Concepts of patchwise mesh refinement in the context of DWR

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Outline

- Adaptivity meets HPC
- Reliability aspects of DWR
- Conclusion and discussion

Motivation

observation 1: Goal-oriented adaptivity and local mesh refinement is mandatory for accurate and efficient computation.

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observation 2: The MFlop/s rates in adaptive FEM calculations today do by far not reach the peak performance.

example: Sparse MV multiplication in **FEATFLOW** (F77-code):

Computer	#Unknowns	**CM**	**TL**	**STO**
	8,320	147	136	**116**
DEC 21264	33,280	125	105	**100**
(667 MHz)	133,120	81	71	**58**
‘EV67’	532,480	60	51	**21**
$\sim$ 1300 MFlop/s	2,129,920	58	47	**13**
	8,519,680	58	45	**10**

Motivation II

reasons for observation 2:

1. on modern computers: CPU speed $\gg$ memory speed

2. current FEM-codes use
   - indirect addressing : many (unaligned) memory accesses
   - global data structures : effective caching prevented

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Elementwise unstructured grids **allow adaptivity** but **prevent direct adressing**;

tensor product grids **allow direct adressing** and (seem to) **prevent adaptivity**.

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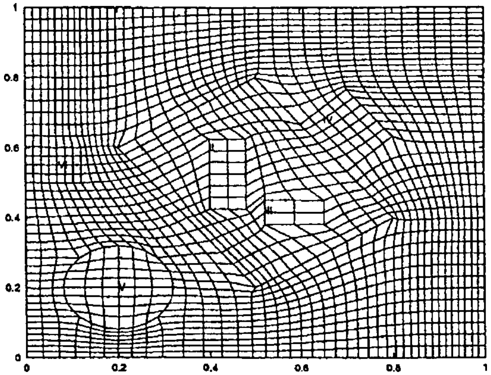
synthesis: FEAST

(grid-related part of the) FEAST concept

definition of a generalized tensor product mesh: Every inner point has exactly 4 neighbours. $\Rightarrow$ linewise regular numbering

example:



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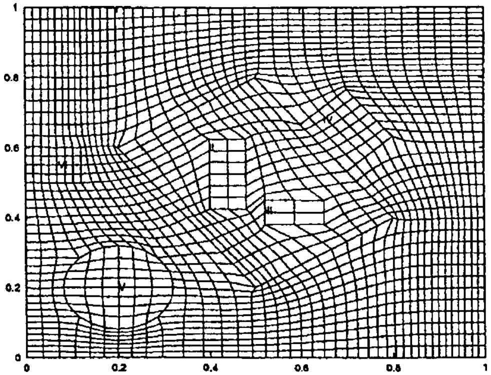

The global grid consists of “many” local generalized tensor product meshes (“macros”).

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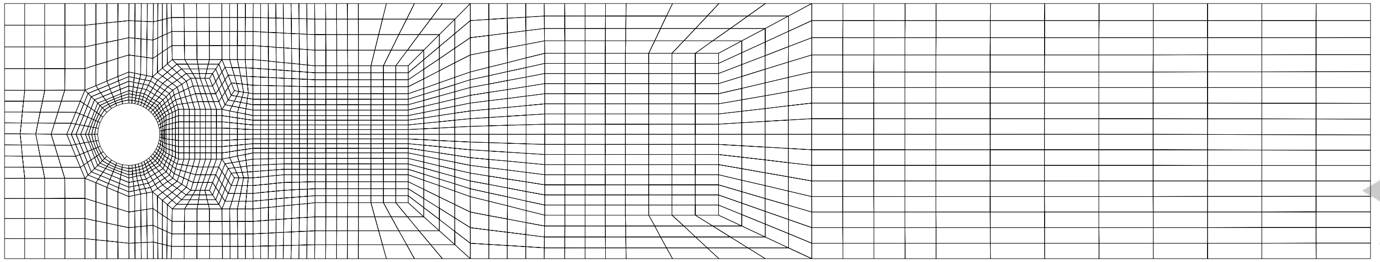

MFLOP-rate preserving adaptivity I

goal: adapted grids with local generalized tensor product structure

approach 1: macro wise adaptive refinement

- hanging nodes on macro edges are allowed
- the level difference of 2 edge-adjacent macros must not exceed 1

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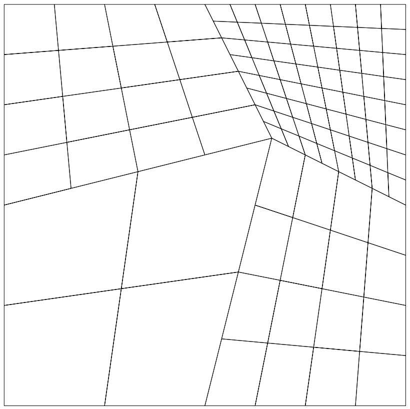
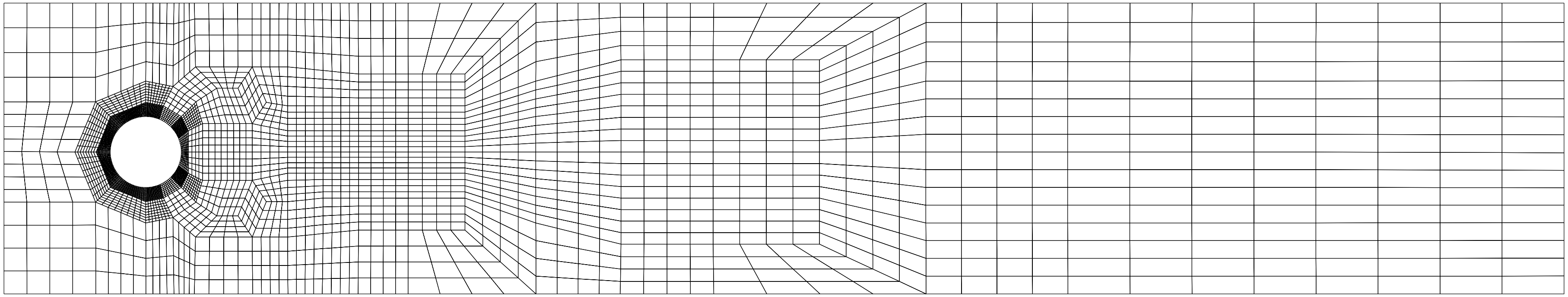


MFLOP-rate preserving adaptivity II

example 2



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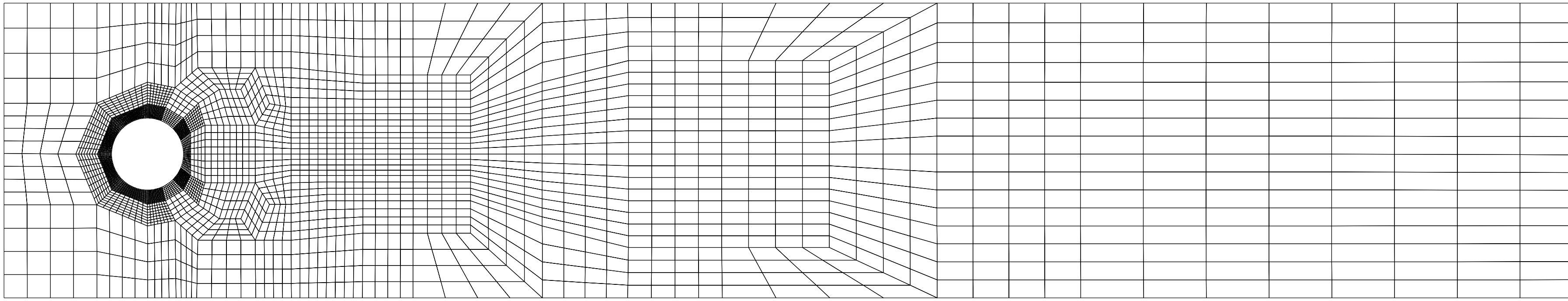

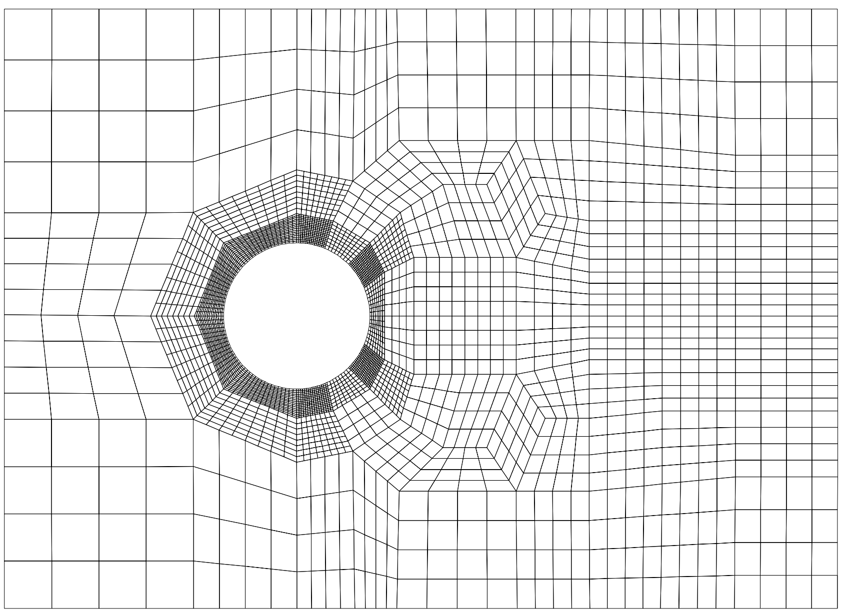

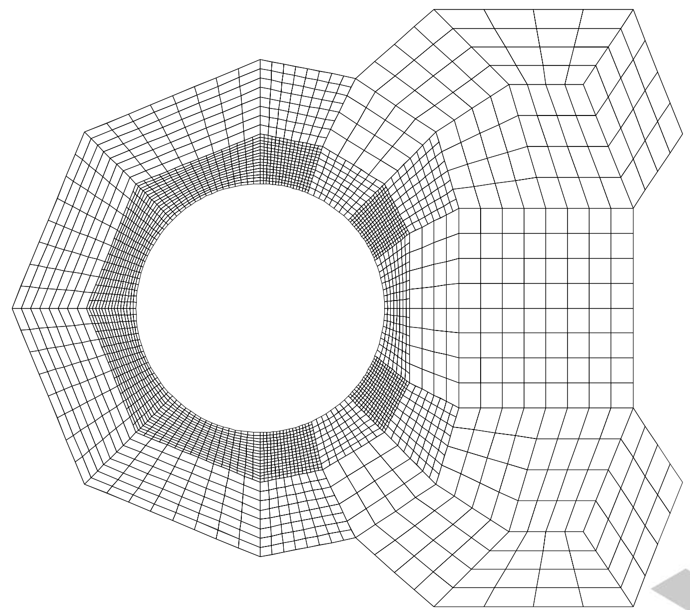

MFLOP-rate preserving adaptivity III

approach 2: grid deformation

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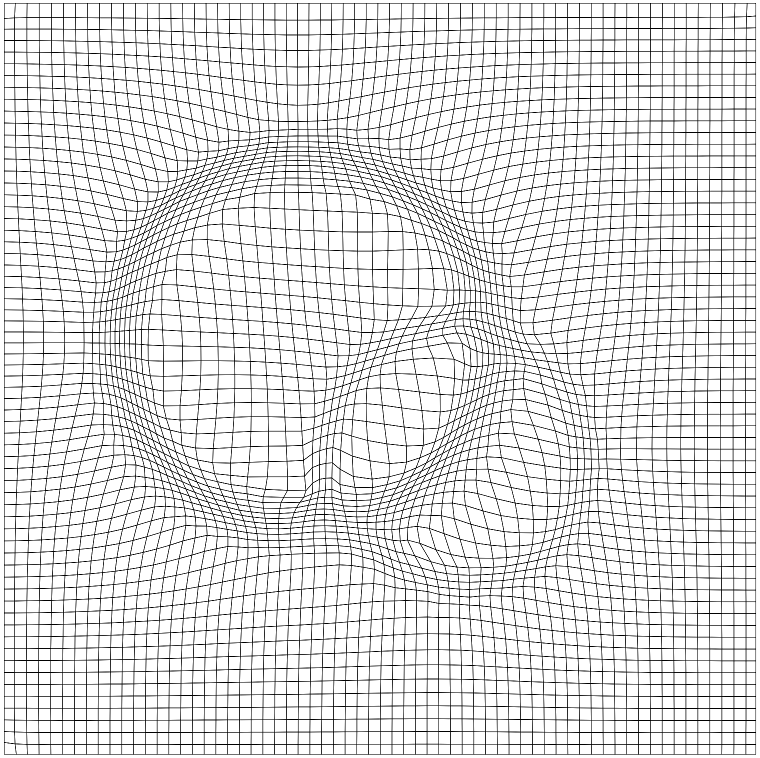

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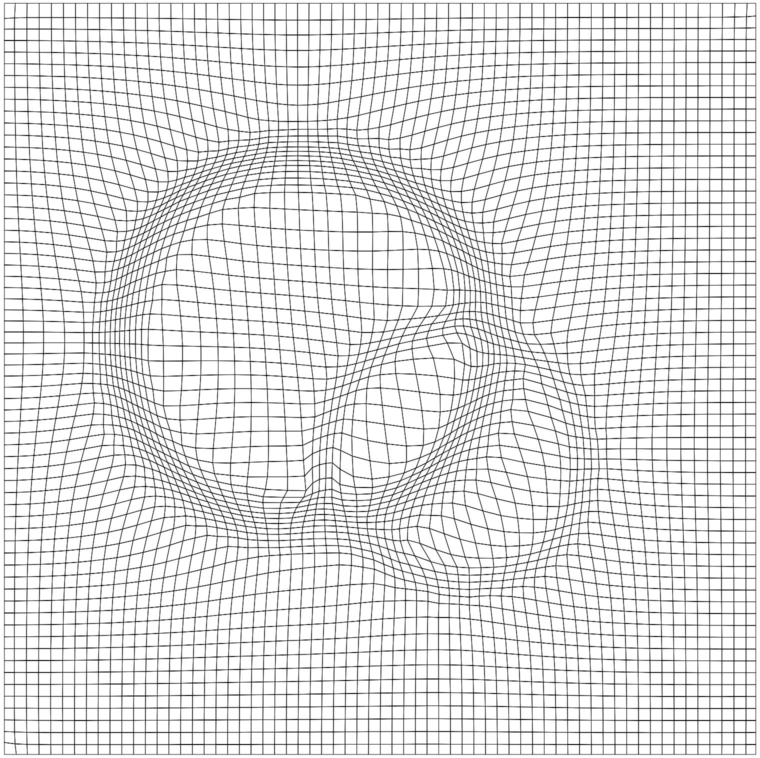

Grid deformation preserves the (local) logical structure of the grid.

Methods for r-adaptivity

1. **location based methods:**
   - Winslow’s method
   - Brackbill’s and Saltzman’s method
   - harmonic mapping

   disadvantages:

   (a) non-linear problems (demanding)
   (b) interaction of monitor function and grid not clear

2. **velocity based methods:**
   - MMPDE/GCL (Cao,Huang, Russell)
   - Deformation method (Liao et al.)

   advantages:

   (a) (several) Laplace problems on fixed mesh (fast)
   (b) monitor function “directly” from error distribution
   (c) mesh tangling prevented

Deformation method (Moser/Liao)

idea : construct transformation $\phi, x = \phi(\xi, t)$ with $\det \nabla \phi = f$
$\Rightarrow$ local mesh area $\approx f$

1. compute monitor function $f(x,t) > 0, f \in \mathcal{C}^1$ and
$\int_\Omega f^{-1}(x,t)\, dx = |\Omega| \quad \forall t \in [0,1]$

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3. solve the ODE system

$$\frac{\partial}{\partial t}\phi(\xi, t) = f\Big(\phi(\xi, t), t\Big)\,\nabla v\Big(\phi(\xi, t), t\Big)$$

new grid points: $x_i = \phi(\xi_i, 1)$

Adaptivity concept

1. adaptive macrowise refinement
2. in the case of localised error sources: grid deformation on (sub)domains

The algorithmic aspects of the grid adaptation process are object of current research.

Reliability of DWR

DWR-method:

$$(\nabla u, \nabla \varphi) = (f, \varphi), \quad (\nabla z, \nabla \varphi) = J(\varphi) \quad \forall \varphi \in V$$

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replace z by an approximation $\tilde{z}$

$$\begin{aligned} |J(u-u_h)| \quad \leq \quad & \Big| \underbrace{(f, \tilde{z} - \varphi_h) - (\nabla u_h, \nabla (\tilde{z} - \varphi_h))}_{:= \eta(u_h)} \Big| \\ & + \Big| \underbrace{(\nabla (u - u_h), \nabla (z - \tilde{z}))}_{:= \eta^*(u_h);\ \text{usually neglected}} \Big| \quad \forall \varphi_h \in V_h \end{aligned}$$

Reliability of DWR II

a-posteriori error correction with DWR (case of linear functional):

$$J(u) = J(u_h) + J(u - u_h) = J(u_h) + \eta(u_h) + \eta^*(u_h)$$

definition (corrected value)

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Is this neglect admissible?

Neglect of η^*

observation The neglect of η^* does not affect the grid steering process.

Are reliability and efficiency of DWR affected ?

definition of the efficiency index I_{eff}:

$$I_{\mathrm{eff}} = \left| \frac{\eta}{J(u) - J(u_h)} \right|$$

Example 1

$$\Omega = [0,1]^2, \text{ ansatz space : } \mathcal{Q}^1$$

$$\Delta u = f, \quad u|_{\partial\Omega} = 0$$

$$u(x,y) = x(x-1)(1-y)y^2 \sin(x+2y)$$

$$J(\varphi) = \varphi(0.5, 0.5)$$

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NEL	$J(u-u_h)$	$I_{\text{eff}}, \mathcal{Q}_1, intp.$	$I_{\text{eff}}, \mathcal{Q}_2$	$\lvert J_{ref} - J^*(u_h)\rvert$
64	$4.05 \cdot 10^{-4}$	2.14	0.99	$1.27 \cdot 10^{-6}$
256	$1.01 \cdot 10^{-4}$	2.10	1.00	$7.87 \cdot 10^{-8}$
1024	$2.52 \cdot 10^{-5}$	2.09	1.00	$4.90 \cdot 10^{-9}$
4096	$6.31 \cdot 10^{-6}$	2.08	1.00	$3.06 \cdot 10^{-10}$
16384	$1.58 \cdot 10^{-6}$	2.08	1.00	$1.91 \cdot 10^{-11}$
65536	$3.94 \cdot 10^{-7}$	2.08	1.00	$1.20 \cdot 10^{-12}$

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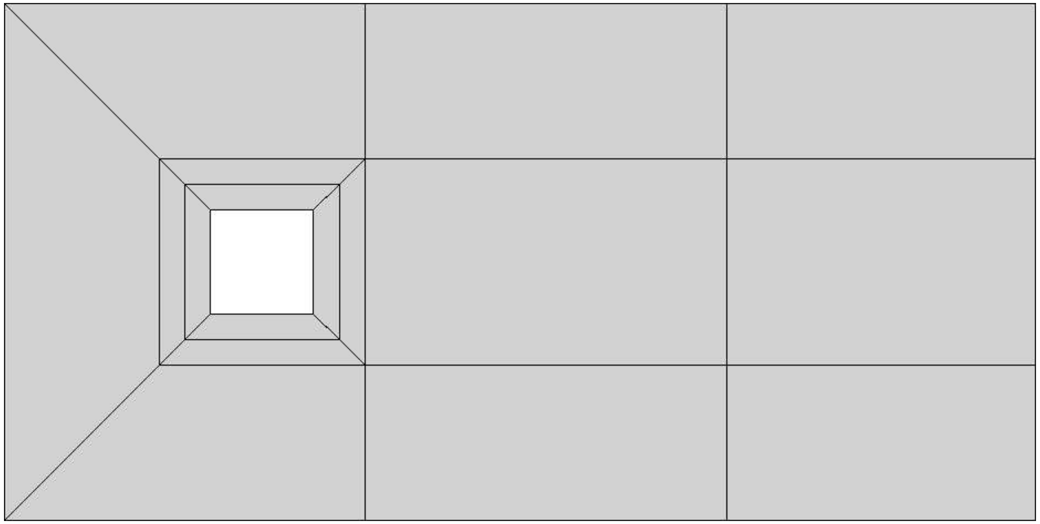
no problems

Example 2, I

harder problem (convection, edge singularities)

$$-\Delta u + (10,0)^\top \cdot \nabla u = 10, \quad u|_{\partial\Omega} = 0$$



$$J(\varphi) = \int_{\partial S} \frac{\partial \varphi}{\partial n}\, ds$$

ansatz space : $\mathcal{Q}_1$

Example 2, II

reference value J_{ref} by “brute force comp.”: 8 regular + 5 local ref. around the square, reference calculation with $\mathcal{Q}_2$

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dual problem with $\mathcal{Q}_1$ and biquadratic interpolation, same mesh

NEL	$\|J_{\mathrm{ref}} - J(u_h)\|$	I_{eff}	$\|J_{\mathrm{ref}} - J^*(u_h)\|$
1088	$1.1386 \cdot 10^{-2}$	1.83	$9.4884 \cdot 10^{-3}$
4352	$1.3793 \cdot 10^{-2}$	0.54	$2.1288 \cdot 10^{-2}$
17408	$1.9698 \cdot 10^{-2}$	0.07	$2.1116 \cdot 10^{-2}$
69632	$1.8048 \cdot 10^{-2}$	0.05	$1.7155 \cdot 10^{-2}$
278528	$1.4222 \cdot 10^{-2}$	0.22	$1.2735 \cdot 10^{-2}$
1114112	$1.0395 \cdot 10^{-2}$	0.21	$8.9900 \cdot 10^{-3}$

Example 2, III

dual problem with $\mathcal{Q}_1$ and biquadratic interpolation, dual mesh one level finer

NEL	$\lvert J_{\mathrm{ref}} - J(u_h)\rvert$	I_{eff}	$\lvert J_{\mathrm{ref}} - J^*(u_h)\rvert$
1088	$1.1386 \cdot 10^{-2}$	3.31	$2.6340 \cdot 10^{-2}$
4352	$1.3793 \cdot 10^{-2}$	0.80	$2.4863 \cdot 10^{-2}$
17408	$1.9698 \cdot 10^{-2}$	0.00	$1.9706 \cdot 10^{-2}$
69632	$1.8048 \cdot 10^{-2}$	0.20	$1.4404 \cdot 10^{-2}$
278528	$1.4222 \cdot 10^{-2}$	0.29	$1.0057 \cdot 10^{-2}$
1114112	$1.0395 \cdot 10^{-2}$	0.34	$6.8202 \cdot 10^{-3}$

Example 2, IV

dual problem with $\mathcal{Q}_2$, same mesh

NEL	$\lvert J_{\mathrm{ref}} - J(u_h)\rvert$	I_{eff}	$\lvert J_{\mathrm{ref}} - J^*(u_h)\rvert$
1088	$1.1386 \cdot 10^{-2}$	2.84	$2.0999 \cdot 10^{-2}$
4352	$1.3793 \cdot 10^{-2}$	0.09	$1.0802 \cdot 10^{-2}$
17408	$1.9698 \cdot 10^{-2}$	0.62	$7.5091 \cdot 10^{-3}$
69632	$1.8048 \cdot 10^{-2}$	0.75	$4.5457 \cdot 10^{-3}$
278528	$1.4222 \cdot 10^{-2}$	0.80	$2.7832 \cdot 10^{-3}$
1114112	$1.0395 \cdot 10^{-2}$	0.83	$1.7166 \cdot 10^{-3}$

Example 2, V

dual problem with $\mathcal{Q}_3$, same mesh

NEL	$\lvert J_{\text{ref}} - J(u_h)\rvert$	I_{eff}	$\lvert J_{\text{ref}} - J^*(u_h)\rvert$
1088	$1.1386 \cdot 10^{-2}$	1.60	$6.7949 \cdot 10^{-3}$
4352	$1.3793 \cdot 10^{-2}$	0.73	$3.7562 \cdot 10^{-3}$
17408	$1.9698 \cdot 10^{-2}$	0.89	$2.1758 \cdot 10^{-3}$
69632	$1.8048 \cdot 10^{-2}$	0.93	$1.2972 \cdot 10^{-3}$
278528	$1.4222 \cdot 10^{-2}$	0.95	$7.8469 \cdot 10^{-4}$
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Example 2, V

dual problem with $\mathcal{Q}_3$, same mesh

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$\Rightarrow$ suffi cient approximation of dual problem absolutely necessary

Remedy (?)

idea (Suttmeier): if $\tilde{z}$ is FEM-solution:

$$|\eta^*| = |(\nabla(u - u_h), \nabla(z - \tilde{z}))| \leq \eta_p \eta_d,$$

where η_p, η_d are energy error estimators

$\Rightarrow$

- reliable error control

$$|J(u - u_h)| \leq |\eta| + \eta_p \eta_d$$

- reliable error control of a-posteriori corrected values

$$|J(u) - J^*(u_h)| \leq \eta_p \eta_d$$

Remedy

example: Laplace-eq., $u(r,\varphi) = r^{2/3}\sin((2\varphi-\pi)/3)$ on L-shaped domain; η_p, η_d: ZZ-estimator, primal and dual problem with $\mathcal{Q}_1$ on same grid

$$J(\varphi) = \int_S \varphi(x)\,dx, \quad S = \big\{(x,y)\in\Omega \,\big|\, x=0\big\}$$

remark: Galerkin-orthogonality : $\eta = 0 \Rightarrow J(u_h) = J^*(u_h)$

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768	$6.53 \cdot 10^{-4}$	4.14
3072	$2.61 \cdot 10^{-4}$	6.48
12288	$1.04 \cdot 10^{-4}$	4.60
49152	$4.15 \cdot 10^{-5}$	7.14
196608	$1.65 \cdot 10^{-5}$	6.81

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reliable

Example (Suttmeier)

same example, but with adaptive steering of

- primal grid according to η_p, η
- dual grid according to η_d

NEL$_p$	NEL$_d$	$\lvert J(u) - J^*(u_h)\rvert$	I_{eff}
483	846	$8,68 \cdot 10^{-4}$	3.35
1101	2034	$3.85 \cdot 10^{-5}$	31.1
2541	4833	$6.27 \cdot 10^{-5}$	5.36
5829	11502	$7.31 \cdot 10^{-5}$	1.54
13146	27060	$2.30 \cdot 10^{-5}$	1.51
29202	62826	$1.17 \cdot 10^{-5}$	0.96
62184	145032	$4.36 \cdot 10^{-6}$	1.03

Conclusion

- for reliable estimation: Suttmeier's method with 2 different grids: primal and dual grid
- MFLOP-rate preserving grid adaptivity by deformation and macro wise refinement

agenda

- extension of Suttmeier's method to more complex problems
- efficient implementation in FEAST
- efficient incorporation of grid deformation in adaptation process

Thank you for your attention!