Hierarchical Solution Concepts for Flow Control Problems

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1st period: Flow Control Model Problem

Distributed Control for the nonstationary Navier-Stokes equation with tracking-type functional for a given z:

$$J(y,u) = \frac{1}{2}||y-z||_Q^2 + \frac{\alpha}{2}||u||_Q^2 + \frac{\gamma}{2}||y(T)-z(T)||_\Omega^2 \quad \rightarrow \quad \text{min!}$$

on $Q = \Omega \times [0,T]$ such that

$$\begin{array}{rcl} y_t - \nu\Delta y + (y\nabla)y + \nabla p & = & u \quad \text{in } Q \\ -\nabla \cdot y & = & 0 \quad \text{in } Q \end{array} \quad + \text{BC}$$

No constraints.

Design goals for optimal control tools

Moderate performance measure; for C not too large (≈ 10):

$$\frac{\text{costs for optimization}}{\text{costs for simulation}} \leq C$$

By modern numerical CFD techniques
($\rightarrow$ special FEM on solution adapted grids, MG+Newton solvers)

$$\text{costs for simulation} = O(N)$$

Aim:
$$\text{costs for optimization} \overset{!}{=} O(N)$$

Corresponding KKT-System (unconstrained case):

$$\begin{aligned} y_t - \nu \Delta y + (y\nabla)y + \nabla p &= u & \text{in } Q \\ -\nabla \cdot y &= 0 & \text{in } Q \\ -\lambda_t - \nu \Delta \lambda - (y\nabla)\lambda + (\nabla y)^t \lambda + \nabla \xi &= (y - z) & \text{in } Q \\ -\nabla \cdot \lambda &= 0 & \text{in } Q \\ u &= -\frac{1}{\alpha}\lambda & \text{in } Q \end{aligned}$$

+ boundary conditions
+ initial condition
+ terminal condition $\lambda(T) = \gamma(y(T) - z(T))$ in Ω

Corresponding KKT-System (unconstrained case):

$$\begin{aligned} y_t + N(y)y + \nabla p + \tfrac{1}{\alpha}\lambda &= 0 && \text{in } Q \\ -\nabla \cdot y &= 0 && \text{in } Q \\ \\ -\lambda_t + N^*(y)\lambda + \nabla \xi - y &= -z && \text{in } Q \\ -\nabla \cdot \lambda &= 0 && \text{in } Q \end{aligned}$$

+ boundary conditions
+ initial condition
+ terminal condition $\lambda(T) = \gamma(y(T) - z(T))$ in Ω

Properties of the KKT system

Observation:

- KKT-system $\longrightarrow$ *elliptic* BVP in space/time

Idea:

- Apply modern $O(N)$ ingredients from CFD (Multigrid + Newton) to this BVP!

Feasible?

Space–time discretization

Discretization in space+time leads to a system

$$A(x)x = b$$

in the form (here e.g. for 2 timesteps):

$$A(x)x =$$

$$\left(\begin{array}{cccc|cccc|cccc}
NST & & -B & & & & & & & & & \\
-M & NST^* & & -B & & -\frac{M}{\Delta t} & & & & & & \\
-B^T & & 0 & & & & & & & & & \\
& -B^T & & 0 & & & & & & & & \\
\hline
-\frac{M}{\Delta t} & & & & NST & \frac{M}{\alpha} & -B & & & & & \\
& & & & -M & NST^* & & -B & & -\frac{M}{\Delta t} & & \\
& & & & -B^T & & 0 & & & & & \\
& & & & & -B^T & & 0 & & & & \\
\hline
& & & & -\frac{M}{\Delta t} & & & & NST & \frac{M}{\alpha} & -B & \\
& & & & & & & & -c(\gamma, \Delta t)M & NST^* & & -B \\
& & & & & & & & -B^T & & 0 & \\
& & & & & & & & & -B^T & & 0
\end{array}\right)
\left(\begin{array}{c}
y_0 \\ \lambda_0 \\ p_0 \\ \xi_0 \\ \hline y_1 \\ \lambda_1 \\ p_1 \\ \xi_1 \\ \hline y_2 \\ \lambda_2 \\ p_2 \\ \xi_2
\end{array}\right)$$

$\rightarrow$ Sparse, (block) tridiagonal system

Design of an efficient optimal control solver

- Nonlinearity: Newton method for quadratic convergence

$$x^{i+1} = x^i + A'^{-1}(x^i)(b - A(x^i)x^i)$$

- Linear subproblems: space-time Multigrid solver

  → using Block-Jacobi/Block-SOR smoothing techniques

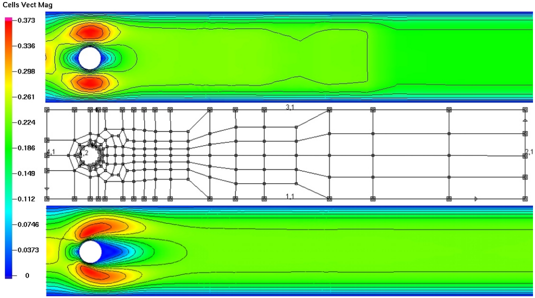
- Linear subproblems in space: Monolithic Multigrid solver

  → using 'local Pressure-Schur complement' techniques
  in each timestep for the coupled Navier–Stokes subproblems

1st period: *Proof of Concept + Prototype for low Re* $\surd$

Flow-around-cylinder

- Target flow z: Stokes flow, $t \in [0, 1]$, starting from rest



Stokes at $t = 1$

Mesh

uncontrolled Nav.St.

- Optimal control problem: Navier–Stokes, $Re = 20$
- Coarse mesh: Standard DFG benchmark
  $\rightarrow$ 1404 DOF's in space, 5 timesteps, $\Delta t := 0.2$
  $\Rightarrow$ 8 424 DOF's, $\times 8$ per level

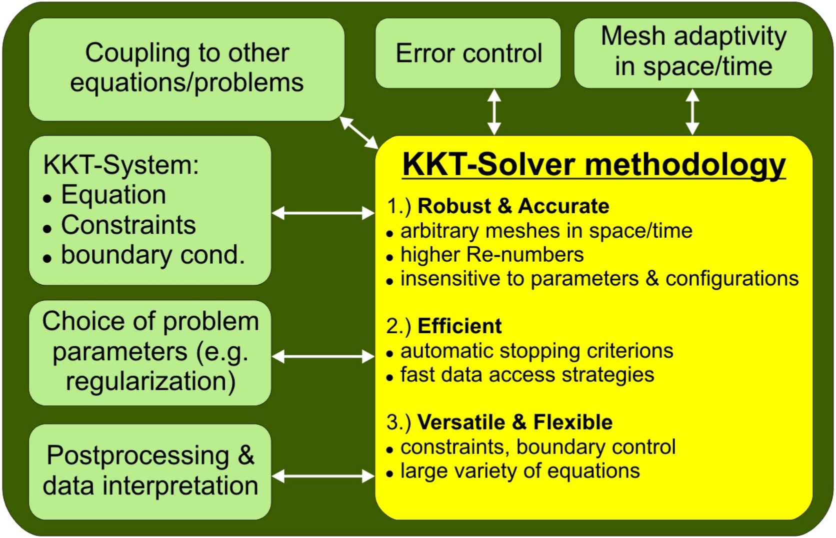
Numerical example

Convergence of the Newton solver

		Simulation				Optimisation	
Δt	Space-Lv.	#NL	#MG	∅#NL	∅#MG	#NL	#MG
1/20	3	63	312	3	16	4	47
1/40	4	123	709	3	18	4	14
1/80	5	246	1589	3	20	4	9

- Nonlinear solver gained 5 digits
- Space-time MG gained 2 digits per step
- Space-preconditioner gained 2 digits per step

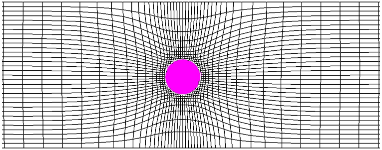
The role of the project in flow control

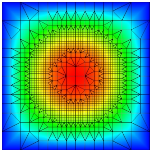


Main goal 1: Higher accuracy + stability

Realization & numerical analysis of

- higher order discretisations in space/time (Q_2/P_1, $\tilde{Q}_2/P_1$, CN, FS-θ)
- r/h-adaptivity in space/time $\rightarrow$ convergence properties?





- stabilization techniques (for higher Re-numbers)
  - e.g. EO-stabilisation (only in space!)

$$j(u,v) = \sum_{\text{edge}E} \gamma |E|^2 \int_E [\nabla u][\nabla v] d\sigma$$

$\rightarrow$ Effect on the solution?

Main goal 2: Improved solver efficiency

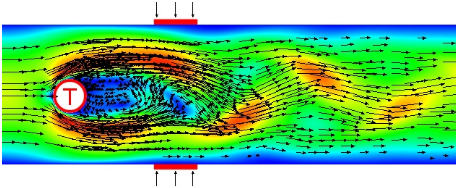
- Design & numerical analysis of space-time smoothers/solvers
  - MG convergence properties for $Q_2/P_1 + CN$ in the Stokes case
  - Coupling between Δt and h?
- Numerical analysis of stopping criteria for inner/outer solvers
- HPC techniques, parallel computing
  - Data management & parallelization; e.g. Block Jacobi preconditioner

$$A(x) = \begin{pmatrix} N(x_0) & M & & \\ M & N(x_1) & M & \\ & M & N(x_3) & M \\ & & & \ddots \end{pmatrix} \Rightarrow \tilde{A}(x) = \begin{pmatrix} N(x_0) & & & \\ & N(x_1) & & \\ & & N(x_3) & \\ & & & \ddots \end{pmatrix}$$

$\Rightarrow$ Simultaneous calculation of timesteps

Main goal 3: Towards real-life problems

- Complex flow geometries + higher *Re* numbers



- Constraints + Boundary Control

- Non-Newtonian + Non-isothermal flow

$$\begin{aligned} y_t - \nabla \cdot (\nu(y,\Theta)\mathbf{D}(y)) + y\nabla y + \nabla p - Gr\Theta g &= f_1 \\ \Theta_t - (1/Pr)\Delta\Theta + y\nabla\Theta &= f_2 \end{aligned}$$

- Preparation for 3D $\longrightarrow$ Basically the same concept, but...

Summary & Outlook

1st period: Proof of concept

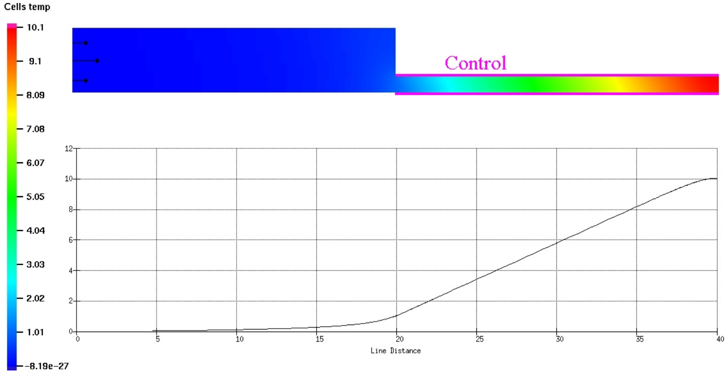
- Design & numerical analysis of a flow solver prototype
- Support for low *Re* number flow problems

2nd period:

- Design & numerical analysis of efficient + robust solver components
- Support for realistic, nonstationary flow problems

Example: Temperature by friction

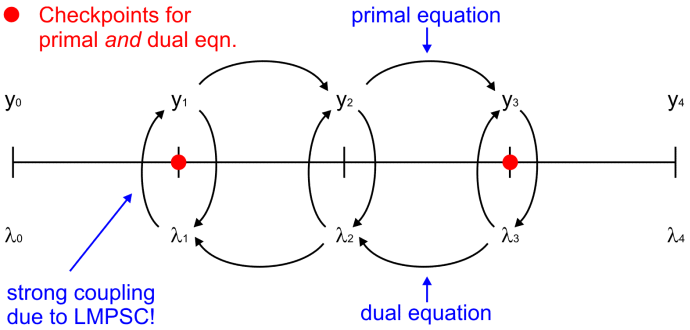
Friction leads to a temperature increase in the rear area of the channel:



$$\begin{aligned} y_t - \nabla \cdot (\nu(y,\Theta)\mathbf{D}(y)) + y\nabla y + \nabla p - Gr\Theta g &= f_1 \\ \Theta_t - (1/Pr)\Delta\Theta + y\nabla\Theta + D(y) : D(y) &= f_2 \end{aligned}$$

$\rightarrow$ How to control to prevent the temperature increase?

Checkpointing in the One-shot approach



- Checkpoints $\rightarrow$ nonlinear subproblems of the same kind.
- High computational costs necessary for recomputation
  $\rightarrow$ due to strong coupling by LPSC!