Hierarchical Solution Concepts for Flow Control Problems

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Flow Control Model Problem

Distributed Control for the nonstationary Navier-Stokes equation with tracking-type functional for a given z:

$$J(y,u) = \frac{1}{2}||y - z||_Q^2 + \frac{\alpha}{2}||u||_Q^2 + \frac{\gamma}{2}||y(T) - z(T)||_\Omega^2 \quad \rightarrow \quad \text{min!}$$

on $Q = \Omega \times [0, T]$ such that

$$\begin{array}{rcll} y_t - \nu\Delta y + (y\nabla)y + \nabla p & = & u & \text{in } Q \\ -\nabla \cdot y & = & 0 & \text{in } Q \end{array} \quad + \text{BC}$$

No constraints.

Corresponding KKT-System (unconstrained case) for (y, p, λ, ξ):

$$
\begin{aligned}
y_t - \nu\Delta y + (y\nabla)y + \nabla p &= -\frac{1}{\alpha}\lambda && \text{in } Q \\
-\lambda_t - \nu\Delta\lambda - (y\nabla)\lambda + (\nabla y)^t\lambda + \nabla\xi &= (y - z) && \text{in } Q
\end{aligned}
$$

+ incompressibility $(-\nabla \cdot y = -\nabla \cdot \lambda = 0)$
+ initial/boundary/terminal cond. $(\lambda(T) = \gamma(y(T) - z(T)))$

Project aim: Solve with $\begin{cases} \text{costs for optimisation} = O(N), \\ \dfrac{\text{costs for optimisation}}{\text{costs for simulation}} \le C \approx 10 - 50 \end{cases}$

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Multigrid solver methodology

technische universität dortmund

Observation:

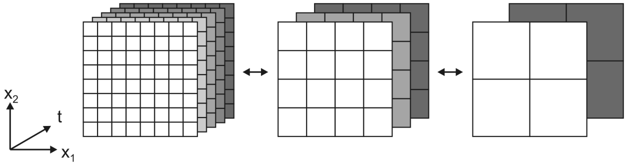
- KKT-system $\rightarrow$ *elliptic* BVP in space/time $(-y_{tt} + \Delta^2 y + \ldots)$

Idea:

- Apply 'optimal' $O(N)$ ingredients from CFD to this BVP!

  $\rightarrow$ unstructured meshes, FEM in space, implicit time stepping
  $\rightarrow$ monolithic Multigrid + Newton solver techniques

- In particular: Solve on a space-time hierarchy
  $\rightarrow$ Space = FE, Time = $\theta-$scheme



Current topics and aims

Overview:

- Higher order discretisation in time: $\tilde{Q}_1$/CN
  $\rightarrow$ Discretisation + Solver components for *Crank-Nicolson*

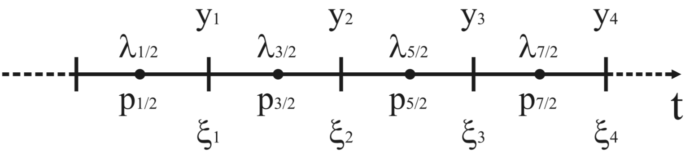
- Increase of Efficiency+Stability
  $\rightarrow$ MG cycle analysis

Crank-Nicolson in time

Special CN discretisation (Stokes case):

$$\frac{y_n - y_{n-1}}{\Delta t} + (-\frac{1}{2}\Delta y_n - \frac{1}{2}\Delta y_{n-1}) + \nabla p_{n-\frac{1}{2}} = -\frac{1}{\alpha}\lambda_{n-\frac{1}{2}}$$

$$\frac{\lambda_{n-\frac{1}{2}} - \lambda_{n+\frac{1}{2}}}{\Delta t} + (-\frac{1}{2}\Delta\lambda_{n-\frac{1}{2}} - \frac{1}{2}\Delta\lambda_{n+\frac{1}{2}}) + \nabla\xi_n = y_n - z_n$$



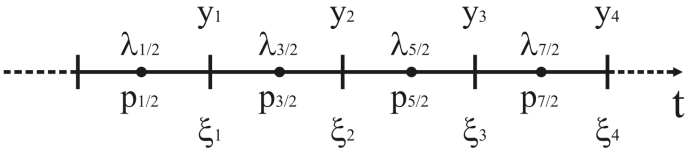
$$\lambda_{n-\frac{1}{2}} \approx \frac{1}{2}\lambda_n + \frac{1}{2}\lambda_{n+1}, \qquad y_n - z_n \approx \frac{1}{2}(y_{n-\frac{1}{2}} - z_{n-\frac{1}{2}}) + \frac{1}{2}(y_{n+\frac{1}{2}} - z_{n+\frac{1}{2}})$$

[Flaig '09]

Crank-Nicolson in time

Special CN discretisation (Stokes case):

$$
\begin{aligned}
\frac{y_n - y_{n-1}}{\Delta t} + \left(-\frac{1}{2}\Delta y_n - \frac{1}{2}\Delta y_{n-1}\right) + \nabla p_{n-\frac{1}{2}} &= -\frac{1}{\alpha}\lambda_{n-\frac{1}{2}} \\
\frac{\lambda_{n-\frac{1}{2}} - \lambda_{n+\frac{1}{2}}}{\Delta t} + \left(-\frac{1}{2}\Delta \lambda_{n-\frac{1}{2}} - \frac{1}{2}\Delta \lambda_{n+\frac{1}{2}}\right) + \nabla \xi_n &= y_n - z_n
\end{aligned}
$$



$$
\lambda_{n-\frac{1}{2}} \approx \frac{1}{2}\lambda_n + \frac{1}{2}\lambda_{n+1}, \qquad y_n - z_n \approx \frac{1}{2}\left(y_{n-\frac{1}{2}} - z_{n-\frac{1}{2}}\right) + \frac{1}{2}\left(y_{n+\frac{1}{2}} - z_{n+\frac{1}{2}}\right)
$$

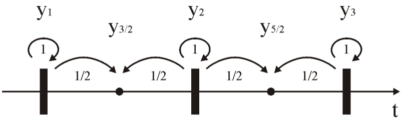
[Flaig '09]

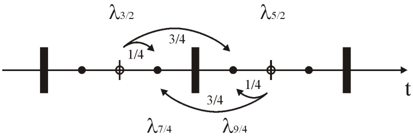
Prolongation/Restriction for Crank-Nicolson

Modified prolongation/restriction

Define prolongation matrices in time:

$$P_p := \begin{pmatrix} \cdots & & & \\ & 1 & & \\ & 1/2 & 1/2 & \\ & & 1 & \\ & & & \cdots \end{pmatrix}, \qquad P_d := \begin{pmatrix} \cdots & & & \\ & 3/4 & 1/4 & \\ & 1/4 & 3/4 & \\ & & 3/4 & 1/4 \\ & & & \cdots \end{pmatrix}$$





→ Linear interpolation in time

Prolongation/Restriction for Crank-Nicolson

Prolongation from time-level l to time-level $l+1$:

$$y^{l+1} := P_p y^l, \qquad \lambda^{l+1} := P_d \lambda^l$$

Restriction from time-level $l+1$ to time-level l:

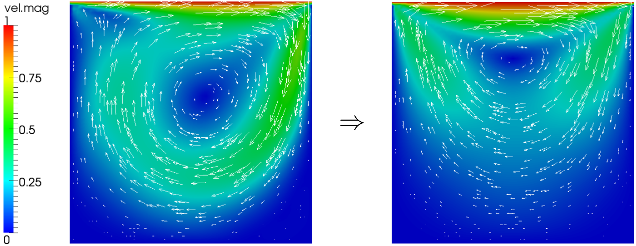
$$d_y^l := R_p d_y^{l+1}, \qquad d_\lambda^l := R_d d_\lambda^{l+1}$$

with

$$R_p := \frac{1}{2} P_d^T, \quad R_d := \frac{1}{2} P_p^T$$

Numerical example

Distributed control of Navier–Stokes:



- Control stationary Nav.-St.-flow (Re=400) to stationary Stokes-flow
- $\Omega = [0,1]^2$, $[0,T] = [0,1]$, $\alpha = 0.01$, $\gamma = 0$

Solver:

- Newton + Space-time-MG applied to the full KKT-system
- BiCGStab(Block-GS)-Smoother
- Accuracy Newton: 8 digits; accuracy MG: 2 digits

Numerical example

Newton with full space-time multigrid (F-cycle):

			$\tilde{Q}_1/Q_0$/IE			$\tilde{Q}_1/Q_0$/CN		
lv.	#steps	#DOF	#NL	#MG	Time	#NL	#MG	Time
2	16	2688	4	7	133	4	7	224
3	32	10496	4	8	642	4	8	830
4	64	41472	4	9	4275	4	8	4732
5	128	164864	4	10	28789	5	11	40875

- Problem size increases by factor ≈ 8
- #nonlinear steps, #multigrid steps $\approx$ constant
- Time increases by factor $\leq$ 8-9

$\Rightarrow$ linear complexity!
$\Rightarrow$ complexity CN $\approx$ complexity IE

The role of the MG cycle

Choice of the MG cycle + hierarchy is crucial for the complexity!

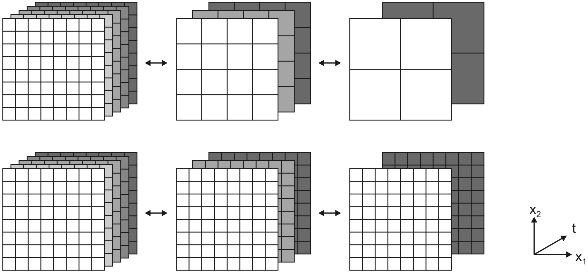
Example (for simplicity): <u>Heat equation</u>

$$y_t - \Delta y = -\frac{1}{\alpha}\lambda \quad \text{in } Q$$
$$-\lambda_t - \Delta\lambda = (y - z) \quad \text{in } Q$$

Discr.: Space=Q_1, Time=IE or CN. Refinement/Coarsening:

1.) Space-time MG

2.) Pure time MG



The role of the MG cycle

- Time for NSM smoothing steps on level n: T_s^n.
- Total time for smoothing per MG step on level 2,...,n:

a) V-cycle: $T_s(2, ..., n) = T_s^n + T_s^{n-1} + T_s^{n-2} + ... + T_s^2$

b) W-cycle: $T_s(2, ..., n) = T_s^n + 2T_s^{n-1} + 4T_s^{n-2} + ... + 2^{n-2}T_s^2$

- Relationship of smoother time between levels (2D):

a) Full space-time multigrid: $T_s^{n-1} \approx 1/8\ T_s^n$

b) Time-multigrid: $T_s^{n-1} \approx 1/2\ T_s^n$

- Total numerical effort for smoothing $T_s(2, ..., n)$:

	space-time MG	pure time-MG
V-cycle	$\leq 8/7\ T_s^n$	$\leq 2\ T_s^n$
W-cycle	$\leq 4/3\ T_s^n$	$\approx (n-1)\ T_s^n$

The role of the MG cycle

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The role of the MG cycle

In practise:

- $\Omega = [0,1]^2$, $[0,T] = [0,1]$, $\alpha = 0.001$, $\gamma = 1000$.
- analyt. solution: $y = t^2(1-t)^2 x_1$; λ, z appropriately.
- space=Q_1, time=IE
- MG-solver with BiCGStab(Block-GS)-smoother.

	Full space-time MG				Time-MG			
	V-cycle		W-cycle		V-cycle		W-cycle	
	#ite	time	#ite	time	#ite	time	#ite	time
$T_s(5,...,5)$	4	245	4	245	3	181	3	182
$T_s(4,...,5)$	4	268	4	291	3	272	3	366
$T_s(3,...,5)$	4	269	4	298	3	320	3	531
$T_s(2,...,5)$	4	270	4	300	3	345	3	663
	$8/7 \approx \frac{270}{245}$		$4/3 \approx \frac{300}{245}$		$2 \approx \frac{345}{181}$		$n-1 \approx \frac{663}{182}$	

Summary & Outlook

Summary:

- CN works fine with space-time MG.
  - special time discretisation $\rightarrow$ special restriction
- MG-cycle analysis: MG-cycle + mesh hierarchy crucial.
  - Do not combine time-MG with W-cycle!

Next steps:

- Constrained control.
- Tests with higher order FEM in space (Q_2,...).
- More complex flow geometries and problems.

The role of the MG cycle

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Example (for simplicity): Heat equation

$$\begin{aligned} y_t - \Delta y &= -\frac{1}{\alpha}\lambda && \text{in } Q \\ -\lambda_t - \Delta\lambda &= (y - z) && \text{in } Q \end{aligned}$$

Discr.: Space=Q_1, Time=IE or CN. Refinement/Coarsening:

Variant 1: Full space-time MG

lv.	#steps	#DOF
1	5	50
2	10	162
3	20	578
4	40	2178
5	80	8450

Variant 2: Pure time MG

lv.	#steps	#DOF
1	5	8450
2	10	8450
3	20	8450
4	40	8450
5	80	8450

The influence of the restriction

Testcase:

- Problem: Optimal control of Navier Stokes
- Driven-Cavity configuration, ν=1/400, $\alpha = 0.01$, $\gamma = 0$
- $\tilde{Q}_1/Q_0$ in space, Crank-Nicolson in time
- Target flow: Stokes
- Pure time multigrid. BiCGStab(Block-GS)-Smoother.

			'Modified' projection			'Intuitive' projection		
lv.	#steps	#DOF	#NL	#MG	Time	#NL	#MG	Time
2	16	41472	4	4	1702	5	6	2426
3	32	41472	4	6	3943	4	7	4786
4	64	41472	4	5	5795	4	7	7960
5	128	41472	4	5	9328	4	7	12868