Hierarchical Solution Concepts for Flow Control Problems

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Overview

- Introduction, aims of the project
- Multigrid in space and time
- Numerical example
- Outlook

Distributed Control of nonstationary flow

Distributed Control for nonstationary Navier-Stokes equation with tracking-type functional:

$$J(y,u) = \frac{1}{2}||y - z||_Q^2 + \frac{\gamma}{2}||y(T) - z(T)||_\Omega^2 + \frac{\alpha}{2}||u||_Q^2 \quad \rightarrow \quad \text{min!}$$

on $Q = \Omega \times [0, T]$ such that

$$\begin{aligned} y_t + (y\nabla)y - \nu\Delta y + \nabla p &= u \quad \text{in } Q \\ -\nabla \cdot y &= 0 \quad \text{in } Q \end{aligned}$$

+ BC. No constraints on the control $u \in Q$.

Distributed Control of nonstationary flow

Corresponding KKT-System:

$$
\begin{aligned}
y_t + (y\nabla)y - \nu\Delta y + \nabla p &= \max\{a, \min\{b, -\tfrac{1}{\alpha}\lambda\}\} && \text{in } Q \\
-\nabla \cdot y &= 0 && \text{in } Q
\end{aligned}
$$

$$
\begin{aligned}
-\lambda_t - \nu\Delta\lambda - (y\nabla)\lambda + (\nabla y)\lambda + \nabla\xi &= (y - z) && \text{in } Q \\
-\nabla \cdot \lambda &= 0 && \text{in } Q
\end{aligned}
$$

$$
\lambda(T) = \gamma(y(T) - z(T)) \quad \text{in } \Omega
$$

+ boundary conditions

Aims of the project

Overall aim of the project:

$$\frac{\text{effort for optimization}}{\text{effort for simulation}} \leq C$$

Idea:

- One shot approach $\rightarrow$ Multigrid in space–time.
- Efficient CFD techniques (FeatFlow) for flow calculation.

Space–time discretisation

- Space discretisation: Finite Elements, 2D
- Time discretisation: Backward Euler (later: CN, dG)

$\Rightarrow$ Space–time system

$$A(x)x = b$$

Here (for n timesteps, $x_i = x(t_i)$):

$$x = (\underbrace{y_0, p_0, \lambda_0, \xi_0,}_{x_0} \ \ldots, \ \underbrace{y_n, p_n, \lambda_n, \xi_n}_{x_n})$$

- Nonlinearity: Newton method
- Linear subproblems: space-time Multigrid solver

Space–time discretisation

$A(x)$ has a very special structure! E.g. for 2 timesteps:

$$
\left(\begin{array}{cc|cc|c|c|c|c}
M & & & & & & & & & & & \\
 & M & & & & & & & & & & \\
\hline
-M & & \frac{M}{\Delta t}+L^* & -B & & & -\frac{M}{\Delta t} & & & & & \\
 & & -B^T & & & & & & & & & \\
\hline
-\frac{M}{\Delta t} & & & & \frac{M}{\Delta t}+L & -B & \frac{1}{\alpha}M & & & & & \\
 & & & & -B^T & & & & & & & \\
\hline
 & & & & -M & & \frac{M}{\Delta t}+L^* & -B & & & -\frac{M}{\Delta t} & \\
 & & & & & & -B^T & & & & & \\
\hline
 & & & & -\frac{M}{\Delta t} & & & & \frac{M}{\Delta t}+L & -B & \frac{1}{\alpha}M & \\
 & & & & & & & & -B^T & & & \\
\hline
 & & & & & & & & -\gamma M & & \frac{M}{\Delta t}+L^* & -B \\
 & & & & & & & & & & -B^T &
\end{array}\right)
$$

$\rightarrow$ Sparse, tridiagonal system

Space–time discretisation

In compressed form for the Stokes equation:

$$
\left(\begin{array}{cc|cc|cc|cc}
M & & & & & & & \\
-M & ST^* & & -\frac{M}{\Delta t} & & & & \\
\hline
-\frac{M}{\Delta t} & & ST & \frac{1}{\alpha}M & & & & \\
 & & -M & ST^* & & -\frac{M}{\Delta t} & & \\
\hline
 & & -\frac{M}{\Delta t} & & ST & \frac{1}{\alpha}M & & \\
 & & & & -M & ST^* & & -\frac{M}{\Delta t} \\
\hline
 & & & & & \dots & & \dots
\end{array}\right)
$$

$\Rightarrow$ Block-Jacobi preconditioner:

$$
P := P_{Jac} := \left(\begin{array}{cc|cc|cc|cc}
M & & & & & & & \\
-M & ST^* & & & & & & \\
\hline
 & & ST & \frac{1}{\alpha}M & & & & \\
 & & -M & ST^* & & & & \\
\hline
 & & & & ST & \frac{1}{\alpha}M & & \\
 & & & & -M & ST^* & & \\
\hline
 & & & & & & & \dots
\end{array}\right)
$$

Space–time discretisation

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-\frac{M}{\Delta t} & & ST & \frac{1}{\alpha}M & & & \\
& & -M & ST^* & & -\frac{M}{\Delta t} & \\
\hline
& & -\frac{M}{\Delta t} & & ST & \frac{1}{\alpha}M & \\
& & & & -M & ST^* & -\frac{M}{\Delta t} \\
\hline
& & & & & \dots & \dots
\end{array}\right)$$

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\hline
& & ST & \frac{1}{\alpha}M & & & \\
& & -M & ST^* & & & \\
\hline
& & & & ST & \frac{1}{\alpha}M & \\
& & & & -M & ST^* & \\
\hline
& & & & & & \dots
\end{array}\right)$$

Space–time multigrid and components

Essential multigrid components:

- An efficient *smoother!*

  $\rightarrow$ e.g. simple defect correction loop:

$$x^{i+1} = x^i + \omega P^{-1}(b - Ax^i), \qquad i = 1, ..., \mathrm{NSM}$$

- Prolongation/Restriction in space/time.

Space–time preconditioner

Essential: To apply P_{Jac}^{-1}, separately apply

$$\begin{pmatrix} ST & \frac{1}{\alpha}M \\ -M & ST^* \end{pmatrix}^{-1}$$

for every timestep!

Use:

- Multigrid in space
- Smoother: mod. LMPSC (VANKA-like) **(NEW!)**

} **FeatFlow!**

Numerical effort

With

- NITT = #global space-time MG iterations
  $\longrightarrow$ mesh independent
- NSM = #smoothing steps in space-time MG

a (pessimistic) upper bound is given by:

$$\frac{\text{effort for optimization}}{\text{effort for simulation}} \leq \text{NITT} \times \; 4 \times \text{NSM}$$

more...

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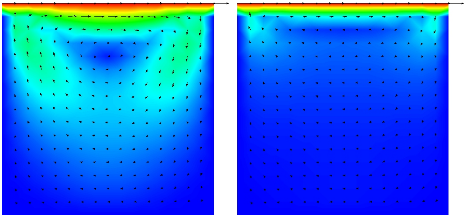
$$\frac{\text{effort for optimization}}{\text{effort for simulation}} \leq (\text{NITT} \times)\ 4 \times \text{NSM}$$

more...

Numerical example: Stokes control

Driven cavity for Stokes

- Uncontrolled flow at $t = 1$:



$\nu = 400$ $\nu = 1/400$

- $z :=$flow at $\nu = 400$. Simulation with $\nu = 1/400$,
- Prolongation/Restriction only in time, NSM=2, $\omega = 0.7$, P^{-1} exact
- Time horizon: $[0, T] = [0, 1]$

Numerical example: Stokes control

Kv-rate space-time MG

time-lv. (it.)	space-lv.	ρ
1..3 (20)	1..3	0.386
1..4 (40)	1..3	0.494
1..5 (80)	1..3	0.527
1..3 (20)	1..4	0.377
1..4 (40)	1..4	0.488
1..5 (80)	1..4	0.520
1..3 (20)	1..5	0.360
1..4 (40)	1..5	0.471
1..5 (80)	1..5	0.504

(SOR-like smoother)

(0.034)
(0.049)
(0.061)

Kv-rate space-MG at t=0.5, 2nd sm.-step

time-lv.	space-lv.	ρ
3	5	0.186
4	5	0.166
5	5	0.063

more...

more...

Numerical example: Stokes control

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1..3 (20)	1..5	0.360	(0.034)
1..4 (40)	1..5	0.471	(0.049)
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Kv-rate space-MG at t=0.5, 2nd sm.-step

time-lv.	space-lv.	ρ
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▸ more...

▸ more...

Outlook

Next steps:

- include Navier-Stokes with Newton method
  $\rightarrow$ change of the diagonal blocks of $A(x)$
- include CN, dG(0) in time, Q_2 in space
- simultaneous prol./rest. in time *and* space
- **improved smoothers for the space-time problem**
- include algorithms for memory management
  $\rightarrow$ solutions on the hard disc
- incorporate constraints
- Newton–multigrid approach for integral equation formulation

Numerical example: Stokes control

Kv-rate space-time MG, SOR-like smoother

time-lv. (it.)	space-lv.	ρ (P^{-1} exact)	ρ (1 MG in P^{-1})
1..3 (20)	1..3	0.040	0.040
1..4 (40)	1..3	0.059	0.059
1..5 (80)	1..3	0.072	0.072
1..3 (20)	1..4	0.038	0.038
1..4 (40)	1..4	0.056	0.056
1..5 (80)	1..4	0.068	0.068
1..3 (20)	1..5	0.034	0.035
1..4 (40)	1..5	0.049	0.049
1..5 (80)	1..5	0.061	0.061

Numerical effort

- effort for 1 space-time smoothing step
  $\hat{=}$ effort for solving 1$\times$ nonstationary coupled Stokes
  $\hat{=}$ effort for solving 2$\times$ nonstationary Stokes
- 1 space time multigrid sweep, NSM smoothing-steps
  $\leq$ 2$\times$NSM smoothing steps on highest level
  $\leq$ 4$\times$NSM nonstationary Stokes

For NITT = #global iterations (level independent):

$$\frac{\text{effort for optimization}}{\text{effort for simulation}} \leq \text{NITT} \times 4 \times \text{NSM}$$

With exactly 1 MG step instead of P^{-1} exact:

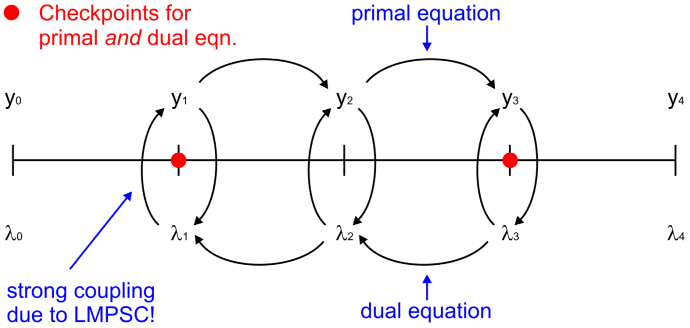
$$\frac{\text{effort for optimization}}{\text{effort for simulation}} \leq 4 \times \text{NSM}$$

Numerical example: Stokes control

Kv-rate space-time MG, SOR-like smoother

time-lv. (it.)	space-lv.	ρ
1..3 (20)	1..3	0.040
1..4 (40)	1..3	0.059
1..5 (80)	1..3	0.072
1..3 (20)	1..4	0.038
1..4 (40)	1..4	0.056
1..5 (80)	1..4	0.068
1..3 (20)	1..5	0.034
1..4 (40)	1..5	0.049
1..5 (80)	1..5	0.061

Checkpointing in the One-shot approach



- Checkpoints $\rightarrow$ nonlinear subproblems of the same kind.
- High computational effort necessary for recomputation $\rightarrow$ due to strong coupling by LMPSC!