technische universität dortmund

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Modified Shallow-Water Equations for Direct Bathymetry Reconstruction

Oberseminar LS3

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Content

The basic equations of fluid dynamics

Navier-Stokes equations

$$
\begin{aligned}
\partial_t \boldsymbol{v} + \boldsymbol{\nabla} \cdot \left(\boldsymbol{v}\, \boldsymbol{v}^T \right) - \nu \boldsymbol{\Delta} \boldsymbol{v} + \frac{1}{\rho} \nabla p + f_c \, \boldsymbol{e}_3 \times \boldsymbol{v} &= \begin{bmatrix} \widehat{\boldsymbol{f}} \\ -g \end{bmatrix} , \\
\nabla \cdot \boldsymbol{v} &= 0 \, .
\end{aligned}
$$

Unknowns: velocity vector $\boldsymbol{v} = \begin{bmatrix} \tilde{u} & \tilde{v} & \tilde{w} \end{bmatrix}^T$ and pressure p ;
parameters: constant fluid density ρ , constant kinematic viscosity ν , Coriolis coefficient f_c , horizontal body forces $\widehat{\boldsymbol{f}}$, and acceleration due to gravity g .

Enriched with a surface boundary condition for the $(x, y)-$dependent variable ξ

$$
\text{at } z = \xi : \quad \frac{\mathrm{d}(\xi - z)}{\mathrm{d}t} = 0 \quad \Rightarrow \quad \frac{\partial \xi}{\partial t} + \tilde{u}\Big|_{\xi} \frac{\partial \xi}{\partial x} + \tilde{v}\Big|_{\xi} \frac{\partial \xi}{\partial y} - \tilde{w}\Big|_{\xi} = 0 \, .
$$

The bottom boundary b has to be known to specify the geometry correctly.

Simplifications of the model

For the purpose of this talk let us assume...

- constant water density,
- no salinity or temperature variations,
- omission of diffusive terms (mainly due to turbulent mixing, not molecular friction).

Shallowness

Water depth $\ll$ horizontal diameter of the domain.
$\Rightarrow$ Analysis of scales: vertical pressure derivative balances the gravity force.
$\Rightarrow$ Hydrostatic pressure assumption (pressure is only due to water weight)

$$\frac{\partial p}{\partial z} = -\, g\, \rho\,,$$

$$\Rightarrow\ p(z) = p(\xi) - \int_z^\xi \frac{\partial\, p}{\partial \zeta}\, \mathrm{d}\zeta \ = \ p_a + g\,\rho\,(\xi - z)\,,$$

$\Rightarrow$ Omission of $z-$component of momentum equations.
$\Rightarrow$ The pressure is no longer an unknown in the system.

3D shallow-water equations

Using the notation $\boldsymbol{u} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}$ and $[\cdot]_{x,y}$ to denote the two horizontal components the system becomes

$$\frac{\partial \xi}{\partial t} + \tilde{u}\Big|_{\xi} \frac{\partial \xi}{\partial x} + \tilde{v}\Big|_{\xi} \frac{\partial \xi}{\partial y} - \tilde{w}\Big|_{\xi} = 0\,, \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \cdot \left(\boldsymbol{v}\,\boldsymbol{u}^T\right) - \nu\,\boldsymbol{\Delta}\,\boldsymbol{u} + g\,\nabla_{x,y}\,\xi + f_c\,[\boldsymbol{e}_3 \times \boldsymbol{v}]_{x,y} = \boldsymbol{f}\,, \tag{2}$$

$$\nabla \cdot \boldsymbol{v} = 0\,. \tag{3}$$

The unknowns are the velocity $\boldsymbol{v}$ as well as the free surface elevation ξ.
The right hand side $\boldsymbol{f}$ consists of $\widehat{\boldsymbol{f}}$ and the atmospheric pressure gradient.

System (1)–(3) is suitable for most applications and can be solved properly with a lower resolution than the full Navier-Stokes equations.

Challenges of solving the 3D shallow-water equations

- The free surface requires a moving 3D grid.
- Equation (1) is solved on the 2D surface boundary.
- Computational cost!

Integration over the total height of water

$$H = \xi - b\,, \qquad u := \frac{1}{H}\int_b^\xi \tilde{u}\,\mathrm{d}z\,, \qquad v := \frac{1}{H}\int_b^\xi \tilde{v}\,\mathrm{d}z\,.$$

$$\begin{aligned}
0 = \int_b^\xi \nabla\cdot\boldsymbol{v}\,\mathrm{d}z &= \int_b^\xi \left(\frac{\partial\tilde{u}}{\partial x} + \frac{\partial\tilde{v}}{\partial y} + \frac{\partial\tilde{w}}{\partial z}\right)\,\mathrm{d}z \\
&= \frac{\partial}{\partial x}\int_b^\xi \tilde{u}\,\mathrm{d}z - \tilde{u}\Big|_\xi \frac{\partial\xi}{\partial x} + \tilde{u}\Big|_b \frac{\partial b}{\partial x} \\
&\quad + \frac{\partial}{\partial y}\int_b^\xi \tilde{v}\,\mathrm{d}z - \tilde{v}\Big|_\xi \frac{\partial\xi}{\partial y} + \tilde{v}\Big|_b \frac{\partial b}{\partial y} \\
&\quad + \tilde{w}\Big|_\xi - \tilde{w}\Big|_b \\
&= \frac{\partial(u\,H)}{\partial x} + \frac{\partial(v\,H)}{\partial y} + \frac{\partial\xi}{\partial t} - \frac{\partial b}{\partial t} \\
&= \frac{\partial(u\,H)}{\partial x} + \frac{\partial(v\,H)}{\partial x} + \frac{\partial H}{\partial t}.
\end{aligned}$$

Depth-averaged shallow-water equations

$$
\begin{aligned}
\frac{\partial H}{\partial t} + \nabla \cdot (\boldsymbol{u}\, H) &= 0, \\
\frac{\partial (u\, H)}{\partial t} + \nabla \cdot (u\, \boldsymbol{u}\, H) + g\, H\, \frac{\partial \xi}{\partial x} + c_f\, |\boldsymbol{u}|\, u - f_c\, v\, H - H\, f_1 &= F_1\,. \\
\frac{\partial (v\, H)}{\partial t} + \nabla \cdot (v\, \boldsymbol{u}\, H) + g\, H\, \frac{\partial \xi}{\partial y} + c_f\, |\boldsymbol{u}|\, v + f_c\, u\, H - H\, f_2 &= F_2\,.
\end{aligned}
$$

Unknowns: $H = \xi - b$ (total height of water), with ξ free surface elevation, b bathymetry; $\boldsymbol{u} = \begin{bmatrix} u, & v \end{bmatrix}^T$ (depth-averaged horizontal velocity);
parameters: g (acceleration due to gravity), c_f (friction coefficient), f_c (Coriolis constant), and force terms $\boldsymbol{f} = \begin{bmatrix} f_1, & f_2 \end{bmatrix}^T$, $\boldsymbol{F} = \begin{bmatrix} F_1, & F_2 \end{bmatrix}^T$.
Further elimination is required:

$$
\begin{aligned}
g\, H\, \nabla \xi &= g\, H\, \nabla (H + b) \; = \; \frac{g}{2} \nabla H^2 + g\, H\, \nabla b \\
&= g(\xi - b) \nabla \xi - g\, \xi \nabla b + g\, \xi \nabla b \\
&= g\, \nabla \left(\xi \left(\frac{\xi}{2} - b \right) \right) + g\, \xi \nabla b.
\end{aligned}
$$

Conservative and well-balanced formulation

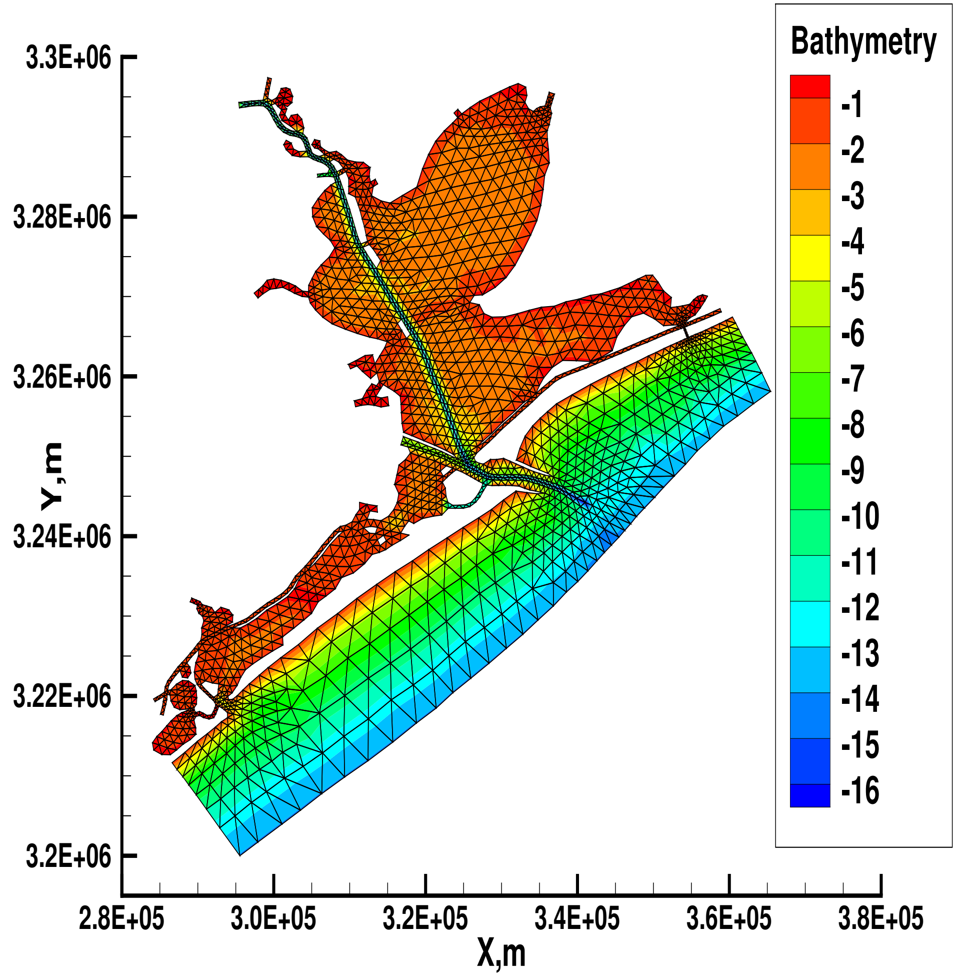
The conserved variables are the height of water and momentum.

$$\frac{\partial(\xi - b)}{\partial t} + \nabla \cdot (\boldsymbol{u}\, H) \;=\; 0\,,$$

$$\frac{\partial(\boldsymbol{u}\, H)}{\partial t} + \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{u}\, H \otimes \boldsymbol{u}\, H}{H} + \frac{g}{2} H^2\, \mathbb{1} \right) + g\, H \nabla b + c_f \left| \frac{\boldsymbol{u}\, H}{H^2} \right| \boldsymbol{u}\, H - f_c\, \boldsymbol{e}_3 \times \boldsymbol{u}\, H \;=\; \boldsymbol{0}\,.$$

flow boundary :	$\boldsymbol{u}\, H = (\boldsymbol{u}\, H)_D\,,$	
land boundary :	$\boldsymbol{u}\, H \cdot \boldsymbol{\nu} = 0\,,$	
outflow boundary :	$-$	
river boundary :	$H = H_D\,,$	$\boldsymbol{u}\, H = (\boldsymbol{u}\, H)_D\,,$
sea boundary :	$H = H_D\,,$	

Examples


Bathymetry
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
-11
-12
-13
-14
-15
-16
3.3E+06
3.28E+06
3.26E+06
3.24E+06
3.22E+06
3.2E+06
Y,m
2.8E+05
3.0E+05
3.2E+05
3.4E+05
3.6E+05
3.8E+05
X,m

Idea

- Standard shallow water equations use bathymetry as a parameter to produce approximations of free surface elevation.
- Idea in [1,2]: switch roles of parameter and solution,
- Benefit: classical inverse problem approach avoided.

Goal of the present work

Generalize the approach in [2] to incorporate **transient** flows, and use an **advanced finite element method** on **unstructured grids**.

[1] A.F. Gessese, M. Sellier, E. Van Houten and G. Smart: *Reconstruction of river bed topography from free surface data using a direct numerical approach in one-dimensional shallow water flow*, Inverse Problems, 2011.

[2] A.F. Gessese, M. Sellier: *A direct solution approach to the inverse shallow-water problem* (two-dimensional analog), Mathematical Problems in Engineering, 2012.

Characterization as PDE systems

Primary unknowns:

$$\boldsymbol{c}^T := \left[H \;\; (\boldsymbol{u}\, H)^T\right]^T =: \left[H \;\; U \;\; V\right]^T .$$

The forward $(\zeta = 1)$ and inverse $(\zeta = 0)$ problems can be expressed as

$$\partial_t \boldsymbol{c} + \boldsymbol{\nabla} \cdot \mathcal{A}^\zeta(\boldsymbol{c}) \;=\; \boldsymbol{Z}^\zeta(\boldsymbol{c}) \qquad \text{where}$$

$$\mathcal{A}^\zeta\left(\begin{bmatrix} H \\ U \\ V \end{bmatrix}\right) = \begin{bmatrix} U & V \\ \dfrac{U^2}{H} + \zeta\,\dfrac{g}{2} H^2 & \dfrac{U\,V}{H} \\ \dfrac{V\,U}{H} & \dfrac{V^2}{H} + \zeta\,\dfrac{g}{2} H^2 \end{bmatrix} ,$$

$$\boldsymbol{Z}^\zeta\left(\begin{bmatrix} H \\ U \\ V \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -g\,H\,\partial_x(\zeta\, b + (1-\zeta)\xi) - \tau_{bf}\, U + f_c\, V \\ -g\,H\,\partial_y(\zeta\, b + (1-\zeta)\xi) - \tau_{bf}\, V - f_c\, U \end{bmatrix} .$$

Characterization of the forward and inverse problem

Eigenvalues and eigenvectors of the first order operator

Let $a = \sqrt{g\,H}$ be the celerity, and let $\sqrt{x^2+y^2} = 1$. The eigenvalues and corresponding eigenvectors of $\frac{\partial}{\partial \boldsymbol{c}}\left(\mathcal{A}^{\zeta}(\boldsymbol{c})\begin{bmatrix} x \\ y \end{bmatrix}\right)$ are

forward $(\zeta = 1)$:

$$
\begin{aligned}
\lambda_0 &= \frac{U}{H}\,x + \frac{V}{H}\,y,\\
\boldsymbol{v}_0 &= \begin{bmatrix} 0, & -y, & x \end{bmatrix}^T,\\
\lambda_{1,2} &= \lambda_0 \pm a,\\
\boldsymbol{v}_{1,2} &= \begin{bmatrix} 1, & \frac{U}{H} \pm a\,x, & \frac{V}{H} \pm a\,y \end{bmatrix},
\end{aligned}
$$

fully hyperbolic

inverse $(\zeta = 0)$:

$$
\begin{aligned}
\lambda_0 &= \frac{U}{H}\,x + \frac{V}{H}\,y,\\
\boldsymbol{v}_0 &= \begin{bmatrix} 0, & -y, & x \end{bmatrix}^T,\\
\lambda_{1,2} &= \lambda_0,\\
\boldsymbol{v}_1 &= \begin{bmatrix} 1, & \frac{U}{H}, & \frac{V}{H} \end{bmatrix},
\end{aligned}
$$

degenerate hyperbolic.

Discretization of the forward problem

Discontinuous Galerkin finite element space

$$\mathbb{V}_h^{DG} := \left\{ \varphi_h \in L^1(\Omega) \,:\, \varphi_h\Big|_T (x_1, x_2) = a\,x_1 + b\,x_2 + c\,,\ a, b, c \in \mathbb{R}\ \forall T \in \mathcal{T}_h \right\}.$$

Seek $\boldsymbol{c}_h(t) \in \left(\mathbb{V}_h^{DG}\right)^3$, such that for almost all $t \in (t_0, t_{end})$, $\forall\, T^- \in \mathcal{T}_h$, with outer unit normal $\boldsymbol{\nu}_{T^-}$, and $\forall\, \boldsymbol{\psi}_h \in \mathbb{P}_1(T^-)^3$, the following holds:

$$\int_{T^-} \boldsymbol{\psi}_h \cdot \partial_t \boldsymbol{c}_h(t)\, \mathrm{d}\boldsymbol{x} = \int_{T^-} \boldsymbol{\nabla}\boldsymbol{\psi}_h : \mathcal{A}^1(\boldsymbol{c}_h(t))\, \mathrm{d}\boldsymbol{x} - \int_{\partial T^-} \boldsymbol{\psi}_h \cdot \widehat{\boldsymbol{A}^{\boldsymbol{\nu}_{T^-}}}(\boldsymbol{c}_h^-(t), \boldsymbol{c}_h^+(t))\, \mathrm{d}s$$
$$+ \int_{T^-} \boldsymbol{\psi}_h \cdot \boldsymbol{Z}^1(\boldsymbol{c}_h(t))\, \mathrm{d}\boldsymbol{x}\,,$$

- $\widehat{\boldsymbol{A}^{\boldsymbol{\nu}_{T^-}}}(\boldsymbol{c}_h^-(t), \boldsymbol{c}_h^+(t))$ describes the advective transport over ∂T^- in direction $\boldsymbol{\nu}_{T^-}$ and is a solution to a Riemann problem.
- Time stepping is performed using the SSP Runge-Kutta method of order 2, i.e. Heun's method.

Flux approximation

Lax-Friedrichs approximation and Roe-Pike mean state

On an edge $E \subset \partial T^-$ of element T^-, let the one-sided limits of $\boldsymbol{c}_h$ be denoted by $\boldsymbol{c}_h^-$, $\boldsymbol{c}_h^+$, and let $\boldsymbol{\nu}_{T^-}$ be the outer unit normal to T^-. We define $\lambda = \lambda(\boldsymbol{c}_h\,, \boldsymbol{\nu}_{T^-}\,, g)$ as the largest absolute eigenvalue of

$$\frac{\partial}{\partial \boldsymbol{c}}(\mathcal{A}^1(\boldsymbol{c})\,\boldsymbol{\nu}_{T^-})\,, \quad \text{which gives} \quad \lambda \;=\; \left|\begin{bmatrix}\frac{U}{H} & \frac{V}{H}\end{bmatrix}\boldsymbol{\nu}_{T^-}\right| + \sqrt{gH}\,.$$

Due to discontinuity we need certain averages of the unknowns

$$\hat{\lambda} \;=\; \hat{\lambda}(\boldsymbol{c}_h\,, \boldsymbol{\nu}_{T^-}\,, g) \;\coloneqq\; \left|\begin{bmatrix}\dfrac{\sqrt{H_h^+}\,U_h^- + \sqrt{H_h^-}\,U_h^+}{H_h^-\sqrt{H_h^+} + H_h^+\sqrt{H_h^-}} \\[2ex] \dfrac{\sqrt{H_h^+}\,V_h^- + \sqrt{H_h^-}\,V_h^+}{H_h^-\sqrt{H_h^+} + H_h^+\sqrt{H_h^-}}\end{bmatrix}\cdot\boldsymbol{\nu}_{T^-}\right| + \sqrt{\frac{g}{2}(H_h^- + H_h^+)}\,.$$

The advective Lax-Friedrichs flux on int(E) in direction $\boldsymbol{\nu}_{T^-}$ is

$$\widehat{\boldsymbol{A}^{\boldsymbol{\nu}_{T^-}}}(\boldsymbol{c}_h^-, \boldsymbol{c}_h^+) \;\coloneqq\; \frac{1}{2}\Big(\big(\mathcal{A}^1(\boldsymbol{c}_h^-) + \mathcal{A}^1(\boldsymbol{c}_h^+)\big)\,\boldsymbol{\nu}_{T^-} + \hat{\lambda}\,\big(\boldsymbol{c}_h^- - \boldsymbol{c}_h^+\big)\Big)\,.$$

Designing the scheme for the inverse problem

- The continuous nature of bathymetry motivates the use of a continuous finite element space for the inverse problem.
- Steady states of the forward problem are desired to be preserved in the inverse problem as well.

Consequences

- A combination of the CG method for bathymetry and a DG method for momentum may be useful.
- As long as the same order of polynomial spaces is used, the subspace property $\mathbb{V}_h^{CG} \subseteq \mathbb{V}_h^{DG}$ holds.
- The flux terms should be the same in the momentum equations of forward and inverse problem.
- Some contributions of the fluxes in the inverse problem consist only of the known free surface elevation.

Discretization of the continuity equation

The primary unknown must be the bathymetry, due to the use of a continuous basis.

$$\partial_t b - \nabla \cdot (\boldsymbol{u}\, H) = \partial_t \xi \,.$$

For a triangular grid with L vertices $\boldsymbol{x}_i$ and K elements let $\varphi_1, \ldots, \varphi_L$ be the piecewise linear nodal basis functions of the linear Lagrange elements, i.e. $\varphi_i(\boldsymbol{x}_j) = \delta_{ij}$ and $\{\psi_{kj} :\ k = 1, \ldots, K,\ j = 1, 2, 3\}$ a basis of the DG space, with supp $\psi_{kj} = T_k$.

The semi-discrete formulation is

$$\sum_{j=1}^{L} \partial_t b_j \int\limits_{\Omega} \varphi_i \varphi_j + \sum_{k=1}^{K} \sum_{l=1}^{3} (\boldsymbol{u}\, H)_{kl} \cdot \int\limits_{T_k} \nabla \varphi_i \, \psi_{kl} - \int\limits_{\partial\Omega} \varphi_i \, \boldsymbol{u}\, H \cdot \boldsymbol{\nu} \ = \ \sum_{k=1}^{K} \sum_{l=1}^{3} \partial_t \xi_{kl} \int\limits_{T_k} \varphi_i \, \psi_{kl} \,,$$

or written in matrix-vector form with rectangular matrices $\mathbf{A}_{1,2}, \mathbf{M}_m \in \mathbb{R}^{L \times 3K}$

$$\mathbf{M}\, \partial_t \boldsymbol{b} + \mathbf{A}_1 \boldsymbol{U} + \mathbf{A}_2 \boldsymbol{V} = \mathbf{M}_m\, \partial_t \boldsymbol{\xi} + \boldsymbol{f}$$

- Interior edge contributions cancel due to the continuity of the test functions.
- $\boldsymbol{f}$ combines the contributions of boundary integrals where $\boldsymbol{u}\, H$ is prescribed.

Solving the continuity equation

- Again, Heun's method is employed for time-stepping.
- Replacing the consistent mass matrix $\mathbf{M}$ by the lumped mass matrix $\mathbf{M}_L$ means no linear system has to be solved.
- The ill-conditioning of the inverse problem requires the use of regularization.
- Adding diffusive terms in the continuity equation results in the desired stabilization.
- The difference between lumped and consistent mass matrix can be seen as a discretization of the Laplace operator.

Hence, the equation becomes

$$\mathbf{M}_L\,\partial_t \boldsymbol{b} + \mathbf{A}_1 \boldsymbol{U} + \mathbf{A}_2 \boldsymbol{V} + \beta(\mathbf{M}_L - \mathbf{M})\,\boldsymbol{b} = \mathbf{M}_m\,\partial_t \boldsymbol{\xi} + \boldsymbol{f}\,,$$

and is solved as follows

$$\begin{aligned}
\boldsymbol{b}^* &= \boldsymbol{b}^n + \Delta t_n \mathbf{M}_L^{-1}\left(\mathbf{M}_m \frac{\boldsymbol{\xi}^* - \boldsymbol{\xi}^n}{\Delta t_n} + \boldsymbol{f}^n - \mathbf{A}_1 \boldsymbol{U}^n - \mathbf{A}_2 \boldsymbol{V}^n - \beta(\mathbf{M}_L - \mathbf{M})\,\boldsymbol{b}^n\right)\,,\\
\boldsymbol{b}^{n+1} &= \frac{1}{2}\boldsymbol{b}^n + \frac{1}{2}\boldsymbol{b}^* + \frac{\Delta t_n}{2}\mathbf{M}_L^{-1}\left(\mathbf{M}_m \frac{\boldsymbol{\xi}^{n+1} - \boldsymbol{\xi}^*}{\Delta t_n} + \boldsymbol{f}^{n+1} - \mathbf{A}_1 \boldsymbol{U}^* - \mathbf{A}_2 \boldsymbol{V}^* - \beta(\mathbf{M}_L - \mathbf{M})\,\boldsymbol{b}^*\right)\,.
\end{aligned}$$

Forward problem setup

space-time cylinder:

$$\Omega = (0,1) \times (0,1) \quad [km] \qquad (t_0, t_{end}) = (0,3) \quad [h]$$

initial conditions:

$$\xi_{(0)} \equiv 0 \quad [m] \qquad (\boldsymbol{u}\,H)_{(0)} \equiv \begin{bmatrix} 4 & 0 \end{bmatrix}^T \quad [m^2/s]$$

boundary conditions:

$$(\boldsymbol{u}\,H)(t) \cdot \boldsymbol{\nu} \equiv 0 \quad [m^2/s] \quad \text{on } (0,1000) \times \{0,1\}$$

$$(\boldsymbol{u}\,H)_D(t) \equiv \begin{bmatrix} 4 & 0 \end{bmatrix}^T \quad [m^2/s] \quad \text{on } \{0\} \times (0,1000)$$

$$\xi(t) \equiv 0 \quad [m] \quad \text{on } \{1\} \times (0,1000)$$

physical parameters:

$$g = 9.81 \quad [m/s^2] \qquad f_c = 3.0 \cdot 10^{-5} \quad [1/s]$$

$$c_f = 10^{-3} \quad [1/s]$$

numerical parameters:

$$h = 40 \quad [m] \qquad \Delta t = 0.1 \quad [s]\,.$$

Spatial grid over the exact bathymetry

$$b(x,y) \;=\; \Pi^{L^2}_{V_h^{CG}(\Omega_h)}\left(-2-\left(1-e^{-2\cdot 10^{-5}\,(x-800)^2-10^{-5}\,(y-700)^2}\right)\,b_r(x,y)\right)\,.$$

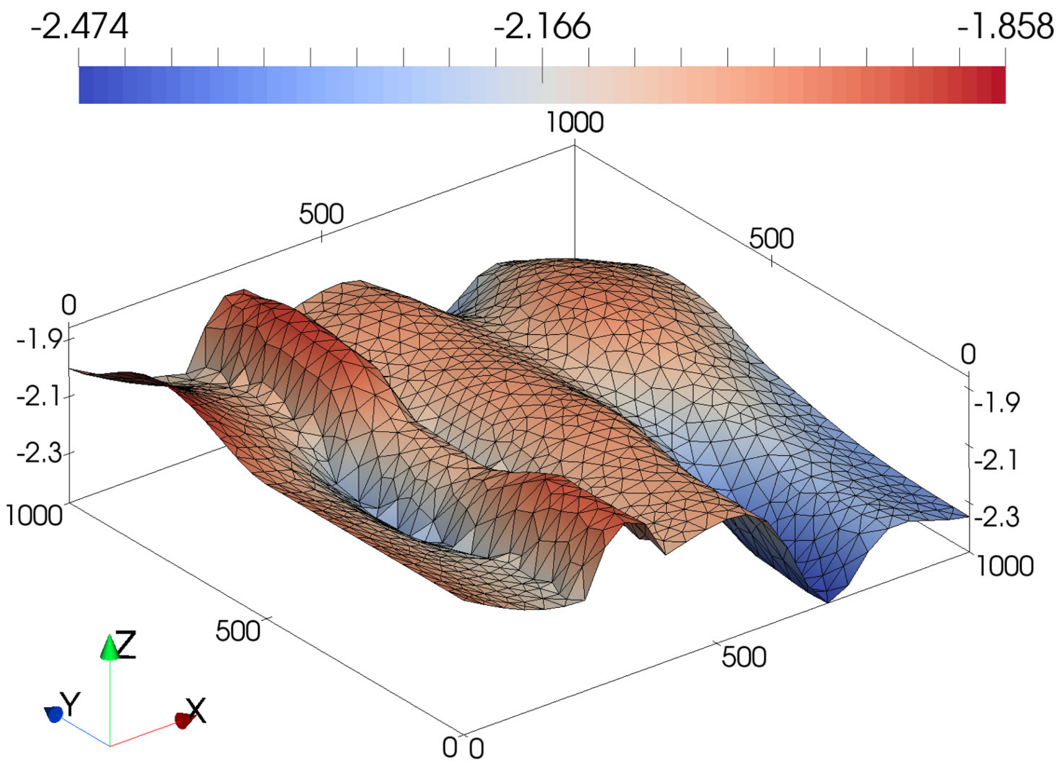
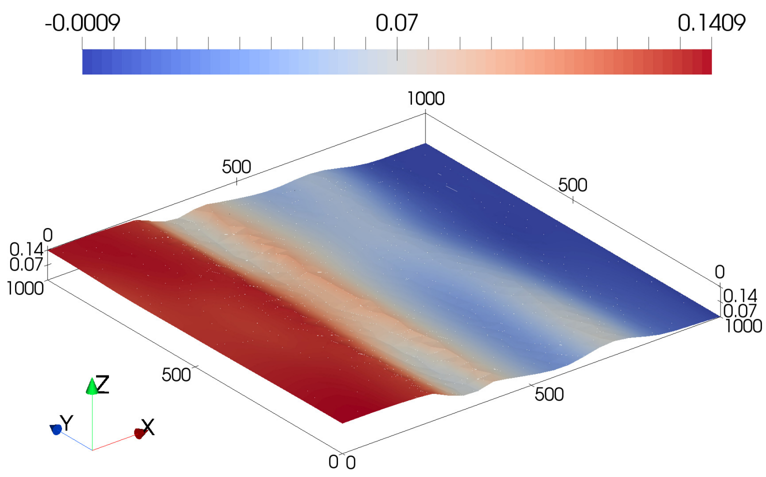


Figure: The analytical solution of the inverse problem

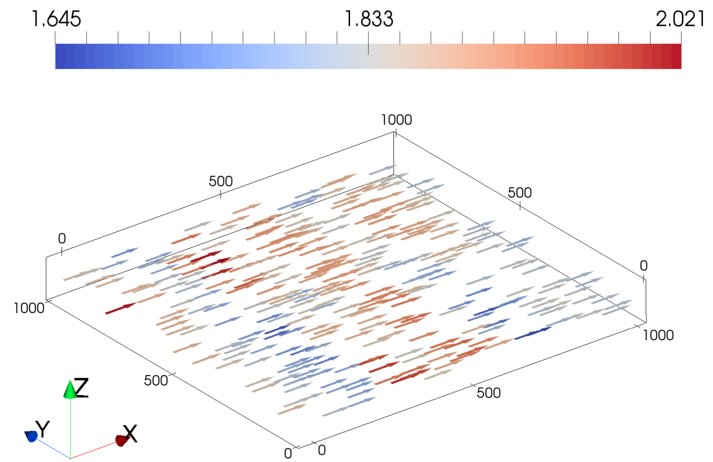
Stationary solution of the forward problem

Convergence criterion:

$$\left\|\xi^{n+1}-\xi^{n}\right\|_{l^2(\mathcal{T}_h)} := \left(\sum_{k=1}^{K}\sum_{j=1}^{3}\left(\xi_{kj}^{n+1}-\xi_{kj}^{n}\right)^2\right)^{\frac{1}{2}} < \varepsilon_f = 10^{-8}\,[m]\,.$$



(a) Steady-state surface elevation



(b) Steady-state velocity (coloring corresponds to the magnitude of the velocity vector)

Solution of the inverse problem

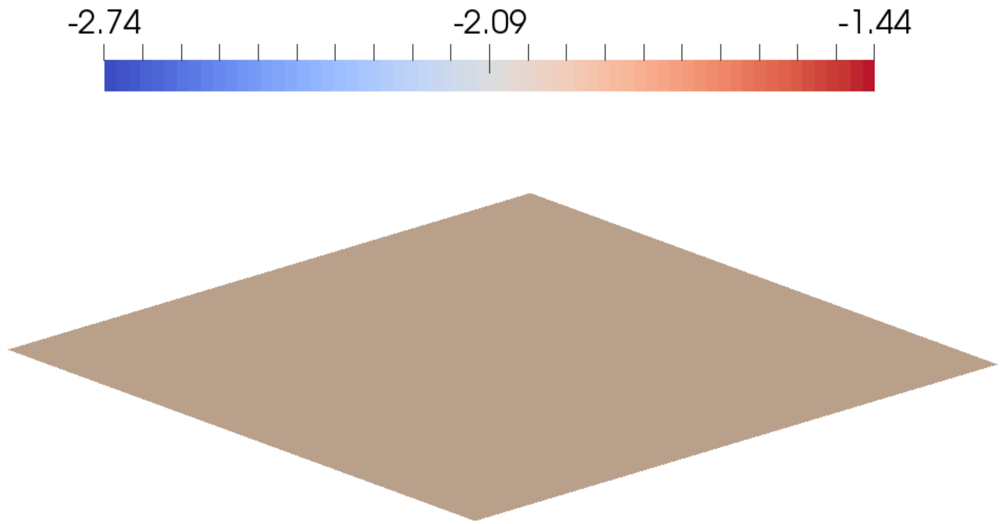


Figure: Bathymetry reconstruction

Convergence criterion:

$$\left\|b^{n+1} - b^n\right\|_{l^2(\mathcal{T}_h)} < \varepsilon_i = 10^{-8}\,[m]\,.$$

Convergence according to this criterion is reached after 246473 iterations. The $l^\infty(\mathcal{T}_h)-$error is $7.85\,\mu m$.

Time-dependent setup and forward problem solution

Change above configuration to include the time-dependent flow boundary condition

$$(uH)_D = \left[0.5\,\cos\left(\frac{\pi\, t}{86400}\right) + 3.5 \;\; 0\right]^T, \quad \text{and use} \quad (\boldsymbol{u}\, H)_{(0)} \equiv \left[4 \;\; 0\right]^T.$$

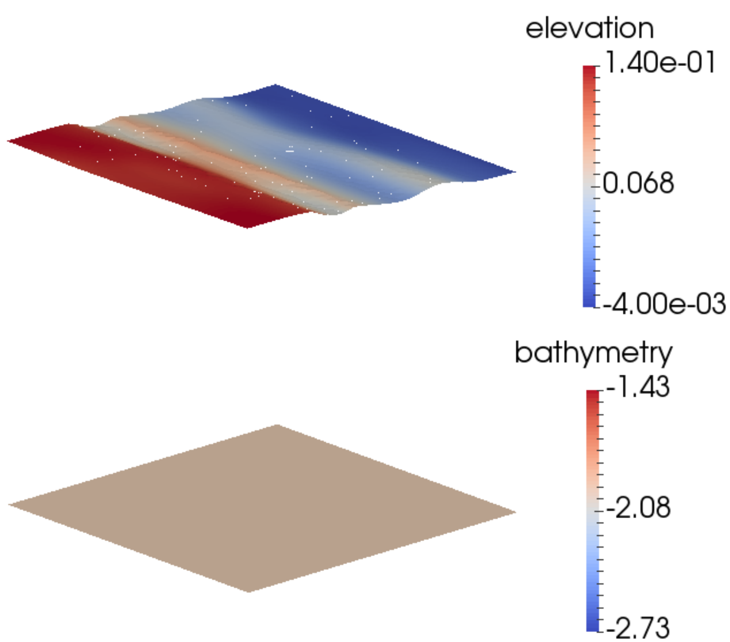
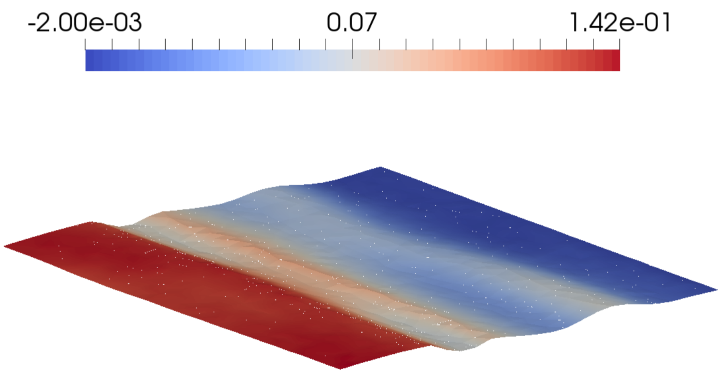


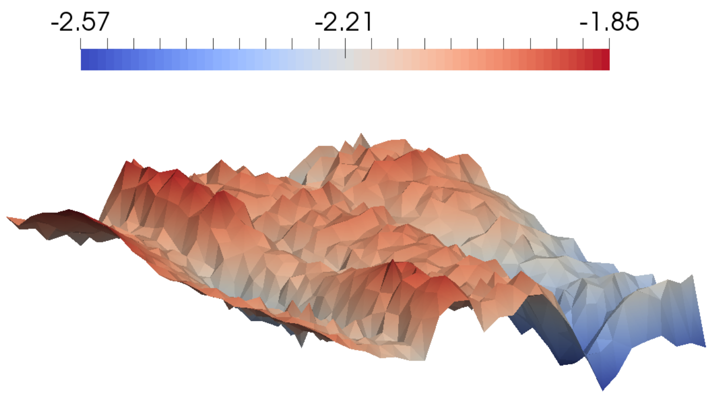
Figure: Evolving free surface (top) and bathymetry (bottom)

Ill-conditioning of the inverse problem

- For absolute an noise level $\iota > 0$, add uniformly distributed random numbers in $(-\iota,\, \iota)$ to the free surface values in the grid points. A representative result of the reconstruction for $\iota = 10^{-4}\, m$ is seen below.



(a) Perturbed solution of the forward problem

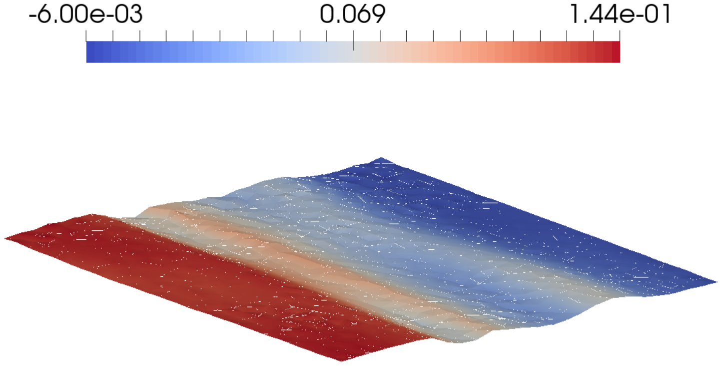


(b) Bathymetry reconstruction from noisy data

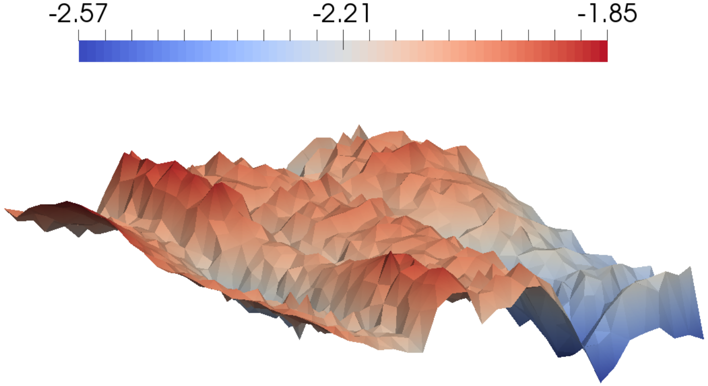
- The amplification of data errors in the reconstruction indicates the ill-posedness of the problem.
- The result gets even worse on refined grids.

Observations on reconstruction from noisy data

- A steady-state is reached if ι is not too large.
- We observe a one-way compatibility consistent to the subspace property $\mathbb{V}_h^{CG} \subseteq \mathbb{V}_h^{DG}$: Using the osciallatory bathymetry as input for the forward problem, we get a different result, than the original perturbed surface elevation.
- This new steady-state reconstructs the osciallatory bathymetry accurately.



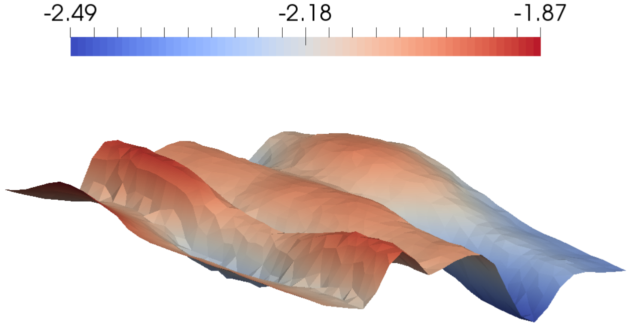
(c) Solution of the forward problem with osciallatory bathymetry



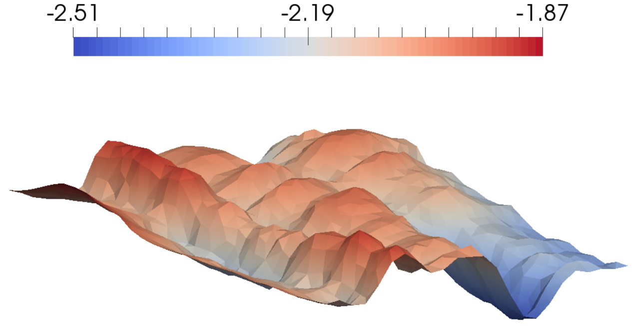
(d) Reconstruction of oscillatory bathymetry

Regularization with artificial diffusion

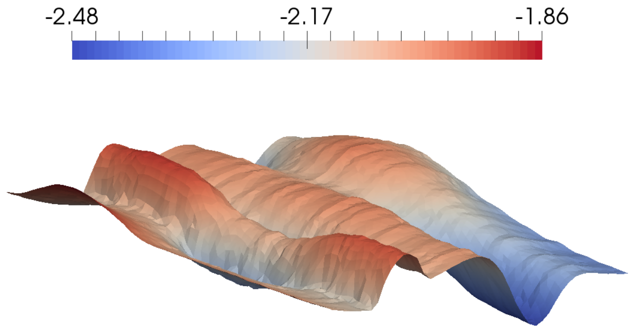
Previous results were computed with regularization parameter $\beta = 0$. Using $\beta = 80\,\frac{m^2}{s}\,\iota$ on the coarse and $\beta = 500\,\frac{m^2}{s}\,\iota$ on the refined grid we get the following results.



(e) Reconstruction on the coarse grid for $\iota = 0.1\,mm\,,\ \beta = 0.008\,m^2/s$



(f) Reconstruction on the coarse grid for $\iota = 1\,mm\,,\ \beta = 0.08\,m^2/s$



(g) Reconstruction on the fine grid for $\iota = 0.1\,mm\,,\ \beta = 0.05\,m^2/s$

case	iterations	$l^\infty(\mathcal{T}_h)$-error in cm
e	11532	0.94
f	13734	7.49
g	19725	1.90

Summary and outlook

Summary:

- Design of a new coupled CG-DG method for bathymetry reconstruction from a modification of the shallow water equations.
- The discretization ensures compatibility to the DG scheme for the forward problem in the case $\mathbb{V}_h^{CG} \subseteq \mathbb{V}_h^{DG}$.
- The examples show the potential of the methodology.

Outlook:

- The optimal choice of regularization parameter β needs to be investigated.
- Different types of artificial diffusion, such as TV-regularization should be compared.
- Investigation of applicability using experiments.