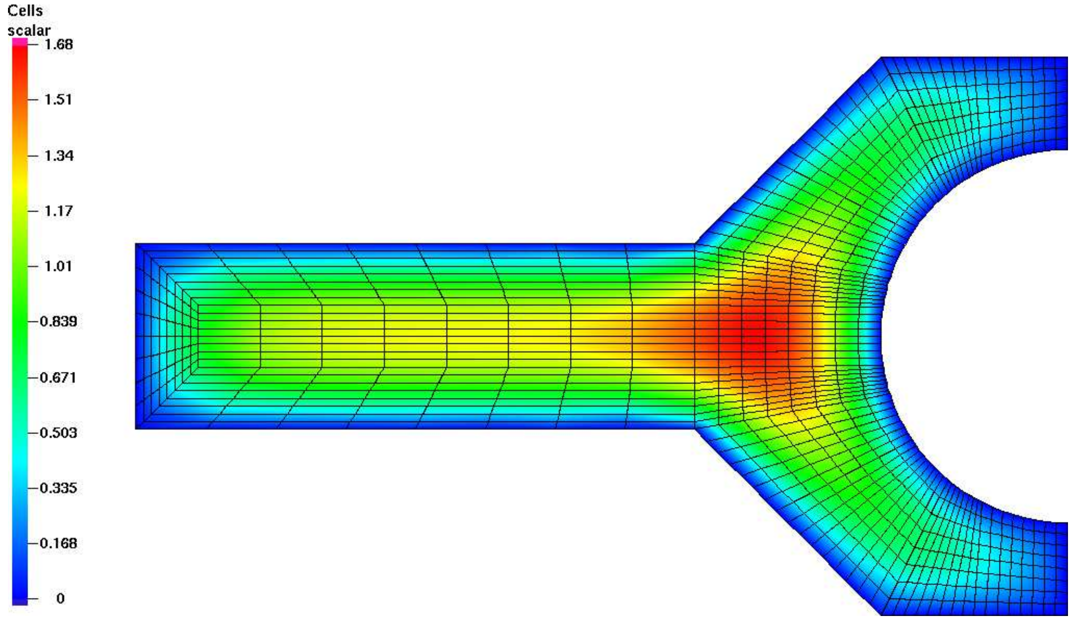

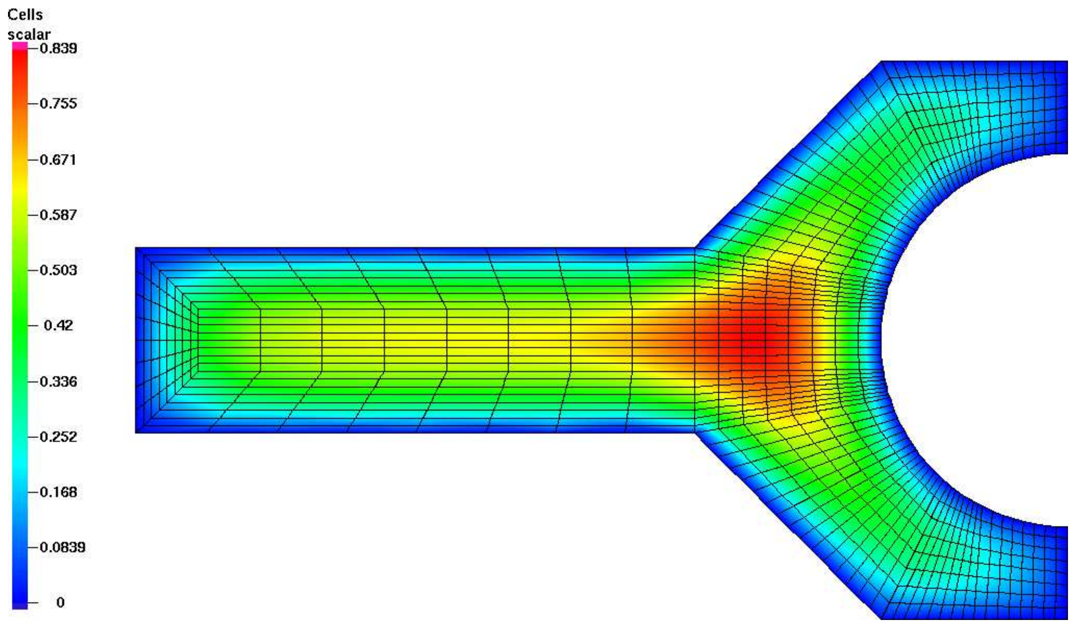

u_h and coefficient vector of u_h

comparison with other elements

Test problem: $-\Delta u = 10, \quad u|_{\partial\Omega} = 0$

$$||e_h||_{l_2} := ||u - u_h||_{l_2} := \left(\frac{1}{N} \sum_{i=1}^{N} |u(v_i) - u_h(v_i)|^2 \right)^{1/2}$$

NEL	$\lvert\lvert e_h \rvert\rvert_{l_2}, Q_1$	red.	$\lvert\lvert e_h \rvert\rvert_{l_2}, \tilde{Q}_1$	red.	$\lvert\lvert e_h \rvert\rvert_{l_2}, \tilde{P}_1$	red.
60	$4.18 \quad 10^{-2}$	-	$6.24 \quad 10^{-2}$	-	$4.07 \quad 10^{-2}$	-
240	$1.62 \quad 10^{-2}$	2.58	$2.58 \quad 10^{-2}$	2.42	$1.96 \quad 10^{-2}$	2.08
960	$5.69 \quad 10^{-3}$	2.85	$8.10 \quad 10^{-3}$	3.19	$6.04 \quad 10^{-3}$	3.25
3840	$1.96 \quad 10^{-3}$	2.90	$2.59 \quad 10^{-3}$	3.13	$1.81 \quad 10^{-3}$	3.34
15360	$6.57 \quad 10^{-4}$	2.98	$8.31 \quad 10^{-4}$	3.12	$5.40 \quad 10^{-4}$	3.35

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observation: multiply connected domains and explicit boundary treatment:
$||e_h||_0 = \mathcal{O}(h)$

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lemma Let us assume that $u = \alpha$. If for one arbitrary $T \in \mathbb{T}$ holds
$c_1 + c_3 = c_2 + c_4$, then:

$c_1 + c_3 = c_2 + c_4 \forall\, T$ and representation is oscillationfree

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$\Rightarrow$ “implicit boundary treatment”:

- for one element on the first boundary component, apply $c_1 + c_3 = c_2 + c_4$
- for all other boundary edges, apply $u(m_j) = c_j + c_{j+1}$

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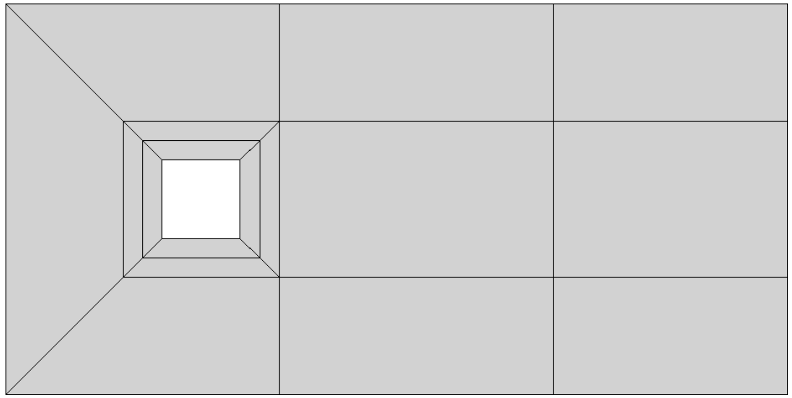
remark:

- on tensor product grids : Lemma even holds for *linear* functions
- with implicit boundary treatment: error of boundary treatment can be $\mathcal{O}(h^3)$

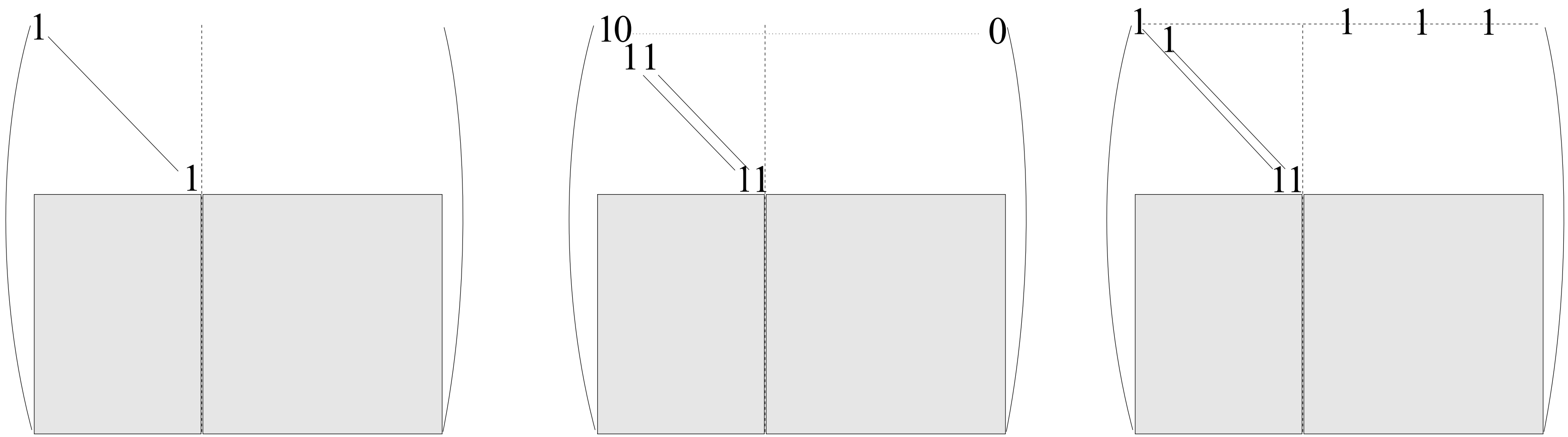
Dirichlet boundary cond. IV



L_2-error of computation of $u = x + 1$ and reduction rates of the error

NEL	implicit	red.	explicit	red.
68	$1.96 \quad 10^{-4}$	-	$3.74 \quad 10^{-3}$	-
272	$2.30 \quad 10^{-5}$	8.52	$1.86 \quad 10^{-3}$	2.01
1088	$2.79 \quad 10^{-6}$	8.24	$9.29 \quad 10^{-4}$	2.00
4352	$3.44 \quad 10^{-7}$	8.11	$4.64 \quad 10^{-4}$	2.00
17408	$4.28 \quad 10^{-8}$	8.04	$2.32 \quad 10^{-4}$	2.00
69632	$5.32 \quad 10^{-9}$	8.05	$1.16 \quad 10^{-4}$	2.00

matrix structure



- standard treatment via elimination (in rows only) in Q_1 and $\tilde{Q}_1$ (left)
- $\tilde{P}_1$ with explicit boundary treatment (middle)
- $\tilde{P}_1$ with implicit boundary treatment (right)

iterative solvers for $\tilde{P}_1$

problem : implicit boundary treatment affects solver behaviour

example : “square in the channel”, $u(x,y) = x(x-1)(1-y)y^2 \sin(x+2y)$,
solver: BiCGStab

NEL	ILU(0)	ILU(0) sort	ILU(1)	ILU(1) sort
68	28	15	12	10
272	63	24	24	17
1088	171	46	48	26
4352	539	110	100	49
17408	2743	1060	292	97
69632	$>$10000	$>$ 10000	1149	236

$\tilde{P}_1$ not LBB-stable

problem: $\tilde{P}_1$ does not fulfill the LBB-condition:

$\Omega = [0,1]^2$, T tensor product grid,

$$u(x,y) = (y,0), p(x,y) = const.$$

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observation: checkerboard oscillation of the pressure

problem: $\tilde{P}_1$ does not fulfill Korn's inequality (analogous to the $\tilde{Q}_1$-case according to Knobloch)

fast matrix assembly

advantage:

- extremely fast matrix assembly possible: 1-point Gauss rule exact
- nonparametric transformation for $\tilde{P}_1$ as fast as parametric one

NEL	$\tilde{P}_1^{\text{par}}$, G_1	$\tilde{P}_1^{\text{par}}$, G_2	$\tilde{P}_1$,G_1	$\tilde{P}_1$, G_2	$\tilde{Q}_1^{\text{par}}$, G_2	$\tilde{Q}_1$, G_2
133,120	1.5 s	2.3	1.6	2.4	2.4	3.1
532,480	6.0 s	9.4	6.5	9.6	9.7	12.3
2,129,920	24.7 s	37.4	25.5	38.6	38.9	48.2

remark:

- for $\tilde{Q}_1$: 2×2-Gauss rule necessary
- for Douglas-element (enhanced $\tilde{Q}_1$): even 3×3-Gauss necessary!

a-posteriori error control (conforming case)

$$a(u,\varphi) := (\nabla u, \nabla\varphi) = (f,\varphi) \quad \forall \varphi \in V := H_0^1(\Omega)$$

wanted: upper bound of the error of an output quantity $J(\cdot)$

Becker, Rannacher: *dual problem*

$$(\nabla\varphi, \nabla z) = J(\varphi) \quad \forall \varphi \in H_0^1(\Omega)$$

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$$|J(e_h)| = |a(e_h, z)|$$

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$$\begin{aligned} |J(e_h)| &= |a(e_h, z)| \\ &= \left| \sum_{T\in\mathbb{T}} (\underbrace{-\Delta u}_{=f} + \Delta u_h, z - z_h)_T - \frac{1}{2}([\partial_{\mathfrak{n}} u_h], z - z_h)_{\partial T} \right| \end{aligned}$$

a posteriori error control (nonconf. case)

problem 1: $e_h \notin V$, as $u_h \notin V \rightsquigarrow J(e_h) =?$
remedy: dual problem defined by

$$(\varphi, -\Delta z) = J(\varphi) \quad \forall \varphi \in L_2(\Omega), z \in H^2(\Omega)$$

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problem 2: Galerkin-orthogonality : $V \cap V_h$ “too small” for z_h

remedy: element enhacing by (parametric) bulb function $\varphi_B(x, y) = xy \rightsquigarrow z_h^C$

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$$\begin{aligned} |J(e_h)| &= \Big| \sum_{T \in \mathbb{T}} \Big\{ (f + \underbrace{\Delta u_h^N}_{=0}, z - z_h^C)_T - \frac{1}{2}([\partial_{\mathfrak{n}} u_h^N], z - z_h^C)_{\partial T} \\ &\qquad - \frac{1}{2}([u_h^N], \partial_{\mathfrak{n}} z)_{\partial T} \Big\} + a_h(e_h, z_h^C) \Big| \end{aligned}$$

practical aspects

- **observation:** consistency error $a_h(e_h, z_h^C) = \mathcal{O}(h^4)$ for Laplace equation
- $\forall u_h^N \in \tilde{P}_1(\mathbb{T}) : \Delta u_h^N = 0$
- replace z (unknown) by z_I obtained by patchwise biquadratic interpolation of z_h
- in nonconformity term $([u_h^N], \partial_{\mathfrak{n}} z)_{\partial T}$: $\partial_{\mathfrak{n}} z$ replaced by $\overline{\partial_{\mathfrak{n}} z} := 0.5(\partial_{\mathfrak{n}} z_h|_{T_1} + \partial_{\mathfrak{n}} z_h|_{T_2})$

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lemma:

$$\forall T \in \mathbb{T} : ([u_h^N], \overline{\partial_\mathfrak{n} z_h^N})_{\partial T} = 0$$

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lemma:

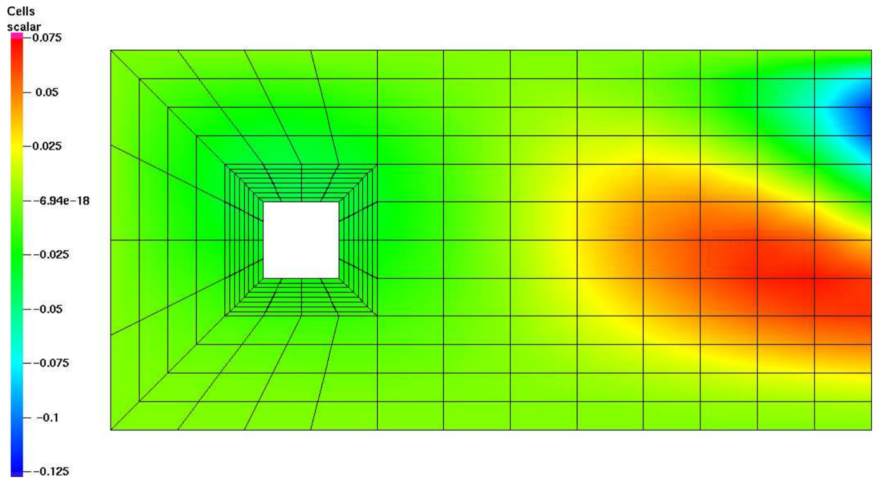
$$\forall T \in \mathbb{T} : ([u_h^N], \overline{\partial_{\mathfrak{n}} z_h^N})_{\partial T} = 0$$

$\Rightarrow$ error estimation:

$$|J(e_h)| \approx \eta(e_h) := \left| \sum_{T \in \mathbb{T}} (f, z_I - z_h^C)_T - \frac{1}{2}([\partial_{\mathfrak{n}} u_h^N], z_I - z_h^C)_{\partial T} \right|$$

numerical test I

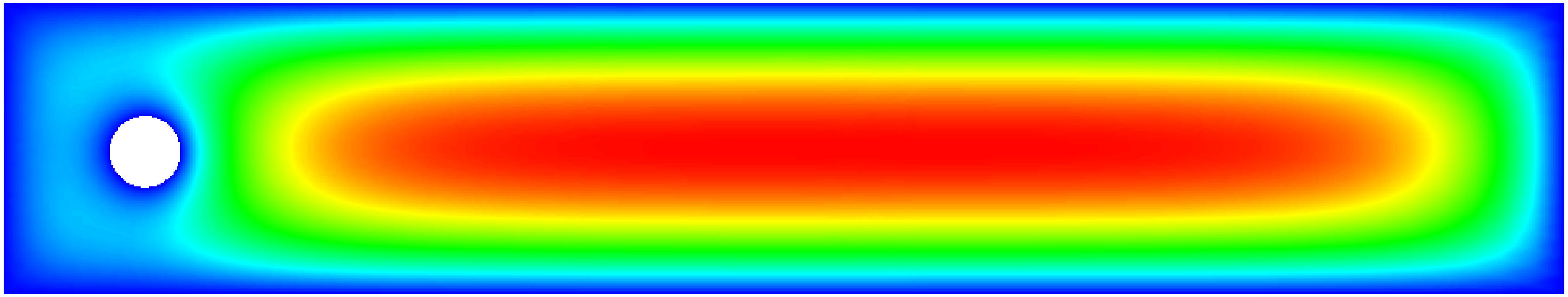
"square in a channel", analytical solution, point error in $(0.35, 0.5)$



NEL	$\left\|J^r_{(0.35,0.5)}(e_h)\right\|$	$\eta(e_h)$	I_{eff}
68	$1.97 \quad 10^{-4}$	$2.13 \quad 10^{-3}$	10.8
272	$2.00 \quad 10^{-4}$	$4.70 \quad 10^{-4}$	2.35
1088	$5.86 \quad 10^{-5}$	$9.65 \quad 10^{-5}$	1.65
4352	$1.51 \quad 10^{-5}$	$2.31 \quad 10^{-5}$	1.53
17408	$3.80 \quad 10^{-6}$	$5.73 \quad 10^{-6}$	1.51
69632	$9.51 \quad 10^{-7}$	$1.43 \quad 10^{-6}$	1.51

numerical test II

DFG-benchmark-grid “cylinder in channel”, $-\Delta u = 10, u|_{\partial\Omega} = 0$,
$J(\varphi) = \int_{\partial\Omega_c} \frac{\partial u}{\partial \mathfrak{n}}\, ds$



NEL	$\lvert J_\Gamma(e_h)\rvert$	$\eta(e_h)$	I_{eff}
520	$4.52 \quad 10^{-2}$	$1.13 \quad 10^{-1}$	2.50
2080	$2.34 \quad 10^{-2}$	$1.53 \quad 10^{-2}$	0.65
8320	$1.14 \quad 10^{-2}$	$2.96 \quad 10^{-3}$	2.61
33280	$5.16 \quad 10^{-3}$	$1.75 \quad 10^{-3}$	3.39

conclusion

advantages of $\tilde{P}_1$

- simplest nonconforming quadrilateral element
- optimal approximation order
- very fast matrix assembly (for parametric and non-parametric trafo)
- fast solvers possible
- rigorous a-posteriori error control

open problems

- multigrid
- stabilization for mixed formulation problems (CFD)
- realistic applications